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FUNCTIONS OF BOUNDED KTH VARIATION AND ABSOLUTELY KTH CONTINUOUS FUNCTIONS

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Functions of bounded kth variation and absolutely kth continuous functions are considered on sets and various properties are studied.

1. INTRODUCTION

Following the approach of Russell [7] we have introduced the concepts of bounded kth variation and of absolutely kth continuity of a function defined on a set. (For a different approach we refer to [6]). The concept of a generalised Lipschitz condition of order k is also introduced. It is shown (Theorem 5) that the family of functions satisfying a generalised Lipschitz condition of order k is a proper subfamily of the family of absolutely kth continuous functions and that the family of absolutely kth continuous functions is a proper subfamily of the family of functions of bounded kth variation. Other properties are also studied. For related work in this area we refer to [2, 10]. It is worth mentioning that [5] studied various properties of functions of bounded kth variation over sets. Also [3] studied properties of kth absolutely continuous functions. Unfortunately the results of these papers have serious deficiencies in their proofs. In fact, in the former paper, Theorem 2.2, on which most of the results depend, is false and in the latter paper all the results depend directly or indirectly on results of another paper which needs correction (see MR 87j : 26011).

2. DEFINITIONS AND NOTATIONS

Let f be a real valued function defined on a set E. The kth divided difference of f at the $(k+1)$ distinct points $x_0, x_1, \ldots, x_k$ in E is defined by

$$Q_k(f; x_0, x_1, \ldots, x_k) = \sum_{i=0}^{k} \frac{f(x_i)}{\omega'(x_i)}$$

where $\omega(x) = \prod_{j=0}^{k} (x - x_j)$.

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From the definition it follows that

$$(x_0 - x_k)Q_k(f; x_0, x_1, \dots, x_k) = Q_{k-1}(f; x_0, \dots, x_{k-1}) - Q_{k-1}(f; x_1, \dots, x_k).$$

Clearly $Q_k(f; x_0, x_1, \dots, x_k)$ is independent of the order of $x_0, x_1, \dots, x_k$.

Let $c, d \in E$, $c < d$. The oscillation of f on $[c, d] \cap E$ of order k is defined to be

$$O_k(f, [c, d] \cap E) = \operatorname{Sup} |(d - c)Q_k(f, c, x_1, \dots, x_{k-1}, d)|$$

where 'Sup' is taken over all possible choices of the points $x_1, x_2, \dots, x_{k-1}$ in $(c, d) \cap E$. We shall take $O_k(f, [c, d] \cap E)$ to be zero, if $(c, d) \cap E$ contains less than $(k - 1)$ points. The weak variation of f on E of order k, denoted by $V_k(f, E)$, is the upper bound of the sums $\sum O_k(f, [c_i, d_i] \cap E)$, the upper bound being taken over all sequences $\{(c_i, d_i)\}$ of nonoverlapping intervals with end points belonging to E.

Definition 1: If $V_k(f, E) < \infty$ then f is said to be of bounded kth variation in the wide sense on E and is written $f \in BV_k(E)$.

Definition 2: If for every $\varepsilon > 0$ there is $\sigma(\varepsilon) > 0$ such that for every sequence of non-overlapping intervals $\{(c_v, d_v)\}$ with end points on E and with $\sum (d_v - c_v) < \sigma$, we have $\sum O_k(f; [c_v, d_v) \cap E) < \varepsilon$ then f is said to be absolutely kth continuous on E in the wide sense and is written $f \in AC_k(E)$.

Definition 3: If $Q_k(f; x_0, \dots x_k)$ remains bounded for all possible choices of points $x_0, x_1, \dots x_k$ on E, then f is said to satisfy a generalised Lipschitz condition of order k and is written $f \in BQ_k(E)$.

To justify Definition 3 note that a function f is said to satisfy a Lipschitz condition of order k on a set E if there is M such that

$$|f(x_1) - f(x_2)| \leqslant M |x_1 - x_2|^k \quad \text{for } x_1, x_2 \in E. \tag{2.1}$$

Suppose that (2.1) holds. Then for any three points x_1, x_2, x_3 of E we have, when $x_1 < x_2 < x_3$,

$$\begin{aligned} |Q_2(f; x_1, x_2, x_3)| &= \left| \frac{Q_1(f; x_1, x_2) - Q_1(f; x_2, x_3)}{x_1 - x_3} \right| \\ &\leqslant \frac{M |x_1 - x_2|^{k-1} + M |x_2 - x_3|^{k-1}}{|x_1 - x_3|} \\ &\leqslant 2M |x_1 - x_3|^{k-2}. \end{aligned} \tag{2.2}$$

Proceeding in this way we get after k steps

$$|Q_k(f; x_1, \dots, x_{k+1})| \leqslant 2^{k-1} M$$

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which shows that $f \in BQ_k(E)$. Thus Definition 3 is a generalisation of the usual concept of a Lipschitz condition of order k.

It is clear from the definitions that if $E \subset F$ and if $f \in BV_k(F)$ (respectively $AC_k(F)$, $BQ_k(F)$), then $f \in BV_k(E)$ (respectively $AC_k(E)$, $BQ_k(E)$).

DEFINITION 4: (see [4, p.280]). Let $x \in E$ be a limit point of E. If there exist real numbers $f_r(x, E)$, $1 \leqslant r \leqslant n$ such that

$$f(x+h) = f(x) + \ldots + \sum_{r=1}^{n} \frac{h^r}{r!} f_r(x, E) + \frac{h^n}{n!} \varepsilon_n(x, h)$$

where $\varepsilon_n(x, h) \to 0$ as $h \to 0$ with $x + h \in E$, then $f_n(x, E)$ is called the Peano derivative of f at x relative to E of order n.

If $f_r(x, E)$ exists we shall write

$$\gamma_r(f, x, t, E) = \frac{r!}{(t-x)^r}\left[f(t) - \sum_{i=0}^{r-1} \frac{(t-x)^i}{i!} f_i(x, E)\right].$$

If $x_1, x_2, \ldots x_s, \ldots x_k$ are distinct points of E and if x_s is a limit point of E then we write

$$Q_k(f; x_1, \ldots x_s, x_s \ldots x_k) = \lim_{\xi \to x_s} Q_k(f; x_1, \ldots x_s, \xi, \ldots x_k)$$

provided the limit exists where the limit is taken over E.

PRELIMINARY RESULTS

THEOREM 1. *For all choices of points $x_0, \ldots x_k$ in E, $(x_k - x_0)Q_k(f; x_0, \ldots x_k)$ remains bounded if and only if $Q_{k-1}(f; x_0, \ldots, x_{k-1})$ is bounded for all choices of points $x_0, \ldots x_{k-1}$ in E.*

PROOF: A proof is given in [7, Theorem 4]. However for completeness we give a different proof.

Let $(x_k - x_0)Q_k(f; x_0, \ldots, x_k)$ be bounded for all choices of points $x_0, \ldots, x_k$ in E. Let $a_0, \ldots, a_{k-1}$ be a fixed collection of points in E and let $A = |Q_{k-1}(f; a_0, \ldots, a_{k-1})|$.

Let M be such that

$$|(x_k - x_0)Q_k(f; x_0, \ldots, x_k)| \leqslant M,$$

for all choices of points $x_0, \ldots, x_k$ in E. Now we claim that

$$|Q_{k-1}(f; x_0, \ldots, x_{k-1})| \leqslant kM + A \tag{3.1}$$

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for all choices of points $x_0, \ldots, x_{k-1}$ in E. In fact, if $x_i = a_i$, for all i, $0 \leqslant i \leqslant k-1$, then (3.1) is clear. If $x_i = a_i$, for all i except $i = j$, then

$$\begin{aligned}|Q_{k-1}(f; x_0, \ldots, x_{k-1}) - Q_{k-1}(f; a_0, \ldots, a_{k-1})| \\ = |(x_j - a_j)Q_k(f; x_0, \ldots, x_{k-1}, a_j)| \leqslant M\end{aligned}$$

and so (3.1) is true. In general if there are m indices $i_1, \ldots, i_m$ such that $a_{i_n} \neq x_{i_n}$, for $n = 1, \ldots, m$, then

$$\begin{aligned}
&|Q_{k-1}(f; x_0, \ldots, x_{i_1}, \ldots, x_{i_m}, \ldots, x_{k-1}) - Q_{k-1}(f; a_0, \ldots, a_{i_1}, \ldots, a_{i_m}, \ldots, a_{k-1})| \\
&\leqslant |Q_{k-1}(f; x_0, \ldots, x_{i_1}, \ldots, x_{i_m}, \ldots, x_{k-1}) - Q_{k-1}(f; x_0, \ldots, a_{i_1}, x_{i_2}, \ldots, x_{i_m}, \\
&\qquad \ldots, x_{k-1})| \\
&+ |Q_{k-1}(f; x_0, \ldots, a_{i_1}, x_{i_2}, \ldots, x_{i_m}, \ldots, x_{k-1}) - Q_{k-1}(f; x_0, \ldots, a_{i_1}, a_{i_2}, x_{i_3}, \\
&\qquad \ldots, x_{i_m}, \ldots, x_{k-1})| \\
&+ \qquad \cdots \qquad \cdots \qquad \cdots \\
&+ \left|Q_{k-1}(f; x_0, \ldots, a_{i_1}, a_{i_2}, \ldots, a_{i_{m-1}}, x_{i_m}, \ldots x_{k-1}) - Q_{k-1}(f; a_0, \ldots, a_{k-1})\right| \\
&\leqslant \sum_{n=1}^{m} |(x_{i_n} - a_{i_n})Q_k(f; x_0, \ldots, x_{i_n}, \ldots, x_{k-1}, a_{i_n})| \\
&\leqslant m\,M,
\end{aligned}$$

and so (3.1) is true.

Since $(x_k - x_0)Q_k(f; x_0, \ldots, x_k) = Q_{k-1}(f; x_0, \ldots, x_{k-1}) - Q_{k-1}(f; x_1, \ldots, x_k)$, the other part is easy. ∎

Corollary. *If $f \in BV_k(E)$, then $Q_{k-1}(f; x_0, \ldots, x_{k-1})$ remains bounded for all choices of the points $x_0, \ldots, x_{k-1}$ in E.*

Proof: Since $f \in BV_k(E)$ implies that $(x_k - x_0)Q_k(f; x_0, \ldots, x_k)$ remains bounded for all choices of the points $x_0, \ldots, x_k$ in E, the result follows from Theorem 1. ∎

Lemma 1. *Let E be a bounded set. If $f \in BV_k(E)$ and c is such that dist.$(E, c) > 0$, then $f \in BV_k(E \cup \{c\})$, $f(c)$ being defined arbitrarily.*

Proof: Since $f \in BV_k(E)$, by the Corollary of Theorem 1, $Q_{k-1}(f; x_0, \ldots, x_{k-1})$ remains bounded for all choices of the points x_i, $0 \leqslant i \leqslant k-1$, on E and hence $f \in BV_{k-1}(E)$. Repeating this argument, $f \in BV_i(E)$, for $i = 1, 2, \ldots, k$. Hence $Q_i(f; x_0, \ldots, x_i)$ remains bounded for all choices of points $x_0, \ldots, x_i$, $1 \leqslant i \leqslant k-1$ and f is bounded on E. Define $f(c)$ arbitrarily. Since dist.$(E, c) = \sigma > 0$,

$$|Q_1(f; x, c)| = \left|\frac{f(x) - f(c)}{x - c}\right| \leqslant \frac{|f(x) - f(c)|}{\sigma}, \text{ for } x \in E.$$

Since f is bounded on E, $Q_1(f;x,c)$ is bounded for $x \in E$. Since

$$|Q_2(f;x_1,x_2,c)| = \left|\frac{Q_1(f;x_1,x_2) - Q_1(f;x_2,c)}{x_1 - c}\right| \leqslant \left|\frac{Q_1(f;x_1,x_2) - Q_1(f;x_2,c)}{\sigma}\right|$$

for all $x_1, x_2 \in E$, and since $Q_1(f;x_1,x_2)$, $Q_1(f;x_2,c)$ are bounded, $Q_2(f;x_1,x_2,c)$ remains bounded.

Repeating this argument, $Q_k(f;x_1,x_2,\ldots,x_k,c)$ is bounded for all $x_1,x_2,\ldots,x_k$ in E. Hence from the definition of BV_k on a set, the lemma follows. □

LEMMA 2. *If $f \in BV_k(E)$ and $c_1, c_2, \ldots, c_n$ are such that* $\inf_i\{\text{dist.}(E, c_i)\} > 0$, *then $f \in BV_k(E \cup \{c_1, \ldots, c_n\})$, f being defined arbitrarily on $\{c_1, c_2, \ldots, c_n\}$.*

This can be deduced by Lemma 1.

LEMMA 3. *Let E and F be bounded sets and let* $\text{dist.}(E,F) > 0$. *If $f \in BV_k(E) \cap BV_k(F)$, then $f \in BV_k(E \cup F)$.*

PROOF: In view of Lemma 2, we suppose that both of E and F are infinite. We may further suppose that each element of E is less than every element of F. Let $\{(c_\nu, d_\nu)\}$ be any sequence of non-overlapping intervals with end points on $E \cup F$. The only case that needs to be considered is $c_\nu \in E$, $d_\nu \in F$ for some ν, since in all other cases

$$\sum_\nu O_k(f;[c_\nu, d_\nu] \cap (E \cup F)) \leqslant V_k(f,E) + V_k(f,F),$$

and hence

$$V_k(f;E \cup F) \leqslant V_k(f,E) + V_k(f,F).$$

So we suppose that $c_p \in E$, $d_p \in F$. We shall prove that $(d_p - c_p)Q_k(f;c_p,\xi_1,\ldots,\xi_{k-1},d_p)$ remains bounded for all choices of the points $\xi_1,\ldots,\xi_{k-1}$ in $(c_p,d_p) \cap (E \cup F)$. Let $\xi_1,\ldots,\xi_{k-1}$ be arbitrarily fixed in $(c_p,d_p) \cap (E \cup F)$ and $\xi_1 < \xi_2 \ldots < \xi_{k-1}$. Let $\xi_i \in E$, $1 \leqslant i \leqslant r$ and $\xi_i \in F$, $r+1 \leqslant i \leqslant k-1$. Choose $\sup E < u_1 < u_2 < \ldots < u_{k-1} < \inf F$ and define $f(u_i)$, $1 \leqslant i \leqslant k-1$ arbitrarily and write $c_p = x_0$; $\xi_i = x_i, 1 \leqslant i \leqslant r$; $u_i = x_{r+i}, 1 \leqslant i \leqslant k-1$; $\xi_i = x_{k-1+i}, r+1 \leqslant i \leqslant k-1$; $d_p = x_{2k-1}$.

Now, by [7, Corollary of Theorem 1], we have

$$\begin{aligned}
&|(d_p - c_p)Q_k(f;c_p,\xi_1,\ldots,\xi_{k-1},d_p)| \\
&= |Q_{k-1}(f;c_p,\xi_1,\ldots,\xi_{k-1}) - Q_{k-1}(f;\xi_1,\ldots,d_p)| \\
&= \left|\sum_{i=0}^{k-1} \alpha_i Q_{k-1}(f;x_i,\ldots,x_{i+k-1}) - \sum_{i=1}^{k} \beta_i Q_{k-1}(f;x_i,\ldots,x_{i+k-1})\right|,
\end{aligned}$$

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where α_i, β_i are positive numbers such that

$$\sum \alpha_i = 1 = \sum \beta_i.$$

Hence

$$\begin{aligned}(d_p - c_p)\,&|Q_k(f; c_p, \xi_1, \ldots, \xi_{k-1}, d_p)| \\ &= |\alpha_0 Q_{k-1}(f; x_0, \ldots, x_{k-1}) + \sum_{i=1}^{k-1}(\alpha_i - \beta_i)Q_{k-1}(f; x_i, \ldots x_{i+k-1}) \\ &\qquad + \beta_k Q_{k-1}(f; x_k, \ldots, x_{2k-1})| \\ &\leqslant |Q_{k-1}(f; x_0, \ldots, x_{k-1})| + \sum_{i=1}^{k-1}|Q_{k-1}(f; x_i, \ldots, x_{i+k-1})| \\ &\qquad + |Q_{k-1}(f; x_k, \ldots, x_{2k-1})| \\ &= \sum_{i=0}^{r}|Q_{k-1}(f; x_i, \ldots, x_{i+k-1})| + \sum_{i=r+1}^{k}|Q_{k-1}(f; x_i, \ldots, x_{i+k-1})|\end{aligned}$$

By Lemma 2, $f \in BV_k(E \cup \{u_1, \ldots u_{k-1}\}) \cap BV_k(F \cup \{u_1, \ldots u_{k-1}\})$. Hence by the Corollary of Theorem 1, $Q_{k-1}(f; x_i, \ldots, x_{i+k-1})$ remains bounded for all choices of points $x_i, \ldots, x_{i+k-1}$ in $E \cup \{u_1, \ldots, u_{k-1}\}$ or in $F \cup \{u_1, \ldots, u_{k-1}\}$ and so there is $M > 0$ such that

$$(d_p - c_p)\,|Q_k(f; c_p, \xi_1, \ldots, \xi_{k-1}, d_p)| \leqslant (r+1)M + (k-r)M = (k+1)M$$

Hence

$$\begin{aligned}\sum_{\nu}(d_\nu - c_\nu)\,&|Q_k(f; c_\nu, x_{\nu,0}, \ldots, x_{\nu,k-2}, d_\nu)| \\ &\leqslant V_k(f, E) + (k+1)M + V_k(f, F).\end{aligned}$$

Hence $f \in BV_k(E \cup F)$. ∎

Lemma 4. *If $f \in BV_k(E \cap [a, c]) \cap BV_k(E \cap [c, b])$, where c is isolated at least from one side of E, then $f \in BV_k(E \cap [a, b])$.*

Proof: Since c is isolated at least from one side, dist.$(E \cap [a, c), E \cap [c, b]) > 0$, and so by Lemma 3,

$$f \in BV_k(E \cap [a, b]).$$

∎

Lemma 5. *Let $a < c < b$, where c is a two sided limit point of E, (c may or may not belong to E). If $f \in BV_k(E \cap [a, c]) \cap BV_k(E \cap [c, b])$ and if there is $\sigma > 0$ such* *that $(x_k - x_0)Q_k(f; x_0, \dots, x_k)$ is bounded for all choices of points $x_0, x_1, \dots, x_k$ in $(c-\sigma, c+\sigma) \cap E$, then $(\xi_k - \xi_0)Q_k(f; \xi_0, \xi_1, \dots, \xi_k)$ remains bounded for all choices of points ξ_i in $E \cap [a, b]$.*

PROOF: We may suppose that $(c-\sigma, c+\sigma) \subset (a, b)$. Let $\xi_0, \xi_1, \xi_2, \dots, \xi_{k-1}\ \xi_k$ be arbitrary points in $E \cap [a, b]$. Suppose that $\xi_r \leqslant c < \xi_{r+1}$. Choose the points x_i, $r+1 \leqslant i \leqslant r+2k-2$ such that x_i's belong to $(c-\sigma, c) \cap E$ for $r+1 \leqslant i \leqslant r+k-1$ and x_i's belong to $(c, c+\sigma) \cap E$ for $r+k \leqslant i \leqslant r+2k-2$.

Since $(y_k - y_0)Q_k(f; y_0, \dots, y_k)$ remains bounded for all choices of points $y_0, \dots, y_k$ in $(c-\sigma, c+\sigma) \cap E$, we conclude from Theorem 1 that there is $M > 0$ such that

$$\sum_{i=r+1}^{r+k-1} |Q_{k-1}(f; x_i, \dots, x_{i+k-1})| \leqslant (k-1)M. \tag{3.2}$$

Writing $\xi_0 = x_0$, $\xi_i = x_i$, $1 \leqslant i \leqslant r$ and $\xi_{r+j} = x_{r+2k+j-2}$, $1 \leqslant j \leqslant k-r-1$, $\xi_k = x_{3k-2}$, and applying [7, Corollary of Theorem 1], we have

$$\begin{aligned} &|(\xi_k - \xi_0)Q_k(f; \xi_0, \xi_1, \dots, \xi_{k-1}, \xi_k)| \\ &= |Q_{k-1}(f; \xi_0, \xi_1, \dots \xi_{k-1}) - Q_{k-1}(f; \xi_1, \dots, \xi_{k-1}, \xi_k)| \\ &= \left| \sum_{i=0}^{2k-2} \alpha_i Q_{k-1}(f; x_i, \dots, x_{i+k-1}) - \sum_{i=0}^{2k-2} \beta_i Q_{k-1}(f; x_{i+1}, \dots, x_{i+k}) \right| \end{aligned} \tag{3.3}$$

where α_i, β_i are positive numbers such that

$$\sum \alpha_i = 1 = \sum \beta_i,$$

$$\begin{aligned} &= |\alpha_0 Q_{k-1}(f; x_0, \dots, x_{k-1}) + \sum_{i=1}^{2k-2} (\alpha_i - \beta_{i-1}) Q_{k-1}(f; x_i, \dots x_{i+k-1}) \\ &\qquad - \beta_{2k-2} Q_{k-1}(f; x_{2k-1}, \dots, x_{3k-2})| \\ &\leqslant \sum_{i=0}^{r} |Q_{k-1}(f; x_i, \dots, x_{i+k-1})| + \sum_{i=r+1}^{r+k-1} |Q_{k-1}(f; x_i, \dots, x_{i+k-1})| \\ &\qquad + \sum_{i=r+k}^{2k-1} |Q_{k-1}(f; x_i, \dots, x_{i+k-1})|. \end{aligned}$$

Since $f \in BV_k(E \cap [a, c]) \cap BV_k(E \cap [c, b])$, by the Corollary of Theorem 1 $Q_{k-1}(f; y_i, \dots, y_{i+k-1})$ is bounded for all choices of points $y_i, \dots, y_{i+k-1}$ in $E \cap [a, c]$ and similarly for all choices of points in $E \cap [c, b]$ and so there is $N > 0$ such that

$$|Q_{k-1}(f; y_i, \dots, y_{i+k-1})| \leqslant N, \tag{3.4}$$

whenever all of $y_i, \dots y_{i+k-1}$ are in $E \cap [a, c]$, or all of them are in $E \cap [c, b]$. Hence from (3.2), (3.3) and (3.4), we have

$$\begin{aligned}(\xi_k - \xi_0)\,|Q_k(f; \xi_0, \xi_1, \dots, \xi_{k-1}, \xi_k)| &\leqslant (r+1)N + (k-1)M + (k-r)N \\ &\leqslant 2kN + kM.\end{aligned}$$

∎

Lemma 6. *If $f \in BV_k(E \cap [a, c]) \cap BV_k(E \cap [c, b])$, where $a < c < b$ and c is a two sided limit point of E, (c may or may not belong to E) and if there is $\sigma > 0$ such that $(x_k - x_0)Q_k(f; x_0, \dots, x_k)$ is bounded for all choices of points $x_0, \dots, x_k$ in $(c - \sigma, c + \sigma) \cap E$, then $f \in BV_k(E \cap [a, b])$.*

Proof: Let $\{(c_\nu, d_\nu)\}$ be any countable collection of non-overlapping subintervals of $[a, b]$ with end points in $E \cap [a, b]$. The only case that needs consideration is $c_\nu \in E \cap [a, c]$, $d_\nu \in E \cap [c, b]$ for some ν. Let $c_p \in E \cap [a, c]$, $d_p \in E \cap [c, b]$.

In view of Lemma 5, $(d_p - c_p)Q_k(f; c_p, x_{p,1}, x_{p,2}, \dots x_{p,k-1}, d_p)$ is bounded for all choices of the points $x_{p,1}, \dots, x_{p,k-1}$ in $(c_p, d_p) \cap E$. The rest is clear. ∎

Lemma 7. *Let f be defined on E and let $x_0 \in E$ be a limit point of E. If f is continuous at x_0 then*

$$\lim_{x_r \to x_0} \cdots \lim_{x_1 \to x_0} Q_r(f; x_1, \dots, x_{r+1}) = \frac{1}{r!}\gamma_r(f; x_0, x_{r+1}, E)$$

provided $f_{r-1}(x_0, E)$ exists finitely where the limits are taken over E.

The proof is in [1, Lemma 4.1] when E is an interval. The same argument will apply here and so the proof is omitted.

Lemma 8. *If $x_1, x_2, \dots, x_k$ are distinct points of E which are also limit points of E and if $f : E \to R$ is differentiable at these points with respect to E, then*

$$Q_{k-1}\left(f^{(1)}; x_1, \dots, x_k\right) = \sum_{h=1}^{k} Q_k(f; x_1, \dots, x_h, x_h, \dots, x_k),$$

where $f^{(1)}$ denotes the derivative of f with respect to E.

The proof is the same as [7, Theorem 8].

Lemma 9. *If $f \in AC_2(E)$, then $f^{(1)}$ exists on E_0, where E_0 is the set of all limit points of E which are in E, the derivative $f^{(1)}$ being taken with respect to E.*

Proof: Let $\varepsilon > 0$ be arbitrary. Since $f \in AC_2(E)$, there is $\sigma = \sigma(\varepsilon) > 0$ such that for every sequence $\{(c_\nu, d_\nu)\}$ of non-overlapping intervals with end points on E and with $\sum (d_\nu - c_\nu) < \sigma$ and for every choice of points x_ν in $(c_\nu, d_\nu) \cap E$, we have

$$\sum_\nu |(d_\nu - c_\nu)Q_2(f; c_\nu, x_\nu, d_\nu)| < \varepsilon.$$

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Let $\xi \in E_0$. If ξ is a two sided limit point of E, then for $x_1, x_2 \in E \cap (\xi - \sigma/2, \xi + \sigma/2)$, $x_1 < \xi < x_2$, we have

$$\left| \frac{f(x_1) - f(\xi)}{x_1 - \xi} - \frac{f(x_2) - f(\xi)}{x_2 - \xi} \right| < \varepsilon.$$

Letting $x_1 \xi -$, $x_2 \to \xi +$ through E independently, it can be shown, as ε is arbitrary, that $f^{(1)}(\xi)$ exists finitely. So $f^{(1)}$ exists on E_0. If ξ is a one sided limit point of E, say from the right, choose $\xi < x_1 < x_2$ such that $x_1, x_2 \in E \cap (\xi, \xi + \sigma)$ and so

$$\begin{aligned} \left| \frac{f(x_1) - f(\xi)}{x_1 - \xi} - \frac{f(x_2) - f(\xi)}{x_2 - \xi} \right| &= |(x_1 - x_2) Q_2(f; \xi, x_1, x_2)| \\ &\leqslant |(\xi - x_2) Q_2(f; \xi, x_1, x_2)| < \varepsilon \end{aligned}$$

and by Cauchy's criterion $f^{(1)+}(\xi)$ exists finitely. ▯

4. Main results

Theorem 2. *If $Q_{k-1}(f; x_0, \ldots, x_{k-1})$ remains bounded for all choices of points x_i, $0 \leqslant i \leqslant k-1$, on E, then $f \in AC_{k-1}(E)$.*

Proof: From the hypopthesis, there is M such that

$$|Q_{k-1}(f; x_0, \ldots, x_{k-1})| \leqslant M$$

for all choices of the points $x_0, \ldots x_{k-1}$ on E.

Let $\{(c_v, d_v)\}$ be any countable collection of non-overlapping intervals with end points on E and let $\varepsilon > 0$ be arbitrary. Let $c_v = x_{v,0}, x_{v,1}, \ldots, x_{v,k-2}, x_{v,k-1} = d_v$ be distinct points on $[c_v, d_v] \cap E$. Then

$$\sum_v |(d_v - c_v) Q_{k-1}(f; x_{v,0}, \ldots, x_{v,k-1})| \leqslant M \sum_v (d_v - c_v) < \varepsilon$$

if $\sum (d_v - c_v) < \varepsilon/M$. Hence $\sum O_{k-1}(f; [c_v, d_v] \cap E) < \varepsilon$, whenever $\sum (d_v - c_v) < \varepsilon/M$. So $f \in AC_{k-1}(E)$. ▯

Theorem 3. *If $f \in AC_k(E)$, where E is a bounded set, then*

$$f \in BV_k(E).$$

Proof: Let $a = \inf E$, $b = \sup E$. Since $f \in AC_k(E)$, there is $\sigma > 0$ such that for every sequence $\{(c_v, d_v)\}$ of disjoint intervals with end points on E,

$$\sum_v (d_v - c_v) |Q_k(f; c_v, x_{v,1}, \ldots, x_{v,k-1}, d_v)| < 1$$

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whenever $\sum(d_v - c_v) < \sigma$ and $x_{v,1}, \dots, x_{v,k-1}$ are in $E \cap (c_v, d_v)$. So $f \in BV_k(E \cap [c, d])$ whenever $(d - c) < \sigma$.

Let $\overline{E}$ be the closure of E. There are only a finite number of contiguous intervals of $\overline{E}$ whose lengths are greater than or equal to $\sigma/2$. Let $(c_1, d_1), \dots, (c_n, d_n)$ be these intervals. We show that $f \in BV_k(E \cap [d_j, c_{j+1}])$, for each $j = 1, \dots, (n-1)$. If $c_{j+1} - d_j < \sigma$, then $f \in BV_k(E \cap [d_j, c_{j+1}])$. If $c_{j+1} - d_j \geqslant \sigma$, divide the interval $[d_j, c_{j+1}]$ by points $d_j = p_{j,1} < p_{j,2} < \dots < p_{j,m} = c_{j+1}$ such that $p_{j,r} - p_{j,r-1} = 3\sigma/4$ for $r = 2, 3, \dots, (m-1)$ and $p_{j,m} - p_{j,m-1} \leqslant 3\sigma/4$. Then $f \in BV_k(E \cap [p_{j,r-1}, p_{j,r}])$ for $r = 2, \dots, m$ and so $f \in BV_k(E \cap [d_j, c_{j+1}])$ by Lemmas 4 and 6. Similarly $f \in BV_k(E \cap [a, c_1])$ and $f \in BV_k(E \cap [d_n, b])$ and so the theorem is proved by Lemma 3. ∎

THEOREM 4. *Let E be a bounded set and let E_0 be a nonempty subset of E such that every point of E_0 is a limit point of E. Let $f^{(1)}$ exist in E_0 where the derivative $f^{(1)}$ is taken with respect to E. Let $k \geqslant 2$.*

(i) *If $f \in AC_k(E)$, then $f^{(1)} \in AC_{k-1}(E_0)$.*

(ii) *If $Q_k(f; x_0, \dots, x_k)$ remains bounded for all choices of points x_i, $0 \leqslant i \leqslant k$, on E, then $Q_{k-1}\left(f^{(1)}; y_0, \dots, y_{k-1}\right)$ remains bounded for all choices of points y_i, $0 \leqslant i \leqslant k-1$, on E_0.*

(iii) *If $f \in BV_k(E)$, then $f^{(1)} \in BV_{k-1}(E_0)$.*

PROOF: (i) Let $\varepsilon > 0$ be arbitrary. Then there is $\sigma = \sigma(\varepsilon) > 0$ such that for every countable collection of non-overlapping intervals $\{(c_v, d_v)\}$ with end points on E,

$$\sum_v |(d_v - c_v) Q_k(f; c_v, y_{v,1}, \dots, y_{v,k-1}, d_v)| < \varepsilon/4(k-1) \tag{4.1}$$

whenever $\sum(d_v - c_v) < \sigma$ and $y_{v,1}, \dots, y_{v,k-1}$ are in $E \cap (c_v, d_v)$. Let $\{(\gamma_r, \delta_r)\}$ be any finite collection of non-overlapping intervals with end points on E_0 such that $\sum_r(\delta_r - \gamma_r) < \sigma/2$ and let $x_{r,1}, \dots, x_{r,k-2}$ be points in $E_0 \cap (\gamma_r, \delta_r)$ and let $\gamma_r = x_{r,0}$, $\delta_r = x_{r,k-1}$. We first suppose that no two intervals of $\{(\gamma_r, \delta_r)\}$ have common end points. To each interval $\{(\gamma_r, \delta_r)\}$ we associate another interval (a_r, b_r) such that $a_r < \gamma_r < \delta_r < b_r$ and $(b_r - a_r) - (\delta_r - \gamma_r) < \sigma/2^{r+1}$ and assume that the intervals $\{(a_r, b_r)\}$ are disjoint. Clearly $\sum_r(b_r - a_r) < \sigma$. By Lemma 8

(4.2)
$$\begin{aligned}
&\sum_r (x_{r,k-1} - x_{r,0}) \left| Q_{k-1}\left(f^{(1)}; x_{r,0}, \dots, x_{r,k-1}\right)\right| \\
&\quad = \sum_r \left| Q_{k-1}\left(f^{(1)}; x_{r,0}, \dots, x_{r,k-2}\right) - Q_{k-2}\left(f^{(1)}; x_{r,1}, \dots x_{r,k-1}\right)\right| \\
&\quad = \sum_r \left| \sum_{t=0}^{k-2} Q_{k-1}(f; x_{r,0}, \dots, x_{r,t}, x_{r,t}, \dots, x_{r,k-2}) \right.
\end{aligned}$$

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$$
\begin{aligned}
&\qquad - \sum_{t=1}^{k-1} Q_{k-1}(f; x_{r,1}, \ldots, x_{r,t}, x_{r,t}, \ldots, x_{r,k-1}) \Bigg| \\
&\leqslant \sum_{t=0}^{k-2} \sum_{r} |Q_{k-1}(f; x_{r,0}, \ldots, x_{r,t}, x_{r,t}, \ldots, x_{r,k-2}) \\
&\qquad\qquad - Q_{k-1}(f; x_{r,1}, \ldots, x_{r,t+1}, x_{r,t+1}, \ldots, x_{r,k-1})| \\
&\leqslant \sum_{t=0}^{k-2} \sum_{r} |Q_{k-1}(f; x_{r,0}, \ldots, x_{r,t}, x_{r,t}, \ldots, x_{r,k-2}) \\
&\qquad\qquad - Q_{k-1}(f; x_{r,0}, \ldots, x_{r,t}, \xi_{r,t}, \ldots, x_{r,k-2})| \\
&\qquad + |Q_{k-1}(f; x_{r,1}, \ldots, x_{r,t+1}, x_{r,t+1}, \ldots, x_{r,k-1}) \\
&\qquad\qquad - Q_{k-1}(f; x_{r,1}, \ldots, x_{r,t+1}, \xi_{r,t+1}, \ldots, x_{r,k-1})| \\
&\qquad + |Q_{k-1}(f; x_{r,0}, \ldots, x_{r,t}, \xi_{r,t}, \ldots, x_{r,k-2}) \\
&\qquad\qquad - Q_{k-1}(f; x_{r,0}, \ldots, x_{r,t}, x_{r,t+1}, \xi_{r,t+1}, \ldots, x_{r,k-2})| \\
&\qquad + |Q_{k-1}(f; x_{r,0}, \ldots, x_{r,t}, x_{r,t+1}, \xi_{r,t+1}, \ldots, x_{r,k-2}) \\
&\qquad\qquad - Q_{k-1}(f; x_{r,1}, \ldots, x_{r,t+1}, \xi_{r,t+1}, \ldots, x_{r,k-1})|
\end{aligned}
$$

where the points $\xi_{r,t}$, $0 \leqslant t \leqslant k-2$ are in $E \cap (a_r, b_r)$ and they are distinct and in the vicinity of $x_{r,t}$ such that

$$
\begin{aligned}
&|Q_{k-1}(f; x_{r,0}, \ldots, x_{r,t}, x_{r,t}, \ldots x_{r,k-2}) \\
&\qquad - Q_{k-1}(f; x_{r,0}, \ldots, x_{r,t}, \xi_{r,t}, \ldots, x_{r,k-2})| \\
&\qquad < \varepsilon / 4.2^r.(k-1)
\end{aligned}
\tag{4.3}
$$

and

$$
\begin{aligned}
&|Q_{k-1}(f; x_{r,1}, x_{r,2}, \ldots, x_{r,t+1}, x_{r,t+1}, \ldots, x_{r,k-1}) \\
&\qquad - Q_{k-1}(f; x_{r,1}, \ldots, x_{r,t+1}, \xi_{r,t+1}, \ldots, x_{r,k-1})| \\
&\qquad < \varepsilon / 4.2^r.(k-1).
\end{aligned}
\tag{4.4}
$$

This is possible since $x_{r,t}$ are limit points of E. Let

$$
\begin{aligned}
T = \sum_r &|Q_{k-1}(f; x_{r,0}, \ldots, x_{r,t}, \xi_{r,t}, \ldots, x_{r,k-2}) \\
&\qquad - Q_{k-1}(f; x_{r,0}, \ldots, x_{r,t+1}, \xi_{r,t+1}, \ldots, x_{r,k-2})| \\
= \sum_r &|(\xi_{r,t+1} - \xi_{r,t}) Q_k(f; \xi_{r,t}, x_{r,0}, \ldots, x_{r,k-2}, \xi_{r,t+1})|.
\end{aligned}
\tag{4.5}
$$

If p_r and q_r are the minimum and maximum of the points $\xi_{r,t}, x_{r,0}, \ldots, x_{r,k-2}, \xi_{r,t+1}$, then

$$
T \leqslant \sum_r (q_r - p_r) |Q_k(f; \xi_{r,t}, x_{r,0}, \ldots, x_{r,k-2}, \xi_{r,t+1})|. \tag{4.6}
$$

Also p_r, $q_r \in E \cap (a_r, b_r)$ and hence $\{(p_r, q_r)\}$ is a countable collection of nonoverlapping intervals with end points on E such that $\sum (q_r - p_r) < \sigma$, and so by (4.1) and (4.6),

$$T \leqslant \varepsilon/4(k-1). \tag{4.7}$$

Hence from (4.5) and (4.7)

$$\begin{aligned}\sum_{t=0}^{k-2}\sum_r |Q_{k-1}(f; x_{r,0}, \ldots, x_{r,t}, \xi_{r,t}, \ldots, x_{r,k-2})\\ -Q_{k-1}(f; x_{r,0}, \ldots, x_{r,k-2}, \xi_{r,t+1})|\\ < [\varepsilon/4(k-1)](k-1) = \varepsilon/4.\end{aligned} \tag{4.8}$$

Similarly

$$\begin{aligned}\sum_{t=0}^{k-2}\sum_r |Q_{k-1}(f; x_{r,0}, \ldots, x_{r,k-2}, \xi_{r,t+1})\\ -Q_{k-1}(f; x_{r,1}, \ldots, x_{r,t+1}, \xi_{r,t+1}, \ldots, x_{r,k-1})|\\ < [\varepsilon/4(k-1)](k-1) = \varepsilon/4.\end{aligned} \tag{4.9}$$

So from (4.2), (4.3), (4.4), (4.8) and (4.9), we get

$$\begin{aligned}\sum_r (x_{r,k-1} - x_{r,0}) \left|Q_{k-1}\left(f^{(1)}; x_{r,0}, \ldots, x_{r,k-1}\right)\right|\\ < \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon.\end{aligned} \tag{4.10}$$

In the general case, that is, when two intervals of $\{(\gamma_r, \delta_r)\}$ have common end points, we can divide $\{(\gamma_r, \delta_r)\}$ into two classes $\{(\gamma_r', \delta_r')\}$ and $\{(\gamma_r'', \delta_r'')\}$ such that no two intervals of $\{(\gamma_r', \delta_r')\}$ or of $\{(\gamma_r'', \delta_r'')\}$ have common end points. Then (4.10) is true for the classes of intervals $\{(\gamma_r', \delta_r')\}$ and $\{(\gamma_r'', \delta_r'')\}$ and hence (4.10) is true for $\{(\gamma_r, \delta_r)\}$ with ε replaced by 2ε. Thus (i) is proved.

(ii) Let y_i, $0 \leqslant i \leqslant k-1$ be arbitrary points on E_0. Then by Lemma 8, we have

$$Q_{k-1}\left(f^{(1)}, y_0, \ldots, y_{k-1}\right) = \sum_{h=0}^{k-1} Q_k(f, y_0, \ldots, y_h, y_h, \ldots, y_{k-1}).$$

Since $Q_k(f; x_0, \ldots, x_k)$ remains bounded for all choices of points x_i, $0 \leqslant i \leqslant k$, on E, there is M such that

$$|Q_k(f; y_0, \ldots, y_h, y_h, \ldots, y_{k-1})| \leqslant M, \quad 0 \leqslant h \leqslant k-1.$$

Hence $|Q_{k-1}(f^{(1)}, y_0, \dots, y_{k-1})| \leqslant kM$. So the result follows.

(iii) The proof is similar to that for AC_k. The only change needed here is to replace $\varepsilon/4(k-1)$ by $V = V(f, E)$ in (4.1), (4.3), (4.4), (4.7), (4.8), (4.9) and (4.10). □

THEOREM 5. *For any bounded set E,*

$$BV_k(E) \subsetneqq BQ_{k-1}(E) \subsetneqq AC_{k-1}(E) \subsetneqq BV_{k-1}(E).$$

PROOF: The inclusions follow from Corollary of Theorem 1, and Theorems 2 and 3. To show that the inclusions are strict, consider the following examples:

EXAMPLE 1: There exists a BV_k function which is not AC_k. Let f be the Cantor singular function on $[0, 1]$, which is of bounded variation but is not absolutely continuous on $[0, 1]$. Let ϕ be the $(k-1)$th repeated integral of f over $[0, 1]$. Then, by [**8**, Corollary 6.2], ϕ is BV_k on $[0, 1]$. But ϕ is not AC_k on $[0, 1]$. For if $\phi \in AC_k([0, 1])$, then since $\phi^{(k-1)}$ exists on $[0, 1]$, by Theorem 4, $\phi^{(k-1)}$ is absolutely continuous on $[0, 1]$ and hence f is absolutely continuous on $[0, 1]$, which is a contradiction.

EXAMPLE 2: There exists an AC_k function on $[0, 1]$ which is not in $BQ_k([0, 1])$.

Let $f(x) = \sqrt{x}$ on $[0, 1]$. Now f is absolutely continuous on $[0, 1]$. But f does not satisfy the Lipschitz condition on $[0, 1]$.

EXAMPLE 3: There exists a function which is in $BQ_{k-1}([0, 1])$ but is not in $BV_k([0, 1])$. Let

$$\begin{aligned} f(x) &= x^2 \sin 1/x, \quad x \neq 0 \\ &= 0, \quad x = 0. \end{aligned}$$

Then f satisfies the Lipschitz condition of order 1 on $[0, 1]$ but does not belong to $BV_2([0, 1])$. For if

$$\begin{aligned} g(x) &= 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \quad x \neq 0 \\ &= 0, \quad x = 0, \end{aligned}$$

then $f'(x) = g(x)$, for $x \in [0, 1]$.

Also $g(x)$ is bounded on $[0, 1]$. Hence f satisfies Lipschitz condition in $[0, 1]$. But $f \notin BV_2([0, 1])$. For if $f \in BV_2([0, 1])$ then since f' exists on $[0, 1]$, by Theorem 4(iii), $g \in BV_1([0, 1])$, which is a contradiction because $g \notin BV_1([0, 1])$. □

THEOREM 6. *Let $k \geqslant 2$ and let E be a bounded perfect set. Let $f^{(r)}$ denote rth successive derivative of f with respect to E.*

(i) *If $f \in AC_{k-1}(E)$ then $f^{(r)}$ exists on E and is in $AC_{k-r-1}(E)$, $0 \leqslant r \leqslant k-2$, and hence $f^{(k-1)}$ exists almost everywhere on E.*

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(ii) *If* $f \in BQ_{k-1}(E)$ *then* $f^{(r)} \in BQ_{k-r-1}(E)$, $0 \leqslant r \leqslant k-2$ *and hence* $f^{(k-2)}$ *satisfies a Lipschitz condition on* E.

(iii) *If* $f \in BV_k(E)$ *then* $f_{ap}^{(k)}$ *exists almost everywhere on* E, *where* $f_{ap}^{(k)}$ *is the approximate derivative of* $f^{(k-1)}$.

PROOF: We first prove the theorem for $k = 2$.

If $f \in AC_1(E)$ then f is absolutely continuous on E and hence $f^{(1)}$ exists almost everywhere on E. In fact, by Theorem 5, $f \in BV_1(E)$ and so by [**9**, Lemma 4.1, p.221], $f^{(1)}$ exists almost everywhere on E.

If $f \in BQ_1(E)$ then $Q_1(f; x_0, x_1)$ remains bounded for $x_0, x_1 \in E$ and so f satisfies a Lipschitz condition on E.

If $f \in BV_2(E)$ then $f \in AC_1(E)$ by Theorem 5 and so $f^{(1)}$ exists almost everywhere on E. Let $S = \{x : x \in E; f^{(1)}(x) \text{ exists}\}$. Then since $f \in BV_2(E)$, $f^{(1)} \in BV_1(S)$ by Theorem 4 and so $\left(f^{(1)}\right)^{(1)}$ exists almost everywhere on S, that is, $f_{\text{ap}}^{(2)}$ exists almost everywhere on E.

Thus the result is true for $k = 2$.

We suppose that the result is true for $k = m \geqslant 2$ and prove it for $k = m + 1$. Then the proof will follow by induction.

Let $f \in AC_m(E)$. Then, by Theorem 5, $f \in AC_2(E)$ and so, by Lemma 9, $f^{(1)}$ exists on E. Hence, by Theorem 4, $f^{(1)} \in AC_{m-1}(E)$. Since the result is true for $k = m$, $f^{(1+r)}$ exists on E and is in $AC_{m-r-1}(E)$, $0 \leqslant r \leqslant m-2$, that is, $f^{(s)}$ exists on E and is in $AC_{m-s}(E)$, $1 \leqslant s \leqslant m-1$. Since this is obviously true for $s = 0$, (i) follows for $k = m + 1$.

Let $f \in BQ_m(E)$. Then, by Theorem 5, $f \in AC_m(E)$ and since (i) is true for $k = m + 1$, $f^{(1)}$ exists on E and by Theorem 4, it is in $BQ_{m-1}(E)$. Since the result is true for $k = m$, $f^{(1+r)} \in BQ_{m-r-1}(E)$, $0 \leqslant r \leqslant m-2$, that is $f^{(s)} \in BQ_{m-s}(E)$, $1 \leqslant s \leqslant m-1$. This being trivially true for $s = 0$, the proof of (ii) for $k = m + 1$ is complete.

Let $f \in BV_{m+1}(E)$. Then $f \in AC_m(E)$ and as above $f^{(1)}$ exists in E and so $f^{(1)} \in BV_m(E)$. Since the result is true for $k = m$, $\left(f^{(1)}\right)_{\text{ap}}^{m}$ exists almost everywhere on E, that is, $f_{\text{ap}}^{(m+1)}$ exists almost everywhere on E and so (iii) is proved for $k = m + 1$. ∎

THEOREM 7. *If* $f \in AC_k(E)$, *then* $f_{k-1}(x, E)$ *exists finitely at every point* x *of* E_1, *where* E_1 *is the set of all points of* E *which are also limit points of* E.

PROOF: For $k = 1$, there is nothing to prove. For $k = 2$, the theorem is true by Lemma 9. We suppose it to be true for $k = m \geqslant 2$. Let $f \in AC_{m+1}(E)$. Then $f \in AC_m(E)$ and so by hypothesis, $f_{m-1}(x, E)$ exists finitely at every point x of E_1.

Let $x_0 \in E$ be a limit point of E, say from the right, and let $\varepsilon > 0$ be given. Then there is $\sigma = \sigma(\varepsilon) > 0$ such that whenever $x_0 < x_1 < \ldots < x_{m+1}$, $|x_0 - x_{m+1}| < \sigma$ and $x_i \in E$, we have

$$|(x_0 - x_{m+1})Q_{m+1}(f; x_0, \ldots, x_{m+1})| < \varepsilon$$

that is,

$$|Q_m(f; x_0, \ldots, x_m) - Q_m(f; x_1, \ldots, x_{m+1})| < \varepsilon.$$

Since $f_{m-1}(x_0, E)$ exists finitely, letting $x_1 \to x_0$ first and then $x_2 \to x_0$ and lastly $x_{m-1} \to x_0$ through E we get

$$\Bigl| \lim_{x_{m-1} \to x_0} \cdots \lim_{x_1 \to x_0} Q_m(f; x_0, x_1, \ldots, x_m) - \lim_{x_{m-1} \to x_0} \cdots \lim_{x_2 \to x_0} Q_m(f; x_0, x_2, \ldots, x_{m+1}) \Bigr| \leqslant \varepsilon,$$

the iterated limits existing by Lemma 7.

Again letting $x_m \to x_0$ and then $x_{m+1} \to x_0$, we have

$$\Bigl| \overline{\lim_{x_m \to x_0}} \lim_{x_{m-1} \to x_0} \cdots \lim_{x_1 \to x_0} Q_m(f; x_0, \ldots, x_m) - \underline{\lim}_{x_{m+1} \to x_0} \lim_{x_m \to x_0} \cdots \lim_{x_2 \to x_0} Q_m(f; x_0, x_2, \ldots, x_{m+1}) \Bigr| \leqslant \varepsilon.$$

Since ε is arbitrary, $\lim\limits_{x_m \to x_0} \ldots . \lim\limits_{x_1 \to x_0} Q_m(f; x_0, \ldots, x_m)$ exists finitely. A similar argument holds if x_0 is a limit point of E from the left or from both sides. Hence, by Lemma 7, $f_m(x_0, E)$ exists finitely., Thus the theorem is true for $k = m + 1$. This completes the proof by induction. ∎

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