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On a Class of Singular Integral Operators With Rough Kernels

Ahmad Al-Salman

Abstract. In this paper, we study the L^p mapping properties of a class of singular integral operators with rough kernels belonging to certain block spaces. We prove that our operators are bounded on L^p provided that their kernels satisfy a size condition much weaker than that for the classical Calderón–Zygmund singular integral operators. Moreover, we present an example showing that our size condition is optimal. As a consequence of our results, we substantially improve a previously known result on certain maximal functions.

1 Introduction and Statement of Results

Let $\mathbf{R}^n$, $n \geq 2$ be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. For nonzero $y \in \mathbf{R}^n$, we shall let $y' = |y|^{-1}y$. Consider the classical Calderón- Zygmund singular integral operator

$$(T_\Omega f)(x) = \text{p.v.} \int_{\mathbf{R}^n} f(x-y)|y|^{-n}\Omega(y')\,dy, \tag{1.1}$$

where Ω is a homogeneous function of degree zero on $\mathbf{R}^n$ and satisfies $\Omega \in L^1(\mathbf{S}^{n-1})$ and

$$\int_{\mathbf{S}^{n-1}} \Omega(y')\,d\sigma(y') = 0. \tag{1.2}$$

In their celebrated paper [7], Calderón and Zygmund proved that T_Ω is bounded on L^p for all $1 < p < \infty$ provided that $\Omega \in L\log L(\mathbf{S}^{n-1})$. It turns out that $\Omega \in L\log L(\mathbf{S}^{n-1})$ is the most desirable size condition for the L^p boundedness of T_Ω to hold. Subsequently, it was proved by Ricci–Weiss [14] and Connett [9] independently that $\mathbf{T}_\Omega$ is bounded in $L^p(\mathbf{R}^n)$ for every Ω in the Hardy space $\mathbf{H}^1(\mathbf{S}^{n-1})$ and $p \in (1,\infty)$.

To improve previously obtained results, Jiang and Lu introduced a special class of block spaces $B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1})$ (see Section 2 for the definition). Jiang and Lu showed that if $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$, $q > 1$, then the operator $\mathbf{T}_\Omega$ is bounded on $L^2(\mathbf{R}^n)$. Subsequently, the L^p boundedness was proved for all $1 < p < \infty$ [1, 2]. In a more recent paper [3], Al-Qassem, Al-Salman, and Pan showed that the L^p boundedness of $\mathbf{T}_\Omega$ may fail at any p if the condition $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$ is replaced by $\Omega \in B_q^{0,\nu}(\mathbf{S}^{n-1})$ for any $-1 < \nu < 0$.

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In [8], Chen and Lin introduced the following maximal function:

$$\mathcal{M}_{\Omega,K}(f)(x) = \sup_{h\in K} |(T_{\Omega,h} f)(x)|, \tag{1.3}$$

where K is the class of all functions $h \in L^2(\mathbf{R}^+, r^{-1}\,dr)$ with $\|h\|_{L^2(\mathbf{R}^+ r^{-1}\,dr)} \le 1$ and

$$(T_{\Omega,h} f)(x) = \text{p.v.} \int_{\mathbf{R}^n} f(x-y)|y|^{-n}\Omega(y')h(|y|)\,dy. \tag{1.4}$$

Chen and Lin proved the following result:

Theorem 1.1 ([8]) *Suppose that $\Omega \in \mathcal{C}(\mathbf{S}^{n-1})$ and satisfies* (1.2). *Then the operator $\mathcal{M}_{\Omega,K}$ is bounded on $L^p(\mathbf{R}^n)$ for any $p > 2n/(2n-1)$.*

It turns out that the condition $\Omega \in \mathcal{C}(\mathbf{S}^{n-1})$ can be substantially weakened; as seen in Theorem 1.4 below.

The main purpose of this paper is studying the L^p mapping properties of the operators $T_{\Omega,h}$ in (1.4) with $h \in L^2(\mathbf{R}^+, r^{-1}\,dr)$ and functions Ω satisfy a condition much weaker than $\Omega \in B_q^{0,0}(\mathbf{S}^{n-1})$. More specifically, we shall show that the operators $T_{\Omega,h}$ in (1.4) do not obey the size condition limitation given by Al-Qassem, Al-Salman, and Pan for the classical Calderón–Zygmund singular integral operators [3]. In order to state our results, we let S_Ω be the operator defined by

$$S_\Omega(f)(x) = \Big(\int_0^\infty \Big|\int_{\mathbf{S}^{n-1}} \Omega(y')f(x-ry')\,d\sigma(y')\Big|^2 r^{-1}\,dr\Big)^{\frac{1}{2}}. \tag{1.5}$$

Clearly, if $h \in L^2(\mathbf{R}^+, r^{-1}\,dr)$, then $|T_{\Omega,h}(x)| \le \|h\|_{L^2(\mathbf{R}^+, r^{-1}\,dr)} S_\Omega(f)(x)$. We have the following:

Theorem 1.2 *Suppose that $\Omega \in B_q^{0,-\frac{1}{2}}(\mathbf{S}^{n-1})$ and satisfies* (1.2). *Then*

$$\|S_\Omega(f)\|_p \le C_p \|f\|_p \quad \text{for } 2 \le p < \infty. \tag{1.6}$$

As a consequence of Theorem 1.2, the observation right after (1.5), and duality, we immediately obtain the following result:

Corollary 1.3 *Suppose that $\Omega \in B_q^{0,-\frac{1}{2}}(\mathbf{S}^{n-1})$ and satisfies* (1.2). *Suppose also that $h \in L^2(\mathbf{R}^+, dr/r)$. Then the singular integral operator $T_{\Omega,h}$ is bounded on $L^p(\mathbf{R}^n)$ for all $1 < p < \infty$.*

By comparing the result in Corollary 1.3 with that given in [3] for the classical Calderón–Zygmund singular integral operator T_Ω, we conclude that the class of the operators $T_{\Omega,h}$ behave quite differently from the class of the classical Calderón–Zygmund singular integral operators.

Concerning the condition $\Omega \in B_q^{0,-\frac{1}{2}}(\mathbf{S}^{n-1})$ in Theorem 1.2, we have the following:

Theorem 1.4 *There exists an* Ω *which lies in* $B_q^{0,-\frac{1}{2}-\varepsilon}(\mathbf{S}^{n-1})$ *for all* $\varepsilon > 0$ *and satisfies* (1.2) *such that the* S_Ω *is not bounded on* $L^2(\mathbf{R}^n)$.

As a consequence of Theorem 1.2, Theorem 1.4, and a duality argument in [8], we obtain the following improvement of Theorem 1.1:

Corollary 1.5 *Suppose that* $\Omega \in B_q^{0,-\frac{1}{2}}(\mathbf{S}^{n-1})$ *and satisfies* (1.2)*. Then the operator* $\mathcal{M}_{\Omega,K}$ *is bounded on* $L^p(\mathbf{R}^n)$ *for any* $p \geq 2$*. Moreover, the condition* $\Omega \in B_q^{0,-\frac{1}{2}}(\mathbf{S}^{n-1})$ *is optimal.*

Throughout this paper the letter C will stand for a constant that may vary at each occurrence, but it is independent of the essential variables. Also, we shall use $\exp(\cdot)$ to denote $e^{(\cdot)}$.

2 Main Lemma and Definition of Block Spaces

Lemma 2.1 *Suppose that* $a \geq 2$, $q > 1$, $\mathbf{b} \in L^1(\mathbf{S}^{n-1})$ *and satisfying* (1.2)*. Suppose also that* $\{\psi_{j,a} : j \in \mathbf{Z}\}$ *is a sequence of radial functions defined on* $\mathbf{R}^n$*. If*

- (i) $\hat{\psi}_j$ *is supported in the interval* $A_{j,a} = \{\xi \in \mathbf{R}^n : 2^{-a(j+1)} \leq |\xi| \leq 2^{-a(j-1)}\}$ *and* $0 \leq \hat{\psi}_j \leq 1$;
- (ii) $\|(\sum_{k\in\mathbf{Z}} |\psi_{j,a} * f|^2)^{\frac{1}{2}}\|_p \leq C_p\|f\|_p$ *for all* $1 < p < \infty$ *with constant* C_p *independent of a;*
- (iii) $\|\mathbf{b}\|_q \leq 2^a$ *and* $\|\mathbf{b}\|_1 \leq 1$.

Then the square function

$$E_{a,j}(f)(x) = \Big(\sum_{k\in\mathbf{Z}} \int_1^{2^a} \Big| \int_{\mathbf{S}^{n-1}} \mathbf{b}(y')(\psi_{j+k,a} * f)(x - 2^{ak} r y')\, d\sigma(y')\Big|^2 r^{-1}\, dr\Big)^{\frac{1}{2}} \tag{2.1}$$

satisfies

$$\|E_{a,j}(f)\|_p \leq \sqrt{a}C_p 2^{-\alpha|j|}\|f\|_p \tag{2.2}$$

for all $2 \leq p < \infty$ *with constants* C_p *and* α *independent of j and the parameter a.*

Proof We shall combine the method developed in [5] with some ideas from [4, 8]. We start by estimating $\|E_{a,j}(f)\|_2$. By Plancherel's theorem and Fubini's theorem, we have

$$\|E_{a,j}(f)\|_2^2 \leq \sum_{k\in\mathbf{Z}} \int_{A_{j,a}} |\hat{f}(\xi)|^2 \mathbf{J}_{a,k}(\xi)\, d\xi, \tag{2.3}$$

where

$$\mathbf{J}_{a,k}(\xi) = \int_1^{2^a} \Big| \int_{\mathbf{S}^{n-1}} \mathbf{b}(y') \exp\big(-i2^{ak}(\xi \cdot y')r\big)\, d\sigma(y')\Big|^2 r^{-1}\, dr. \tag{2.4}$$

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By the cancellation property of **b** and (iii), we immediately obtain

$$\mathbf{J}_{a,k}(\xi) \le 2^{2a} a|2^{ak}\xi|^2; \tag{2.5}$$

which when interpolated with the trivial estimate $\mathbf{J}_{a,k}(\xi) \le a$, implies that

$$\mathbf{J}_{a,k}(\xi) \le 4a|2^{ak}\xi|^{\frac{2}{a}}. \tag{2.6}$$

On the other hand, by (iii), it is easy to see that

$$\mathbf{J}_{a,k}(\xi) \le \sup_{z'\in \mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} |\mathbf{b}(y')| \Big| \int_1^{2^a} \exp\Big(-i2^{ak}\big(\xi\cdot(y'-z')r\big)\Big) r^{-1}\,dr\Big|\, d\sigma(y'). \tag{2.7}$$

Now, it is straightforward to show that

$$\Big|\int_1^{2^a} \exp\Big(-i2^{ak}\big(\xi\cdot(y'-z')r\big)\Big) r^{-1}\,dr\Big| \le a\min\Big\{1, \big|2^{ak}\big(\xi\cdot(y'-z')\big)\big|^{-1}\Big\}. \tag{2.8}$$

This implies that

$$\Big|\int_1^{2^a} \exp\Big(-i2^{ak}\big(\xi\cdot(y'-z')r\big)\Big) r^{-1}\,dr\Big| \le a\big|2^{ak}\big(\xi\cdot(y'-z')\big)\big|^{-\frac{1}{2q'}}. \tag{2.9}$$

Therefore, by (2.9), (2.7), Hölder's inequality, and (iii), we get

$$\mathbf{J}_{a,k}(\xi) \le a2^a C|2^{ak}\xi|^{-\frac{1}{2q'}}; \tag{2.10}$$

which when interpolated with the estimate $\mathbf{J}_{a,k}(\xi) \le a$ implies that

$$\mathbf{J}_{a,k}(\xi) \le 2aC|2^{ak}\xi|^{-\frac{1}{2aq'}}. \tag{2.11}$$

Combining (2.6) and (2.11) along with the support property in (i), (2.3) immediately implies that

$$\|E_{a,j}(f)\|_2 \le \sqrt{a}C2^{-|j|}\|f\|_2. \tag{2.12}$$

Next, for $p \ge 2$, there exists $g \in L^{(p/2)'}$ with $\|g\|_{(p/2)'} = 1$ such that

$$\begin{aligned}
&\|E_{a,j}(f)\|_p^2 \\
&\quad = \int_{\mathbf{R}^n} \sum_{k\in\mathbf{Z}} \int_1^{2^a} \Big|\int_{\mathbf{S}^{n-1}} \mathbf{b}(y')(\psi_{j+k,a} * f)(x-2^{ak}ry')\,d\sigma(y')\Big|^2 r^{-1}\,dr|g(x)|\,dx \\
&\quad \le \|\mathbf{b}\|_1 \sum_{k\in\mathbf{Z}} \int_{\mathbf{R}^n} |(\psi_{j+k,a} * f)(z)|^2 \Big\{\sup_{k\in\mathbf{Z}} \int_{2^{ak}<|y|\le 2^{a(k+1)}} |\mathbf{b}(y)|\,|g(z+y)|\frac{dy}{|y|^n}\Big\}\,dz \\
&\quad \le C\Big\|\Big(\sum_{k\in\mathbf{Z}} |\psi_{j,a} * f|^2\Big)^{\frac{1}{2}}\Big\|_p^2 \Big\|\sup_{k\in\mathbf{Z}} \int_{2^{ak}<|y|\le 2^{a(k+1)}} |\mathbf{b}(y)|\,|g(z+y)|\frac{dy}{|y|^n}\Big\|_{(p/2)'};
\end{aligned}$$

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which when combined with (ii), (i), and a theorem in [16, p. 477], implies that

$$\|E_{a,j}(f)\|_p \le C\sqrt{a}\|f\|_p. \tag{2.13}$$

Hence the proof is complete by (2.12), (2.13), and an interpolation argument. ■

Now, we recall the definition of block spaces introduced by Jiang and Lu [13]:

Definition 2.2 (1) For $x_0' \in \mathbf{S}^{n-1}$ and $0 < \theta_0 \le 2$, the set $B(x_0', \theta_0) = \{x' \in \mathbf{S}^{n-1} : |x' - x_0'| < \theta_0\}$ is called a *cap on* $\mathbf{S}^{n-1}$.

(2) For $1 < q \le \infty$, a measurable function b is called a *q-block on* $\mathbf{S}^{n-1}$ if b is a function supported on some cap $I = B(x_0', \theta_0)$ with $\|b\|_{L^q} \le |I|^{-\frac{1}{q'}}$ where $|I| = \sigma(I)$ and $1/q + 1/q' = 1$.

(3) $B_q^{\kappa,\upsilon}(\mathbf{S}^{n-1}) = \{\Omega \in L^1(\mathbf{S}^{n-1})$: $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where each c_μ is a complex number; each b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$; and $M_q^{\kappa,\upsilon}(\{c_\mu\}, \{I_\mu\}) = \sum_{\mu=1}^{\infty} |c_\mu|\big(|1+\phi_{\kappa,\upsilon}(|I_\mu|)\big) < \infty$, where $\phi_{\kappa,\upsilon}(t) = \int_t^1 u^{-1-\kappa} \log^{\upsilon}(u^{-1})\, du$ if $0 < t < 1$ and $\phi_{\kappa,\upsilon}(t) = 0$ if $t \ge 1\}$.

Notice that $\phi_{\kappa,\upsilon}(t) \sim t^{-\kappa} \log^{\upsilon}(t^{-1})$ as $t \to 0$ for $\kappa > 0$, $\upsilon \in \mathbf{R}$, and $\phi_{0,\upsilon}(t) \sim \log^{\upsilon+1}(t^{-1})$ as $t \to 0$ for $\upsilon > -1$. Moreover, among many properties of block spaces [12], we cite the following which are closely related to our work:

$$B_q^{0,0} \subset B_q^{0,-\frac{1}{2}} \quad (q > 1);$$

$$B_{q_2}^{0,\upsilon} \subset B_{q_1}^{0,\upsilon} \quad (1 < q_1 < q_2);$$

$$L^q(\mathbf{S}^{n-1}) \subseteq B_q^{0,\upsilon}(\mathbf{S}^{n-1}) \quad (\text{for } \upsilon > -1);$$

$$\bigcup_{q>1} B_q^{0,\upsilon}(\mathbf{S}^{n-1}) \not\subseteq \bigcup_{p>1} L^p(\mathbf{S}^{n-1}) \quad \text{for any } \upsilon > -1.$$

3 Proof of Main Results

Proof of Theorem 1.2 Assume that $\Omega \in B_q^{0,-\frac{1}{2}}(\mathbf{S}^{n-1})$, $q > 1$. Then $\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu$ where each c_μ is a complex number; each b_μ is a q-block supported on a cap I_μ on $\mathbf{S}^{n-1}$; and

$$M_q^{0,-\frac{1}{2}}(\{c_\mu\}, \{I_\mu\}) = \sum_{\mu=1}^{\infty} |c_\mu|\big(1 + \log^{\frac{1}{2}}(|I_\mu|^{-1})\big) < \infty. \tag{3.1}$$

For each block function $b_\mu(\cdot)$, let $\bar{b}_\mu(x) = b_\mu(x) - \int_{\mathbf{S}^{n-1}} b_\mu(u)\, du$. Then it is straightforward to show that $\bar{b}_\mu$ satisfies the cancellation property (1.2) and condition (ii) in Lemma 2.1. Moreover, $\Omega = \sum_{\mu=1}^{\infty} c_\mu \bar{b}_\mu$, which immediately implies

$$S_\Omega f(x) \le \sum_{\mu=1}^{\infty} c_\mu S_{\bar{b}_\mu} f(x), \tag{3.2}$$

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where $S_{\bar{b}_\mu}$ is given by (1.5) with Ω is replaced by $\bar{b}_\mu$. Thus, by (3.1) and (3.2), it suffices to prove the following inequality:

$$\|S_{\bar{b}_\mu} f\|_p \le \big(1 + \log^{\frac{1}{2}}(|I_\mu|^{-1})\big) C_p \|f\|_p \tag{3.3}$$

for all $2 \le p < \infty$ with constant C_p independent of μ. However, this follows by applying Lemma 2.1. We argue as follows:

Given $\bar{b}_\mu$, let $a = 2$ if $|I_\mu| \ge 2^{q'} e^{-2q'}$ and $a = \log 2 |I_\mu|^{-\frac{1}{q'}}$ if $|I_\mu| < 2^{q'} e^{-2q'}$. By an elementary procedure [4], choose a collection of $\mathcal{C}^\infty$ functions $\{\omega_{j,a}\}_{j\in\mathbf{Z}}$ on $(0,\infty)$ with the properties: $\operatorname{supp}(\omega_{j,a}) \subseteq [2^{-a(j+1)}, 2^{-a(j-1)}]$, $0 \le \omega_{j,a} \le 1$, $\sum_{j\in\mathbf{Z}} \omega_{j,a}(u) = 1$, and $|\frac{d^s\omega_{j,a}}{du^s}(u)| \le C_s u^{-s}$ with constants C_s independent of a. Therefore,

$$\begin{aligned} S_{\bar{b}_\mu}(f)(x) &\le \Big(\sum_{k\in\mathbf{Z}} \int_1^{2^a} \Big| \sum_{j\in\mathbf{Z}} \int_{\mathbf{S}^{n-1}} \bar{b}_\mu(y')(\psi_{j+k,a} * f)(x - 2^{ak} r y')\, d\sigma(y') \Big|^2 r^{-1}\, dr \Big)^{\frac{1}{2}} \\ &\le \sum_{j\in\mathbf{Z}} E_{a,j}(f)(x), \end{aligned} \tag{3.4}$$

where $E_{a,j}$ is the operator given in (2.1) with $\mathbf{b}$ is replaced by $\bar{b}_\mu$. Moreover, by the properties of $\{\omega_{j,a}\}_{j\in\mathbf{Z}}$, it follows that condition (iii) holds by Littlewood–Paley theory with L^p constants independent of the parameter a (for details see [4], [15]). Hence, by Lemma 2.1 and (3.4), we obtain (3.3). This completes the proof.

Now, we prove Theorem 1.4.

Proof of Theorem 1.4 By Plancherel's theorem, it is easy to see that S_Ω is bounded on L^2 if the multiplier

$$m_\Omega(\xi) = \int_0^\infty \Big| \int_{\mathbf{S}^{n-1}} \exp(-ir\xi \cdot y')\Omega(y')\, d\sigma(y') \Big|^2 r^{-1}\, dr$$

is uniformly bounded. By the cancellation property of Ω and a simple limiting process, it can be easily seen that

$$\begin{aligned} m_\Omega(\xi) = \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} & \Omega(y')\overline{\Omega(z')} \\ & \Big\{ \log|\xi' \cdot (y' - z')|^{-1} - i\frac{\pi}{2} \operatorname{sgn}\big(\xi' \cdot (y' - z')\big) \Big\}\, d\sigma(y')\, d\sigma(z'). \end{aligned}$$

By restricting Ω to be real, we obtain

$$\mathcal{R}(m_\Omega)(\xi) = \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \Omega(y')\Omega(z') \log|\xi' \cdot (y' - z')|^{-1}\, d\sigma(y')\, d\sigma(z'),$$

where $\mathcal{R}(m_\Omega)$ denotes the real part of m_Ω. Therefore, to prove the result of Theorem 1.4, it suffices to construct a real $\Omega \in B_q^{0,-1/2-\varepsilon}(\mathbf{S}^{n-1})$ for all $\varepsilon > 0$ and satisfies

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(1.2) such that $\mathcal{R}(m_\Omega)$ is not an L^∞ function. For sake of simplicity, we shall construct Ω on $\mathbf{S}^1$ and assuming $q = \infty$. Also, we shall work on the interval $[-1, 1]$ and follow the similar ideas developed in ([4]).

For $k \in \mathbf{N}$, let $I_k = \big[1/(k+1), 1/k\big)$ and let $C_\Omega = \sum_{k=3}^{\infty}(k+1)^{-1}(\log k)^{-\frac{3}{2}}$. Define Ω on $[-1, 1]$ by

$$\Omega(u) = \sum_{k=3}^{\infty} k(\log k)^{-\frac{3}{2}} \chi_{I_k} - C_\Omega \chi_{[-1,0]}, \tag{3.5}$$

where χ_{I_k} is the characteristic function of the interval I_k. Then, clearly

$$\Omega \in B_{\infty}^{0,-1/2-\varepsilon}([-1, 1])$$

for all $\varepsilon > 0$. Moreover, the following holds:

$$\int_{-1}^{1} \Omega(u)\, du = 0. \tag{3.6}$$

On the other hand, by noticing that the sum $\sum_{k=3}^{\infty} k(\log k)^{-\frac{3}{2}}(1 + \log^{\frac{1}{2}}(|I_k|^{-1}))$ is divergent, one can easily verify that $\Omega \notin B_{\infty}^{0,-1/2}$.

Finally, we show that $|\mathcal{R}(m_\Omega)(\xi)| = \infty$, *i.e.*,

$$\iint_{[-1,1]^2} \Omega(u)\Omega(v) \log|u - v|^{-1}\, du\, dv = \infty. \tag{3.7}$$

To this end, we break the integral over $[-1, 1]^2$ into two terms: the first is the integral over $[-1, 1]^2 \backslash [0, 1]^2$ and the second one is the integral over $[0, 1]^2$. Since the integral over $[-1, 1]^2 \setminus [0, 1]^2$ is clearly finite, we conclude that (3.7) holds if and only if the integral over $[0, 1]^2$ is infinite. But, the latter is indeed infinite. To see this, notice that

$$\begin{aligned}
&\iint_{[0,1]^2} \Omega(u)\Omega(v) \log|u - v|^{-1}\, du\, dv \\
&\qquad \geq C \sum_{k=3}^{\infty} k(\log k)^{-\frac{3}{2}} \Big\{ \sum_{j=k+1}^{\infty} j(\log j)^{-\frac{3}{2}} \int_{D_k}\int_{D_j} \log|u - v|^{-1}\, du\, dv \Big\} \\
&\qquad \geq \sum_{k=3}^{\infty} (k + 1)^{-1}(\log k)^{-1} = \infty.
\end{aligned}$$

This completes the proof.

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Department of Mathematics
Yarmouk University
Irbid
Jordan
email: alsalman@yu.edu.jo