Canad. Math. Bull. Vol. **19** (4), 1976

WEIGHTED MAXIMAL INEQUALITIES FOR ℓ^r-VALUED FUNCTIONS

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1. C. Feffermann and E. M. Stein [2] have shown that the continuity property of the Hardy-Littlewood maximal functions between L^p-spaces, $1<p<\infty$, extends to ℓ^r-valued functions on $\mathbb{R}^n$. Specifically, if $f=(f_1, f_2, \cdots)$ is a sequence of functions defined on R^n, let for $1<r<\infty$, $\|f(x)\|_r$ be given by

$$\|f(x)\|_r = \left\{\sum_{k=1}^{\infty} |f_k(x)|^r\right\}^{1/r}.$$

If f^* denotes the sequence of functions whose kth term is the maximal function of f, that is

$$f_k^*(x) = \sup \frac{1}{|Q|}\int_Q |f_k(t)|\, dt,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ centered at x and $|Q|$ denotes the Lebesque measure of $Q \subset \mathbb{R}^n$, then their result is:

THEOREM 1. *If* $1<r,\ p<\infty$, *then*

$$\int_{\mathbb{R}^n} \|f^*(x)\|_r^p\, dx \le A_{r,p}\int_{\mathbb{R}^n} \|f(x)\|_r^p\, dx. \tag{1}$$

Moreover,

$$|\{x \in \mathbb{R}^n : \|f(x)\|_r > y\}| \le \frac{A_r}{y}\int_{R^n} \|f(x)\|_r\, dx \tag{2}$$

where $|\{\ \}|$ *denotes Lebesgue measure of the set* $\{\ \}$.

For $0<p<1$, Theorem 1 fails of course, although there are well-known results that show that integrability of a function insures the existence of the Hardy-Littlewood maximal function in L^p for $0<p<1$ over a set of finite measure (See e.g. [4, §21.80]). Similarly, $f \in (L \log^+ L)$ implies integrability of the maximal function over a set of finite measure.

Recently, a number of estimates involving scalar valued maximal functions between weighted L^p-spaces appeared in the literature (see e.g. [5], [6], [7]). A particularly elegant characterization of weight functions for which such a norm estimate holds was given by B. Muckenhoupt [6]. However, unlike the scalar

Received by the editors February 10, 1975 and, in revised form, October 6, 1975.

valued functions considered there, it is the purpose of this paper to extend Theorem 1 to a weighted L^p-estimate and consider further the case when $0<p\leq 1$ for the weighted vector valued case. Our results depend on two lemmas which may be of independent interest.

It is the purpose of this paper to extend these results to ℓ^r-valued functions on $\mathbb{R}^n$. In fact, we will extend the results to ℓ^r-valued functions which belong to weighted Lebesgue spaces whose weights satisfy a somewhat special condition. The main result is obtained via two lemmas which are also used to extend Theorem 1 to weighted L^p-spaces.

Throughout, A, A_p, $A_{p,r}$, are constants dependent only on the indicated parameters, but may be different at different occurrences. χ_E denotes as usual the characteristic function of a set E.

2. The weights we consider in the sequel satisfy

Definition 1. A non-negative weight function w defined on R^n is said to satisfy condition M, if there exist a constant A, such that

$$w^*(x)\leq Aw(x),$$

where w^* is the maximal function of w.

For convenience, we state the Calderon-Zygmund-Lemma as we apply it in the arguments below.

Lemma. (*Calderón-Zygmund*; [6; §3.2]). *If $\|f(x)\|_r$ is integrable over R^n, then for any $y>0$, there exists a collection of disjoint cubes $\{Q_j\}$ in $\mathbb{R}_n$ satisfying*

(a) $$|Q_j|<\frac{1}{y}\int_{Q_j}\|f(x)\|_r\,dx$$

(b) $$\|f(x)\|_r\leq y \quad \text{if} \quad x\notin\Omega\equiv\bigcup_j Q_j$$

(c) $$\frac{1}{|Q_j|}\int_{Q_j}\|f(x)\|_r\,dx\leq A_y \quad \text{for each } Q_j.$$

We introduce the following notation: By E_y and E_y^f we denote the sets

$$\{x\in\mathbb{R}^n:|f_k(x)|>y\},\ y>0,\qquad k=1,2,\ldots;$$

and

$$\{x\in\mathbb{R}^n:\|f(x)\|_r>y\}\qquad y>0,$$

respectively.

Let w be a weight function on $\mathbb{R}^n$. If $E\subset\mathbb{R}^n$ we write

$$\mu(E)=\int_E w(x)\,dx.$$

The μ-measure of the sets E_y and E_y^f are denoted by

$$D_{f_k}^w(y) \quad \text{and} \quad D_{\|f\|_r}^w(y),$$

respectively.

Our first result extends (2) of Theorem 1.

LEMMA 1. *If w satisfies condition M, then*

$$D_{\|f^*\|_r}^w(y) \le \frac{A_r}{y} \int_{\mathbb{R}^n} \|f(x)\|_r w(x)\, dx, \qquad 1<r<\infty,$$

provided the right side is finite.

Proof. Let $f=(f_1, f_2, \ldots)$ be a sequence of continuous functions with compact support in $\mathbb{R}^n$ satisfying

$$\int_{\mathbb{R}^n} \|f(x)\|_r\, w(x)\, dx < \infty.$$

Then $\|f(x)\|_r$ is integrable so that the Calderón-Zygmund Lemma applies.

For each k, let

$$f'_k = f_k \cdot \chi \mathbb{R}^n \backslash \Omega' \quad \text{and} \quad f''_k = f_k - f'_k,$$

then by Minkowski's inequality

$$\|f^*(x)\|_r \le \|f'^*(x)\|_r + \|f''^*(x)\|_r.$$

It suffices, therefore, to prove

$$D_{\|f'^*\|_r}^w(y) \le \frac{A_r}{y} \int_{\mathbb{R}^n} \|f(x)\|_r\, w(x)\, dx \tag{3}$$

$$D_{\|f''^*\|_r}(y) \le \frac{A_r}{y} \int_{\mathbb{R}^n} \|f(x)\|_r\, w(x)\, dx. \tag{4}$$

for then by the usual density argument the Lemma follows.

To prove (3), observe that by [2, Lemma 1] and (b)

$$\begin{aligned}
y^r D_{\|f'^*\|_r}^w(y) &\le r \int_0^y D^w{}_{\|f'^*\|_r}(t) t^{r-1}\, dt \\
&\le r \int_0^\infty \mu\left\{x \in \mathbb{R}^n : \sum_{k=1}^\infty |f_k'^*(x)|^r > t^r\right\} t^{r-1}\, dt \\
&= \int_0^\infty \mu\left\{x \in \mathbb{R}^n : \sum_{k=1}^\infty |f_k'^*(x)|^r > s\right\} ds \qquad (s = t^r) \\
&= \int_{\mathbb{R}^n} \sum_{k=1}^\infty |f_k'^*(x)|^r\, w(x)\, dx
\end{aligned}$$

$$= \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k'^*(x)|^r w(x)\, dx \le A_r \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k'(x)|^r w^*(x)\, dx$$

$$\le A_r \int_{\mathbb{R}^n} \sum_{k=1}^{\infty} |f_k'(x)|^r w(x)\, dx$$

$$\le A_r y^{r-1} \int_{R^n} \|f'(x)\|_r\, w(x)\, dx \le A_r y^{r-1} \int_{\mathbb{R}^n} \|f(x)\|_r\, w(x)\, dx.$$

It remains, therefore, to prove (4).

Define

$$\bar{f}_k, \quad k = 1, 2, \ldots; \quad \text{by}$$

$$\bar{f}_k(x) = \begin{cases} \dfrac{1}{|Q_j|} \displaystyle\int_{Q_j} |f_k(t)|\, dt & \text{if } x \in Q_j \\ 0 & \text{if } x \notin Q_j. \end{cases}$$

For $x \in Q_j$, Minkowski's inequality and (c) show that

$$\|\bar{f}(x)\|_r = \left\{ \sum_{k=1}^{\infty} \left[\frac{1}{|Q_j|} \int_{Q_j} |f_k(t)|\, dt \right]^r \right\}^{1/r}$$

$$\le \frac{1}{|Q_j|} \int_{Q_j} \|f(t)\|_r\, dt \le Ay.$$

while $x \notin \Omega$ implies $\bar{f}_k = 0$, for all k, that is $\|\bar{f}(x)\|_r = 0$ if $x \notin \Omega$. Now by (a) we obtain

$$\int_{\mathbb{R}^n} \|\bar{f}(x)\|_r^r\, w(x)\, dx \le Ay^r \int_Q w(x)\, dx$$

$$= Ay^r \sum_j \int_{\Omega} w(x)\, dx$$

$$= Ay^r \sum_j |Q_j| \cdot \left(\frac{1}{|Q_j|} \int_{Q_j} w(x)\, dx \right)$$

$$\le Ay^r \sum_j \frac{1}{y} \int_{Q_j} \|f(x)\|_r\, dx \left(\frac{1}{|Q_j|} \int_{Q_j} w(x)\, dx \right)$$

$$\le Ay^{r-1} \sum_j \int_{Q_j} \|f(x)\|_r\, w^*(x)\, dx$$

$$\le Ay^{r-1} \int_{\Omega} \|f(x)\|_r\, w(x)\, dx$$

$$\le Ay^{r-1} \int_{\mathbb{R}^n} \|f(x)\|_r\, w(x)\, dx.$$

Using as before [2, Lemma 1] and the fact that $f''_k(x) \le A\bar{f}_k(x)$, shown in the proof of [2, Lemma 1]

$$y^r D^w_{\|f''^*\|_r}(y) \le \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f''^*_k(x)|^r \, w(x)\, dx$$

$$\le A_r \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f''_k(x)|^r \, w^*(x)\, dx$$

$$\le A_r \int_{\mathbb{R}^n} \|f''(x)\|_r^r \, w(x)\, dx$$

$$\le A_r \int_{\mathbb{R}^n} \|\bar{f}(x)\|_r^r \, w(x)\, dx$$

$$\le A_r y^{r-1} \int_{\mathbb{R}^n} \|f(x)\|_r \, w(x)\, dx$$

which implies the lemma.

The next lemma sharpens the previous result.

LEMMA 2. *If w satisfies condition M and $0<j<1$, then*

$$D^w_{\|f^*\|_r}(y) \le \frac{A_r}{(1-j)y} \int_{E_{yj}^f} \|f(x)\|_r \, w(x)\, dx,$$

provided the integral is finite.

Proof. For $k=1,2,\ldots;$ define

$$g_k(x) = \begin{cases} f_k(x) & \text{if} \quad |f_k(x)| > yj2^{-k}, \quad y>0 \\ 0 & \text{otherwise,} \end{cases}$$

so that

$$\sum_{k=1}^{\infty} |g_k(x)|^r = \begin{cases} \sum_{k=1}^{\infty} |f_k(x)|^r & \text{if} \quad x \in \bigcap_{k=1}^{\infty} \{x : |f_k(x)|^r > (yj)^r 2^{-kr}\}, \\ 0 \quad \text{otherwise,} \end{cases}$$

Let

$$\theta = \bigcap_{k=1}^{\infty} \{x : |f_k(x)|^r > (yj)^r \cdot 2^{-kr}\},$$

then

$$\theta \subset \left\{ x \in \mathbb{R}^n : |f(x)|_r > y \cdot \frac{j}{2^{r-1}} \right\} \subset E^f_{yj}.$$

https://doi.org/10.4153/CMB-1976-067-3 Published online by Cambridge University Press

But since

$$f_k^*(x) = \sup \frac{1}{|Q|} \int_Q f_k(t)\, dt$$
$$= \sup \left\{ \frac{1}{|Q|} \int_Q f_k(t) \chi_{E_{yj2^{-k}}}(t)\, dt + \frac{1}{Q} \int_Q f_k(t)[1 - \chi_{E_{yj2^{-k}}}(t)]\, dt \right\}$$
$$\le g_k^*(x) + yj2^{-k},$$

Minkowski's inequality yields

$$|f^*(x)|_r \le |g^*(x)|_r + yj$$

from which $E_y^{f*} \subset E_{y(1-j)}^{g*}$ follows. Therefore, by Lemma 1

$$yD_{|f^*|_r}^w(y) = y \int_{E_y^{f*}} w(x)\, dx \le y \int_{E_y^{g*}(1-j)} w(x)\, dx = yD_{|g^*|_y}^w(y(1-j))$$
$$\le \frac{A_r}{1-j} \int_{R^n} |g(x)|_r\, w(x)\, dx \le \frac{A_r}{1-j} \int_\theta |f(x)|_r\, w(x)\, dx$$
$$\le \frac{A_r}{1-j} \int_{E_{yj}^f} |f(x)|_r\, w(x)\, dx.$$

which proves the result.

3. We are now in the position to prove the main result.

THEOREM 2. *Let w satisfy condition M and*

$$\int_E w(x)\, dx < \infty$$

for $E \subseteq \mathbb{R}^n$. If $0 < p < 1 < r < \infty$, then

$$\int_E |f^*(x)|_r^p\, w(x)\, dx \le A_{r,p} \left[\int_E w(x)\, dx \right]^{1-p} \left[\int_{\mathbb{R}^n} |f(x)|_r\, w(x)\, dx \right]^p \tag{5}$$

Moreover, if $0 < j < 1$ then

$$\int_E |f^*(x)|_r\, w(x)\, dx \le \frac{1}{j} \int_E w(x)\, dx + \frac{A_r}{1-j} \int_{\mathbb{R}^n} (|f(x)|_r \log_+ |f(x)|_r)\, dx \tag{6}$$

Proof. *Let* $\gamma > 0$, then by Lemma 2

$$\int_E |f(x)|_r^p\, w(x)\, dx = p \int_0^\infty y^{p-1} D_{|f^*|_r}^w(y)\, dy$$
$$= p \left\{ \int_0^{\gamma/j} + \int_{\gamma/j}^\infty \right\} y^{p-1} D_{|f^*|_r}^w(y)\, dy$$

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$$\leq p\int_0^{\gamma/j} y^{p-1}\left\{\int_{E_y{}^{f*}} w(x)\,dx\right\}dy$$

$$+\frac{A_r p}{1-j}\int_{\gamma/j}^{\infty} y^{p-2}\left\{\int_{E_{yj}{}^f} \|f(x)\|_r\, w(x)\,dx\right\}dy$$

$$\leq\left(\frac{\gamma}{j}\right)^p\int_E w(x)\,dx$$

$$+\frac{A_r p}{1-j}\int_{\mathbb{R}^n} \|f(x)\|_r\, w(x)\left\{\int_{\gamma/j}^{\infty} y^{p-2}\chi_{E_{yj}{}^f}(x)\,dy\right\}dx$$

$$\leq\left(\frac{\gamma}{j}\right)^p\int_E w(x)\,dx+\frac{A_r p}{1-j}\int_{\mathbb{R}^n} \|f(x)\|_r\, w(x)\int_{r/j}^{\|f(x)\|_r/\gamma} y^{p-2}\,dy\,dx$$

$$\leq\left(\frac{\gamma}{j}\right)^p\int_E w(x)\,dx+\frac{A_r p}{(1-j)(1-p)}\left(\frac{r}{j}\right)^{p-1}\int_{\mathbb{R}^n} \|f(x)\|_r\, w(x)\,dx$$

from which (5) follows on minimizing the last expression with respect to γ.

To prove (6) Lemma 2 applies again, so that

$$\int_E \|f^*(x)\|_r\, w(x)\,dx=\int_0^{\infty} D^w_{\|f^*\|_r}(y)\,dy=\left\{\int_0^{1/j}+\int_{1/j}^{\infty}\right\}D^w_{\|f^*\|_r}(y)\,dy$$

$$\leq\int_0^{1/j}dy\int_{E_y{}^{f*}} w(x)\,dx$$

$$+\frac{A_r}{1-j}\int_{1/j}^{\infty}\frac{dy}{y}\int_{E_{yj}{}^f} \|f(x)\|_r\, w(x)\,dx$$

$$\leq\frac{1}{j}\int_E w(x)\,dx$$

$$+\frac{A_r}{1-j}\int_{\mathbb{R}^n} \|f(x)\|_r\, w(x)\,dx\int_{1/j}^{\infty}\chi_{E_{yj}{}^f}(x)\frac{dy}{y}$$

$$\leq\frac{1}{j}\int_E w(x)\,dx+\frac{A_r}{1-j}\int_{\mathbb{R}^n} \|f(x)\|_r\, w(x)\times\left(\int_{1/j}^{\|f(x)\|_{r/j}}\frac{dy}{y}\right)dx.$$

$$=\frac{1}{j}\int_E w(x)\,dx+\frac{A_r}{1-j}\int_{\mathbb{R}^n}\left(\|f(x)\|_r\log_+\|f(x)\|_r\right)w(x)\,dx$$

which completes the proof of the theorem.

Lemma 2 may also be applied to extend (1) of Theorem 1 to weighted spaces, provided the weights satisfy condition M.

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THEOREM 3. *If w satisfies condition M and $1<p,\ r<\infty$, then*

$$\int_{\mathbb{R}^n} \|f^*(x)\|_r^p\, w(x)\, dx \le A_{r,p} \int_{\mathbb{R}^n} \|f(x)\|_r^p\, w(x)\, dx$$

Proof. By Lemma 2

$$\begin{aligned}\int_{\mathbb{R}^n} \|f^*(x)\|_r^p\, w(x)\, dx &= p\int_0^\infty y^{p-1} D^w_{\|f^*\|_r}(y)\, dy \\ &\le \frac{A_{r,p}}{1-j}\int_0^\infty y^{p-2}\left[\int_{E_{yj}^f} \|f(x)\|_r\, w(x)\, dx\right] dy \\ &= \frac{A_{r,p}}{1-j}\int_{\mathbb{R}^n} \|f(x)\|_r\, w(x)\left\{\int_0^\infty y^{p-2}\chi_{E_{yj}^f}(x)\, dy\right\} dx \\ &= \frac{A_{r,p}}{1-j}\int_{R^n} \|f(x)\|_r\, w(x)\left\{\int_0^{\|f(x)\|_{r/j}} y^{p-2}\, dy\right\} dx \\ &\le \frac{A_{r,p}\, j^{p-1}}{(1-j)(p-1)}\int_{\mathbb{R}^n} \|f(x)\|_r^p\, w(x)\, dx\end{aligned}$$

which is the result.

As in [2] we observe that if $f_k = \chi_{Q_k}$, where Q_k are disjoint cubes in $\mathbb{R}^n$, then

$$\sum_{k=1}^\infty |f_k^*(x)|^r \approx \sum_{k=1}^\infty \frac{|Q_k|}{(|x-y_k|^n + |Q_k|^r)} \equiv \Gamma_r(x)$$

where y_k is the centre of Q_k. $\Gamma_r(x)$ is a modification of the Marcinkiewicz integral of order r, corresponding to $\{Q_k\}$. Lemma 1, then asserts that for $1<r<\infty$

$$D^w_{\Gamma_r}(x) \le \frac{A_r}{y^{1/r}} \sum_{k=1}^\infty \mu(Q_k),$$

where μ is defined above and w satisfies condition M, while Theorem 2 states that for $0<p<1<r<\infty$

$$\int_{\mathbb{R}^n} \Gamma_r(x)^{p/2} w(x)\, dx \le A\left\{\int_{\mathbb{R}^n} w(x)\, dx\right\}^{1-p}\left\{\sum_{k=1}^\infty \mu(Q_k)\right\}^p.$$

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https://doi.org/10.4153/CMB-1976-067-3 Published online by Cambridge University Press