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DIVISOR SUMS OF GENERALISED EXPONENTIAL POLYNOMIALS

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ABSTRACT. A study is made of sums of reciprocal norms of integral and prime ideal divisors of algebraic integer values of a generalised exponential polynomial. This includes the important special cases of linear recurrence sequences and general sums of S-units. In the case of an integral binary recurrence sequence, similar (but stronger) results were obtained by P. Erdős, P. Kiss and C. Pomerance.

1. **Introduction.** Let $E(\underline{h})$ denote an algebraic integral exponential polynomial in the variable $\underline{h} \in \mathbf{Z}^r$. Write $\underline{h} = (h_1, \ldots, h_r)$ then $E(\underline{h})$ is a finite expression of the form

$$E(\underline{h}) = \sum_{i=1}^{m} A_i(\underline{h})\alpha_{i1}^{h_1} \cdots \alpha_{ir}^{h_r},$$

where $A_i(\underline{h})$ denotes a polynomial in $\underline{h}$. Assume that the α_{ij}, together with the coefficients of the A_i all lie in the ring of integers of an algebraic number field L of finite degree $d = [L : \mathbf{Q}]$ over $\mathbf{Q}$. Assume that $E(\underline{h})$ has only finitely many zeros $\underline{h} \in \mathbf{N}^r$. This will be the case if for each pair of distinct indices k and l, the numbers $\alpha_{k1}/\alpha_{l1}, \ldots, \alpha_{kr}/\alpha_{lr}$ are multiplicatively independent (see [5]). Assume also that for some $i = 1, \ldots, r$ any specialisation in the remaining variables yields an exponential polynomial $E(h_i)$ with the property that the coefficients of its monomials are coprime algebraic integers. Define this condition by saying $E(\underline{h})$ is *factor-free** (in h_i). Assume that $E(\underline{h})$ is *non-degenerate* in the sense that no pairwise quotient α_{ij}/α_{kj}, $i, j = 1, \ldots, m$, $k = 1, \ldots, r$ is a root of unity. In the case where $r = 1$, the exponential polynomial $E(h)$ denotes the h-th term of a linear recurrence sequence and the notion of degeneracy is the usual one, see [9].

Our results apply also to a general sum of S-units. For example, over $\mathbf{Q}$, let S consist of the primes p, q and infinity. Consider a general sum of S-units as in [3]. This is an expression

$$E(p,q) = a_0 + a_1 p^{h_1} q^{k_1} + \cdots + a_n p^{h_n} q^{k_n},$$

where the a_i, $i = 0, \ldots, n$ are non-zero coprime integers with a_0 coprime to p and q. If one of the variables is zero, say h_1, then it is clear that $E(p,q)$ is factor-free in k_1. Looking ahead to Theorem 2 and using induction, we deduce that the contribution from vectors with a zero entry lies in the error term. If $h_1 \neq 0$, the only specialisations which fail to yield an exponential polynomial with coprime coefficients are those with $k_j = 0$ for some

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j. For the reasons given above, we may ignore these. Thus, although $E(p, q)$ is not necessarily factor-free, all the relevant specialisations yield exponential polynomials with the coprimality condition. Therefore we may deduce a result of the shape of Theorem 2.

Write O_L for the ring of algebraic integers in L and let $N_L = N_{L|\mathbf{Q}}: L \longrightarrow \mathbf{Q}$ denote the usual field norm. This can be defined upon the ideals of O_L by letting $N_L(M) = [O_L : M]$, the group index. This definition is compatible because $|N_L(\mu)|$ turns out to be the same as the norm of the principal ideal generated by μ. Let $\nu_L(M)$ and $\rho_L(M)$ denote the number of distinct prime ideal divisors and the sum of reciprocal norms of prime ideal divisors of an ideal M:

$$\nu_L(M) = \sum_{P|M} 1, \tag{1.1}$$

$$\rho_L(M) = \sum_{P|M} 1/N_L(P). \tag{1.2}$$

Also, let $\theta_L(M)$ denote the sum of reciprocal norms of ideal divisors of M,

$$\theta_L(M) = \sum_{Q|M} 1/N_L(Q). \tag{1.3}$$

These functions satisfy the following well-known bounds:

$$\nu_L(M) = O\big(\log N_L(M)/ \log\log N_L(M)\big), \tag{1.4}$$

$$\rho_L(M) = O\big(\log\log\log N_L(M)\big), \tag{1.5}$$

$$\theta_L(M) = O\big(\log\log N_L(M)\big). \tag{1.6}$$

When $L = \mathbf{Q}$, many results are known for the average values of these functions for specific sequences over intervals. For example, the sequence of integers or the values of a polynomial. In [1] and [6], average values are obtained for the sequence of Mersenne numbers $M(h) = 2^h - 1$ over an interval. It was shown that,

$$H^{-1} \sum_{h=N+1}^{N+H} \rho\big(M(h)\big) = \gamma + o(1), \tag{1.7}$$

for all N where $H \longrightarrow \infty$ with $H/\log\log\log N \longrightarrow \infty$. In (1.7), γ denotes an absolute constant. To state our theorems, suppose first that $r = 1$ and let N and H denote non-negative integers with $H > 0$, letting $\sigma_E(N, H)$ denote the sum

$$\sigma_E(N,H) = \sum_{N<h\leq N+H} \rho\big(E(h)\big). \tag{1.8}$$

Also, let $S_E(N, H)$ denote the sum,

$$S_E(N,H) = \sum_{N<h\leq N+H} \theta\big(E(h)\big). \tag{1.9}$$

In definitions (1.8) and (1.9), we assume that only those h are considered which do not cause $E(h)$ to vanish. Let $1 < n$ denote the order of the minimal recurrence equation satisfied by $E(h)$.

THEOREM 1. *Assume $E(h)$ is a non-degenerate exponential polynomial and consider those $h \in \mathbf{N}$ for which $E(h) \neq 0$. There are positive constants α and β such that the following asymptotic formulae hold as $H \longrightarrow \infty$;*

$$\sigma_E(N,H)/H = \alpha + O(H^{-1}\log\log N + H^{-2/(n^2-2n+2)}), \tag{1.10}$$

$$S_E(N,H)/H = \beta + O(H^{-1}\log\log N + \log\log H/\log H). \tag{1.11}$$

In (1.10) and (1.11), the error terms are effective and uniform in the sense that they depend only upon the degree of L over $\mathbf{Q}$ and n. If $n = 2$ then the error term in (1.10) is excellent. This arises in the case of a binary recurrence sequence. Also, α and β are given explicitly by the formulae in (2.12) and (2.15). An application of the Theorem is that α (resp. β) is the mean value of $\rho_L\big(E(h)\big)$ (resp. $\theta_L\big(E(h)\big)$) as h varies over the integers in a box $0 < h \leq H$. The interest of the formula lies in the case where $H/\log\log N \longrightarrow \infty$. This is a very weak condition so the Theorem makes a very strong statement about the divisibility of recurrence sequences.

In the general case, define

$$\sigma_E(H) = \sum_{|\underline{h}|<H} \rho_L\big(E(\underline{h})\big), \tag{1.12}$$

$$S_E(H) = \sum_{|\underline{h}|<H} \theta_L\big(E(\underline{h})\big). \tag{1.13}$$

We let $E(h_i)$ denote $E(\underline{h})$ viewed as an exponential polynomial in the single variable h_i. Let n_i denote the order of the minimal recurrence equation satisied by $E(h_i)$ and let $n = \min_i\{n_i\}$, where the minimum runs over those variables for which $E(\underline{h})$ is factor-free in h_i.

THEOREM 2. *Assume $E(\underline{h})$ is a non-degenerate, factor-free generalised exponential polynomial with only finitely many zeros in $\mathbf{N}^r$. The following effective asymptotic formulae are valid as $H \longrightarrow \infty$,*

$$\sigma_E(H)/H^r = \gamma + O(H^{-2/(n^2-2n+2)}), \tag{1.14}$$

$$S_E(H)/H^r = \delta + O(\log\log H/\log H). \tag{1.15}$$

The constants are given by the formulae in (3.12) and (3.15). These formulae generalise those in (1.10) and (1.11) for a box about the origin. The methods in this paper could probably be extended to give results over intervals, but this would impose a great strain upon the notation in the proofs and probably the patience of the reader. These results make effective statements about the divisibility of some generalised exponential polynomials. The results in [2] also make statements about divisibility. The difference is that in [2], the concern is with 'local height', that is, divisibility by a fixed prime ideal. Here we are concerned more with 'length', that is, divisibility by all prime ideals. It seems very likely that an archimedean version of the results in [2] holds; we believe that

$$H^{-r} \sum_{|\underline{h}| \leq H} \log N_L\big(E(\underline{h})\big) = \nu + o(1),$$

for $\nu > 0$, where the error term is effective. The emergent picture is the existence of effective mean value results for an important class of arithmetic functions evaluated on generalised exponential polynomials.

The background to these Theorems is as follows. As we mentioned, for the case of Mersenne numbers, a form of (1.10) was proved in [1] which is valid for $H/\log\log\log N \longrightarrow \infty$ and this is stronger than our result. In [13], the second author considered the following average of a recurrence sequence $u(h)$,

$$\sum_{h=1}^{H} \phi\big(|u(h)|\big)/|u(h)|. \tag{1.16}$$

In (1.16), ϕ denotes the Euler ϕ-function. The sum in (1.16) is rather similar to the sum $S(0, H)$ and indeed, it is the methods of [13] which are generalised here to obtain our Theorem. There exist many results about the arithmetic of recurrence sequences and their generalisations. The reader is recommended to consult [2],[3],[4],[7]–[14].

2. **Congruences for linear recurrence sequences.** Let M denote an ideal of O_L and suppose H and N are non-negative integers with H positive. Suppose $u(h)$ denotes a non-degenerate, exponential polynomial in the single variable h, with order $n > 1$ and with all its coefficients in O_L. Suppose that $u(h)$ is factor-free, meaning that the g.c.d. of the coefficients of its monomials is 1. Let $R_M(N, H)$ denote the number of rational integer solutions of the congruence,

$$u(h) \equiv 0 \pmod{M}, \qquad N < h \leq N + H. \tag{2.1}$$

The sequence $u(h)$ is periodic modulo M with some minimal period τ_M, beginning at a point l_M. That is,

$$u(h + \tau_M) \equiv u(h) \pmod{M}, \qquad h > l_M.$$

Now $u(h)$ satisfies a linear recurrence relation of order n so any n consecutive values determine the entire sequence. The total number of such sequences modulo M is at most $N_L(M)^n$. It follows that the quantities above are related by the inequality,

$$\tau_M + l_M \leq N_L(M)^n. \tag{2.2}$$

Define R_M to be $R_M(l_M, \tau_M)$. The following bound is trivial,

$$R_M(N, H) = HR_M/\tau_M + O\big(N_L(M)^n\big). \tag{2.3}$$

The results of [11] give an improvement of (2.3) when the underlying quotient O_L/M is a finite field. If P is a prime ideal of O_L then O_L/P is a field with $N_L(P)$ elements. Theorem 1 of [11] is quoted as the following Lemma.

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LEMMA 1. *If P denotes a prime ideal of O_L then the following formula holds, with an error term depending upon d and n only,*

$$R_P(N,H) = HR_P/\tau_P + O\big(N_L(P)^{n/2-1}\log N_L(P)\big). \tag{2.4}$$

The monomials in $u(h)$ are constant multiples of the functions $\alpha_i^h h^{j-1}$, $i = 1,\ldots,m$, $j = 1,\ldots,n_i$, where $\sum_{i=1}^m n_i = n$. Enumerate these n functions in any order and denote the k-th function by $\psi_k(h)$. Suppose s is a positive integer $s \le n$, and $h_0,\ldots,h_{s-1}$ are non-negative integers. Let $D(h_0,\ldots,h_{s-1})$ denote the determinant of the matrix $\big(\psi_k(h_l)\big)_{k,l=0}^{s-1}$. If $D(0,\ldots,h_{s-1}) \equiv 0 \bmod M$ for every sequence of integers with $0 = h_0 < h_1 < \cdots < h_{n-1}$, define t_M to be 1. In this case, the entire sequence reduces to zero modulo M. Otherwise, define t_M to be the maximal integer T with the property that,

$$D(0,h_1,\ldots,h_{n-1}) \not\equiv 0 \pmod M,$$

for any $0 = h_0 < h_1 < \cdots < h_{n-1} \le T$. We can make some trivial estimates for t_M. Fermat's little Theorem implies $t_M = O\big(N_L(M)\big)$. On the other hand, the norm of the determinant is bounded by $O(C^{h_{n-1}})$ for some constant $C > 0$. Thus, if the determinant is divisible by M then $t_M \le h_{n-1}$ and taking the norm and the logarithm of both sides gives $\log N_L(M) = O(t_M)$.

In the 3 lemmas that follow, the error terms are uniform in the sense that they depend upon d and n only. The bound (2.5) that follows was proved in [12].

LEMMA 2. *In the following upper bound, the implied constant depends upon n and d only,*

$$R_M(N,H) = O(H/t_M + 1). \tag{2.5}$$

The essential idea in the proof is that if solutions exist to the congruence $u(h_j) \equiv 0 (\bmod M)$ for $j = 0,\ldots,s$ then the corresponding system of linear equations (mod M) can be inverted by the factor free assumption. Thus, in a range of length t_M, the number of contributions to $R_M(N,H)$ is bounded uniformly. Bound (2.6) was proved in [13], in the rational case.

LEMMA 3.

$$\sum_{N_L(M)\ge X} 1/t_M N_L(M) = O(\log\log X/\log X). \tag{2.6}$$

Firstly, we are going to prove a version of (2.6) when the sum is restricted to prime ideals then make the adjustments necessary to deduce (2.6).

LEMMA 4.

$$\sum_{N_L(P)\geq X} 1/t_P N_L(P) = O(X^{-1/n}). \tag{2.7}$$

PROOF. Given $t > 0$, define

$$W(t) = \prod_{0<h_1<\cdots<h_{n-1}\leq t} \left|N_L\big(D(0,h_1,\ldots,h_{n-1})\big)\right|.$$

It follows from the definition of t_P that $P \mid W(t)$ whenever $t_P < t$. Therefore,

$$\begin{aligned}
\sum_{N_L(P)\geq X} 1/t_P N_L(P) &= \sum_{t=1}^{\infty} t^{-1} \sum_{\substack{N_L(P)\geq X\\ t_P=t}} 1/N_L(P)\\
&= \sum_{t=1}^{\infty}\big(1/t - 1/(t+1)\big) \sum_{\substack{N_L(P)\geq X\\ t_P<t}} 1/N_L(P)\\
&= O\Big(\sum_{t=1}^{\infty} t^{-2} \sum_{\substack{N_L(P)\geq X\\ t_P<t}} 1/N_L(P)\Big) = O\Big(\sum_{t=1}^{\infty} t^{-2} \sum_{\substack{N_L(P)\geq X\\ P\mid W(t)}} 1/N_L(P)\Big).
\end{aligned}$$

Using the bound $N_L(P) \geq X$ together with the definition in (1.1), shows that the inner sum is bounded by $\nu_L\big(W(t)\big)/X$. On the other hand we can use the definition in (1.2) to bound this sum by $\rho_L\big(W(t)\big)$. It is clear that $\log W(t) = O(t^n)$. Therefore, the bounds (1.4) and (1.5) now imply an upper bound for the inner sum of the form,

$$\sum_{\substack{N_L(P)\geq X\\ P\mid W(t)}} 1/N_L(P) = O(\min\{t^n/X\log t, \log\log t\}). \tag{2.8}$$

Set $T = (X\log X)^{1/n}$. Apply the first bound in (2.8) when $t < T$ and the second bound otherwise, to obtain

$$\begin{aligned}
\sum_{N_L(P)\geq X} 1/t_P N_L(P) &= O\Big(\sum_{t<T} t^{n-2}/X\log t + \sum_{t\geq T} t^{-2}\log\log t\Big)\\
&= O(X^{-1}T^{n-1}/\log T + T^{-1}\log\log T).
\end{aligned}$$

and the estimate follows. ∎

Now we prove (2.6). Use the remark before Lemma 2 stating the bound $\log N_L(M) = O(t_M)$. Thus, setting $T = c\log X$ for some constant $c > 0$ we obtain

$$\sum_{N_L(M)\geq X} 1/t_M N_L(M) \leq \sum_{t_M\geq T} 1/t_M N_L(M). \tag{2.9}$$

Using the definition of $W(t)$, and arguing as before, the right hand side of (2.9) is

$$\sum_{t=T}^{\infty} t^{-1} \sum_{t_M=t} 1/N_L(M) = O\Big(\sum_{t=T}^{\infty} t^{-2} \sum_{M\mid W(t)} 1/N_L(M)\Big).$$

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Using the definition (1.3) and the bound (1.6) gives the following upper bound for (2.9),

$$O\Big(\sum_{t=T}^{\infty} \log t/t^2\Big) = O(\log T/T),$$

which proves (2.6).

PROOF OF THEOREM 1. Define,

$$U(N,H) = E(N+1)\cdots E(N+H).$$

We may write σ_E as follows,

$$\sigma_E(N,H) = \sum_{P|U(N,H)} R_P(N,H)/N_L(P). \tag{2.10}$$

Now fix some $X > 1$, to be determined in terms of H. Apply Lemma 1 to (2.10) for $N_L(P) < X$. Apply Lemmas 2 and 4 otherwise, to obtain

$$\sigma_E(N,H) = \sum_{N_L(P)<X} \Big(HR_P/\tau_P N_L(P) + O\big(N_L(P)^{n/2-2}\log N_L(P)\big)\Big)$$
$$+ O\Big(HX^{-1/n} + \rho\big(U(N,H)\big)\Big). \tag{2.11}$$

In (2.11), the error terms are uniform. In the leading term of (2.11), the partial sum differs from the full expression,

$$\alpha = \sum_{\exists h, P|E(h)} R_P/\tau_P N_L(P), \tag{2.12}$$

by an amount equal to,

$$\sum_{N_L(P)\geq X} R_P/\tau_P N_L(P). \tag{2.13}$$

Comparing (2.4) with (2.5) shows that if $R_P > 0$ then $t_P R_P = O(\tau_P)$. Now Lemma 4 shows that the expression in (2.13) is $O(X^{-1/n})$, uniformly. Also, the second term in (2.11) is $O(X^{n/2-1})$, uniformly. This shows that (2.10) may be written as,

$$\sigma_E(N,H) = \alpha H + O\Big(HX^{-1/n} + X^{n/2-1} + \rho_L\big(U(N,H)\big)\Big). \tag{2.14}$$

Apply (1.5) and write $\log|U(N,H)| = O\big(H(N+H)\big)$. The first two expressions in the error term in (2.14) are equal when $X = H^{2n/(n^2-2n+2)}$ so take this to define X. If $N \leq H$ then the error term is independent of N so assume not and replace the first part of the error term by $\log\log N$. This completes the proof of (1.10).

Now we prove formula (1.11). This is very similar so retain the notation. Then

$$S_E(N,H) = \sum_{M|U(N,H)} R_M(N,H)/N_L(M).$$

Fix $X > 1$, to be determined as a function of H. Apply Lemma 1 for $N_L(M) < X$ and Lemmas 2 and 3 otherwise to write this in the form,

$$\begin{aligned} S_E(N,H) &= \sum_{N_L(M)<X} \Big(HR_M/\tau_M N_L(M) + O\big(N_L(M)^n\big)\Big) \\ &\quad + O\Big(H \log\log X/\log X + \theta_L\big(U(N,H)\big)\Big) \\ &= \beta H + O\Big(H\log\log X/\log X + X^{n+1} + \theta_L\big(U(N,H)\big)\Big). \end{aligned}$$

This time, choose $X = H^{1/2(n+1)}$. Also (6) applies in tandem with the trivial bound $\log|U(N,H)| = O\big(H(N+H)\big)$. The error terms are all uniform so formula (1.11) follows just as before. To finish, we record the form of β. Namely,

$$\beta = \sum_{\exists h, M | E(h)} R_M/\tau_M N_L(M) \tag{2.15}$$

and the Theorem is proved. ∎

3. **Congruences for generalised exponential polynomials.** Suppose $E(\underline{h})$ denotes a non-degenerate, factor-free generalised exponential polynomial in the variable $\underline{h}$, with all coefficients in O_L. As before, M denotes an ideal of O_L and $H > 0$. Let $R_M(H)$ denote the number of rational integer solutions of the congruence,

$$E(\underline{h}) \equiv 0 \pmod M, \qquad 0 \le h_i \le H, i = 1, \ldots, r. \tag{3.1}$$

By induction, the exponential polynomial $E(\underline{h})$ is periodic modulo M with some minimal period τ_M with respect to every variable. That is,

$$E(h_1, \ldots, h_i + \tau_M, \ldots, h_r) \equiv E(\underline{h}) \pmod M, \qquad i = 1, \ldots, r. \tag{3.2}$$

Recall that $E(h_i)$ denotes $E(\underline{h})$ viewed as an exponential polynomial in the single variable h_i. For each i, let n_i denote the order of the minimal recurrence equation satisfied by $E(h_i)$. Amongst all those i for which $E(\underline{h})$ is factor-free in h_i, choose i such that n_i is minimal. After relabelling if necessary, we suppose $i = 1$ and write $n = n_1$. Write $R_M = R_M(\tau_M)$. The following inequalities are evident,

$$(H/\tau_M)^r R_M \le R_M(H) \le (H/\tau_M + 1)^r R_M, \qquad R_M \le \tau_M^r.$$

From these, together with (3.2), we obtain

LEMMA 5.

$$R_M(H) = (H/\tau_M)^r R_M + O\big(H^{r-1} N_L(M)^n\big). \tag{3.3}$$

PROOF. The main term arises from the inequalities before (3.3) so it suffices to count within a single period to achieve the error term. We appeal to a specialisation argument.

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Let $r-1$ of the variables be assigned to yield a factor-free exponential polynomial in the single variable h_1. The total number of possibilities for solutions to the congruence is $O\big(N_L(M)^n\big)$ uniformly, just as in (2.3). This needs to be multiplied by the total number of possibilities for the specialisation, which is $O(H^{r-1})$. Notice that the uniformity of the error term relies upon a crucial independence from the specialisation. ∎

We could argue directly without recourse to (2.3) and obtain a better error term of the form $O\big(N_L(M)^{nr}+H^{r-1}\big)$. However, this does not yield an improvement to (1.15), rather, it makes the deduction of (1.15) more difficult. Also, this line of proof sets the tone for the following two lemmas. Firstly, an improvement of (3.3) when $M=P$ is a prime ideal of O_L. We appeal to a specialisation argument to derive the following Lemma from Theorem 1 of [11].

LEMMA 6. *If P denotes a prime ideal of O_L then the following formula holds,*

$$R_P(H)=(H/\tau_P)^r R_P+O\big(H^{r-1}N_L(P)^{n/2-1}\log N_L(P)\big). \tag{3.4}$$

PROOF. The main term follows from (3.3). The error term comes by counting within a single period, just as for (3.3). Specialise $r-1$ of the variables to yield an exponential polynomial in the single variable h_1. The factor-free assumption guarantees that no auxilliary factorisation is gained by this process. Theorem 1 of [11] applies just as in (2.4) to show that the number of solutions of the congruence is $O\big(N_L(P)^{n/2-1}\log N_L(P)\big)$ uniformly. The error term is independent from the specialisation so the total bound for the error in (3.4) is obtained by multiplying this by the number of specialisations. ∎

The monomials in $E(h_1)$ are constant multiples of n functions of the form the $\alpha_j^{h_1}h_1^k$. Enumerate these n functions in any order and denote the l-th function by $\psi_{1l}(h_1)$. Suppose s is a positive integer $s\le n$, and $h_{11},\ldots,h_{1s}$ are non-negative integers. Let $D(h_{11},\ldots,h_{1s})$ denote the determinant of the matrix $\big(\psi_{1l}(h_{1m})\big)_{l,m=1}^{s}$. Define t_M to be the maximal integer T with the property that

$$D(0,h_{11},\ldots,h_{1n-1})\not\equiv 0 \pmod{M}$$

for all $0=h_{10}<h_{11}\cdots<h_{1n-1}\le T$. As before, if the determinant is always zero modulo M then t_M is defined to be 1. Once again, there are trivial bounds $\log N_L(M)=O(t_M)$ and $t_M=O\big(N_L(M)\big)$. The uniformity of the error term in (2.5), together with the factor-free assumption gives

LEMMA 7.

$$R_M(H)=O(H^r/t_M+H^{r-1}). \tag{3.5}$$ ∎

The shape of the error terms in Theorem 2 is now determined by the following Lemmas.

LEMMA 8.

$$\sum_{N_L(M)\geq X} 1/t_M N_L(M) = O(\log\log X/\log X). \tag{3.6}$$

As we have done before with Lemmas 3 and 4, we shall prove first the next estimate.

LEMMA 9.

$$\sum_{N_L(P)\geq X} 1/t_P N_L(P) = O(X^{-1/n}). \tag{3.7}$$

PROOF. Given $t > 0$, define

$$W(t) = \prod_{i=1}^{r} \prod_{0<h_{i1}<\ldots<h_{in_i-1}\leq t} \left|N_L\big(D(0,h_{i1},\ldots,h_{in_i-1})\big)\right|.$$

Notice that the bound $\log W(t) = O(t^n)$ still holds. Arguing just as before, we find

$$\begin{aligned}\sum_{N_L(P)\geq X} 1/t_P N_L(P) &= \sum_{t=1}^{\infty} t^{-1} \sum_{N_L(P)\geq X, t_P=t} 1/N_L(P)\\ &= O\Big(\sum_{t=1}^{\infty} t^{-2} \sum_{N_L(P)\geq X, P\mid W(t)} 1/N_L(P)\Big).\end{aligned}$$

Using the bound $N_L(P) \geq X$ together with the definitions in (1.1) and (1.2) shows that the inner sum is bounded by

$$\min\big\{\nu_L\big(W(t)\big)/X, \rho_L\big(W(t)\big)\big\}.$$

Now (1.4) and (1.5), together with the bound on $\log W(t)$ imply an upper bound for the inner sum of the form,

$$\sum_{N_L(P)\geq X, P\mid W(t)} 1/N_L(P) = O(\min\{t^n/X\log t, \log\log t\}). \tag{3.8}$$

The remainder of the proof of (3.7) follows verbatim that of (2.7). The proof of (3.6) also follows that for (2.6), the crucial estimate being that for $\log W(t) = O(t^n)$. ■

PROOF OF THEOREM 2. Define,

$$U(H) = \prod_{|\underline{h}|<H} E(\underline{h}). \tag{3.9}$$

We may write the sum defining $\sigma_E(H)$ in the form

$$\sum_{P\mid U(H)} R_P(H)/N_L(P). \tag{3.10}$$

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Now fix some $X > 1$, to be determined in terms of H. Apply Lemma 6 to (3.10) for $N_L(P) < X$. Apply Lemmas 7 and 9 otherwise, to obtain

$$\sigma_E(H) = \sum_{\substack{N_L(P)<X \\ P|U(H)}} \left(H^r R_P/\tau_P^r N_L(P) + O\left(H^{r-1} N_L(P)^{n/2-1} \log N_L(P)\right)\right)$$

(3.11) $$+ O\left(H^r X^{-1/n} + H^{r-1}\rho\left(U(H)\right)\right)$$

In the first term of (3.11), the partial sum differs from the full expression,

(3.12) $$\gamma = \sum_{\exists \underline{h}, P|E(\underline{h})} R_P/\tau_P^r N_L(P),$$

by an amount equal to,

(3.13) $$\sum_{N_L(P)\geq X} R_P/\tau_P^r N_L(P).$$

Comparing (3.4) with (3.5) shows that if $R_P > 0$ then $t_P R_P = O(\tau_P^r)$. Now Lemma 4 shows that the expression in (3.13) is $O(X^{-1/n})$. This shows that (3.10) may be written as,

(3.14) $$\gamma H^r + O\left(H^r X^{-1/n} + H^{r-1} X^{n/2-1} + H^{r-1}\rho_L\left(U(H)\right)\right).$$

Apply (1.5) and write $\log|U(H)| = O(H^n)$. Set the first two terms in (3.14) equal. Then (3.14) becomes,

$$\gamma H^r + O\left(H^{r-1} \log\log H + H^{r-1+\frac{n^2-2n}{n^2-2n+2}}\right).$$

This completes the proof of (1.14).

Now prove formula (1.15). Write the sum defining $S_E(H)$ as follows,

$$S_E(H) = \sum_{M|U(H)} R_M(H)/N_L(M).$$

Fix $X > 1$, to be determined as a function of H. Apply Lemma 5 for $N_L(M) < X$ and Lemmas 7 and 8 otherwise to give

$$S_E(H) = \sum_{\substack{M|U(H) \\ N_L(M)<X}} \left(H^r R_M/\tau_M^r N_L(M) + O\left(H^{r-1} N_L(M)^n\right)\right)$$

$$+ O\left(H^r \log\log X/\log X + H^{r-1}\theta_L\left(U(H)\right)\right)$$

$$= \delta H^r + O\left(H^r \log\log X/\log X + H^{r-1} X^{n+1} + H^{r-1}\theta_L\left(U(H)\right)\right).$$

Now (1.6) applies in tandem with the trivial bound $\log|U(H)| = O(H^n)$ to give an error term of

$$O(H^r \log\log X/\log X + H^{r-1} X^{n+1} + H^{r-1}\log H).$$

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Setting the first two terms equal gives the form of the error in (1.15). Finally, we record the form of δ,

$$\delta = \sum_{\exists \underline{h}, M | E(\underline{h})} R_M / \tau_M^r N_L(M) \tag{3.15}$$

and the theorem is proved. ∎

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