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NON COMMUTATIVE CONVOLUTION MEASURE ALGEBRAS WITH NO PROPER L-IDEALS

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We study non-commutative convolution measure algebras satisfying the condition in the title and having an involution with a non-degenerate finite dimensional *-representation. We show first that the group algebra $L^1(G)$ of a locally compact group G satisfies these conditions. Then we show that to a given algebra $\mathcal{A}$ with the above conditions there corresponds a locally compact group G such that $\mathcal{A}$ is a * and L-subalgebra of $M(G)$ and such that the enveloping C^*-algebra of $\mathcal{A}$ is *-isomorphic to $C^*(G)$. Finally we show for certain groups that $L^1(G)$ is the only example of such algebras, thus giving a characterisation of $L^1(G)$.

INTRODUCTION

In [**6**, 7.6.3] Taylor gives the following characterisation of the group algebra $L^1(G)$ of a locally compact abelian ($l.c.a$) group G. A commutative convolution measure algebra is isomorphic to $L^1(G)$ for some $l.c.a$ group G is and only if it is semisimple and has no non-zero proper L-ideals. The proof of this theorem uses the full machinery developed by Taylor for the study of $\triangle M(G)$. This theorem improves an earlier result [**7**] in which Taylor studied commutative convolution measure algebras with group maximal ideal spaces.

It is this latter paper we claim to generalise here, while we believe that a generalisation of [**6**, 7.6.3] is still far from our reach.

The crucial rôle of the dual group $\hat{G}$ of G which is heavily used in Taylor's papers is replaced here by the algebra $B(G)$ or the algebra $A(G)$. Recall that a non-abelian $l.c.$ group G may be recovered as a topological group - from $\triangle B(G)$ or $\triangle A(G)$. This is the starting point in our analysis here.

Let $\mathcal{A}$ be as in the title and assume that $\mathcal{A}$ has an involution with a non-degenerate finite dimensional *-representation. Let $\overline{B}$ be the subalgebra of $\mathcal{A}^*$ which is generated by the positive functionals on $\mathcal{A}$. We let G be the maximal subgroup at the identity of $\triangle\overline{B}$. We then show that $\mathcal{A}$ shares many of the properties of $L^1(G)$, for example, we show that $\mathcal{A}$ is a $\sigma(M(G), C_b(G))$ - dense $^*L$-subalgebra of $M(G)$, the enveloping C^*-algebra of $\mathcal{A}$ is *-isomorphic to $C^*(G)$ and the algebra of almost periodic functionals on

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$\mathcal{A}$ is isomorphic to that of $L^1(G)$. With the extra condition $L^1(G) \cap \mathcal{A} \neq (0)$ we show that $\mathcal{A} = L^1(G)$ and for certain groups we show that this condition is automatically satisfied.

Preliminaries

We shall make the following conventions, G will always denote a locally compact group. All topological groups and semigroups we use are Hausdorff. If A is a commutative Banach algebra $\triangle A$ is its maximal ideal space.

For results and notation for C^*-algebras and their duals we refer to Dixmier [**1**]. We follow Eymard [**2**] and denote by $B(G)$, $A(G)$, $C^*(G)$, $W^*(G)$, the Fourier-Stilltjes algebra, the Fourier algebra, the C^*-algebra and the W^*-algebra of G.

We shall need the following facts about $B(G)$ and $A(G)$ which are either results in [**2**] or easy consequences of results in [**2**].

(i) Let K be compact in G and U open in G with $K \subseteq U$ then there is a function $w \in A(G)$ with $w(k) = 1\,(k \in K)$ and $w(x) = 0\,(x \in G \setminus U)$.

(ii) $\triangle B(G)$ is a *-semigroup in $W^*(G)$ and $\triangle A(G) = G$.

(iii) If A is a closed subspace of $B(G)$ which is invariant under left translation by elements of G then A is a left $W^*(G)$-module, hence A is generated by its positive definite elements, see [**1**, 12.2.4].

For definition and fundamental results on convolution measure algebras ($C.M.A.$) we refer to [**5**] and [**6**]. We shall state briefly here some of the results we need.

Recall that a $C.M.A.$ is a Banach algebra M which is a complex L-space such that the multiplication map

$$M \otimes M \to M$$

is an L-homomorphism. M^* is then a commutative von Neumann algebra, hence $M^* = C(X)$ for some compact X. We say that M is a $C.M.A.$ with involution if M is a Banach * algebra and $\mu \to \overline{\mu^*}$ is an L-homomorphism.

Let A be a closed *-subalgebra of M^* such that A contains the identity of M^*, A is invarinat under translantion by elements of M and $A \subseteq W$ (the space of weakly almost periodic functionals on M). Then $S = \triangle A$ is a compact topological *-semigroup and $\mu \to \mu_S$ is a * and L-homomorphism of M onto a w^*-dense subalgebra of $M(S)$. S is jointly continuous if $A \subseteq A_p$ (the space of almost periodic functionals on M). $\mu \to \mu_s$ is an isomorphism if A is w^*-dense in M^*. Finally M has a bounded approximate identity if and only if S has an identity. [**5**, Theorem 3.1, Lemma 3.3 and p.825].

Our first lemma may be thought of as a generalisation of the Fourier algebra of a non-abelian group, see [**6**, 4.2.3].

Let A be a closed subspace of $B(G)$ we say that A is left (right) invariant if ${}_xu \in A(u_x \in A)$ for all $u \in A$ and all $x \in G$ where ${}_xu(y) = u(xy)$.

Lemma 1. *$A(G)$ has no non-zero proper left (right, two sided) -invariant ideals.*

Proof: Suppose that A is a proper left invariant ideal in $A(G)$. Since $A(G)$ is Tauberian and $\triangle A(G) = G$, there is an element $x_0 \in G$ such that $u(x_0) = 0$ for all $u \in A$. Since A is left invariant this would imply that $u(e) = 0$ for all $u \in A$. Now since left invariant subspaces of $B(G)$ are generated by their positive definite elements this implies that $A = (0)$ as required. ∎

Theorem 1. *$L^1(G)$ has no non-zero proper left (right, two sided) L-ideals.*

Proof: Let A be a non-zero left L-ideal in $L^1(G)$. Then for $f \in A$ we get that $e_x * f \in A$ for all $x \in G$. Hence A is left invariant. Since A is a L-subspace we have that $uf \in A$ for all $f \in A$ and all $u \in C_b(G)$. Let $0 \neq f \in A$ then $g \in A$ for any $g \in L^1(G)$ which vanishes wherever f does. If follows that there is a non-zero bounded compactly supported function $k \in A$. Let $h = k^*k$ then $h \in A(G)$ and $h \neq 0$. Now the above observations imply that $A \cap A(G)$ is a non-zero left invariant ideal in $A(G)$. Hence the closure in $B(G)$ of $A \cap A(G)$ is $A(G)$ (Lemma 1). Given $\varepsilon > 0$ and $f \in L^1(G)$ let $u \in A(G) \cap C_b(G)$ be such that $\|f - u\|_1 < \varepsilon/2$. Now choose $g \in A \cap A(G) \cap C_b(G)$ such that $\|u - g\|_{A(G)} < \varepsilon/(2\,|K|)$ where K is the compact support of $u + g$ and $|K| = m(K)$. It follows that

$$\begin{aligned} \|g - f\|_1 &\leqq \|f - u\|_1 + \|u - g\|_1 \\ &< \frac{\varepsilon}{2} + \int_k |u - g|\,(x)dx \\ &< \varepsilon; \end{aligned}$$

that is we can approximate functions in $L^1(G)$ by elements in A. Since A is closed we have that $A = L^1(G)$. ∎

Blanket assumption

Throughout $\mathcal{A}$ is a convolution measure algebra with involution, has no non-zero proper left L-ideals and $\mathcal{A}$ admits a non-degerate finite dimensional *-representation.

Let A_p be the space of almost periodic function in $\mathcal{A}^*$. Then A_p is a *-subalgebra of $\mathcal{A}^*$ and A_p is invariant under translation by elements of $\mathcal{A}$. Let A be the closed *-subalgebra of A_p which is generated by the positive functionals on $\mathcal{A}$ associated with finite dimensional representations. Here positive means $f(\mu * \mu^*) \geqq 0\,(\mu \in \mathcal{A})$.

Lemma 2. *A_p is ω^*-dense in $\mathcal{A}^*$.*

Proof: We show first that A is invariant under translation by elements of $\mathcal{A}$. Let $f \in A$ be of the form

$$f(\mu) = \langle \pi(\mu)\varepsilon, \eta \rangle$$

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and let $\nu \in \mathcal{A}$ then

$$\begin{aligned} \nu.f(\mu) = f(\nu * \mu) &= \langle \pi(\nu)\pi(\mu)\varepsilon, \eta \rangle \qquad (\mu \in \mathcal{A}) \\ &= \langle \pi(\mu)\varepsilon, \tau \rangle \end{aligned}$$

where $\tau = \mu(\nu)^* \in H_\pi$.

Let $I = \{\mu \in \mathcal{A} \mid \mu(A) = 0\}$, then I is clearly a closed left ideal in $\mathcal{A}$. Let $\mu \in I$ and $\nu \in \mathcal{A}$ be such that $0 \leqq \nu \leqq \mu$ then $\nu(f) = 0$ for each positive function f with $\mu(f) = 0$. Since A is a *-subalgebra of $\mathcal{A}^* = C(X)$, it is generated by its positive functions, hence $\nu \in I$ and I is an L-ideal in $\mathcal{A}$. Now the assumption that $\mathcal{A}$ admits a non-degenerate finite dimensional *-representation means that $A \neq (0)$, hence $I \neq \mathcal{A}$. Thus $I = (0)$ and A is w^*-dense in $\mathcal{A}$. ∎

Let Γ denote the maximal ideal space of A and $\mu \to \mu_\Gamma$ be the canonical embedding of $\mathcal{A}$ in $M(\Gamma)$. Then we have.

COROLLARY. *$\mu \to \mu_\Gamma$ is an isometry.*

PROPOSITION 1. *Γ is a group.*

PROOF: It follows from [**5**, Theorem 2.2 and Lemma 3.3] that Γ is a jointly continuous *-semigroup. Let π be the representation of Γ into $B(H_\pi)$ which is defined by the positive functionals in A_p then π separates the points of Γ.

If Γ is not a subset of the unitary group of $B(H_\pi)$ then we may find an element $x_0 \in \Gamma$ such that $\pi(x_0^* x_0) < I$ (the identity element in $B(H_\pi)$). Let $T = \{y \in \Gamma \mid \pi(y^* y) \leqq (x_0^* x_0)\}$, then T is non-empty and T is proper in Γ. If $x \in \Gamma$ then $\pi(x^* x) \leqq I$, hence

$$\pi\big((xy)^* xy\big) \leqq \pi(y^* y) \leqq \pi(x_0^* x_0) \quad (y \in T).$$

Thus T is a left ideal in Γ. Finally since Γ is jointly continuous we have that the map $x \to x^* x \to \pi(x^* x)$ is continuous. Thus T is a proper closed ideal in Γ with non-empty interior.

Now $\mathcal{A} \cap M(T)$ would be a non-zero proper left L-ideal in $\mathcal{A}$. Hence π maps Γ into the unitary group of $B(H_\pi)$ but since Γ is a *-subsemigroup of $B(H_\pi)$, we have that Γ is a group. ∎

COROLLARY. *$\mathcal{A}$ has a bounded approximate identity.*

Let B be the linear span of all positive functionals in $\mathcal{A}^*$. Then B is a *-subalgebra of $\mathcal{A}^*$ and each $f \in B$ is of the form

$$f(\mu) = \langle \pi(\mu)\varepsilon, \eta \rangle \qquad (\varepsilon, \eta \in H_\pi)$$

for some non-degenerate *-representation π of $\mathcal{A}$ on H_π. That B is invariant under translation by elements of $\mathcal{A}$ follows as in Lemma 2. Let $\overline{B}$ be the closure of B in $\mathcal{A}^*$. Then $\overline{B}$ is a translation invariant closed *-subalgebra of $\mathcal{A}^*$, and $\overline{B} \supset A$. Let $S = \triangle\overline{B}$, then S is a compact separately continuous *-semigroup with identity (see [**5**, Theorem 3.4]).

LEMMA 3.

(i) *The positive functionals in B separate the points in $\mathcal{A}$.*

(ii) *$\mathcal{A}$ is a ω^*-dense aubalgebra of $M(S)$.*

PROOF: Let $I = \{\mu \in \mathcal{A} \mid \mu(P) = 0\}$, where P is the set of positive functionals in B. Then

$$\begin{aligned} I &= \{\mu \in \mathcal{A} \mid \mu(B) = 0\} \\ &= \{\mu \in \mathcal{A} \mid \mu(\overline{B}) = 0\} \\ &= (0) \end{aligned}$$

since $B \supseteq A$, (see Lemma 2), and (i) follows. (ii) follows from ([**5**, Theorem 3.4] and (i). ∎

Remarks.

(i) We note here that the condition that $\mathcal{A}$ is a semisimple in [**6**, 7.6.3] may be replaced by the weaker condition that $\mathcal{A}$ is non-radical.

(ii) $\|\mu\|_* = \sup\limits_{\substack{f \in P \\ \|f\| \leqq 1}} |f(\mu)|$ is a norm, and the enveloping C^*-algebra $C^*(\mathcal{A})$ is actually the completion of $\mathcal{A}$ in this norm.

(iii) Each element $\phi \in S = \triangle\overline{B}$ defines a complex homomorphism on B bounded with respect to dual $C^*(\mathcal{A})$ norm on B. Thus we may (and do) regard S as a *-subsemigroup of the unit sphere of $W^*(A)$ the second dual of $C^*(\mathcal{A})$.

(iv) Since S separates the points in B, S is a generating a subset of $W^*(A)$, therefore the identity e of S is that of $W^*(\mathcal{A})$.

Let G be the maximal subgroup of S at e.

The following lemma is valid for the more general case that $\mathcal{A}$ is a *C.M.A.* with involution and $\mathcal{A}$ separates the points in B.

LEMMA 4.

(i) $G = S \cap W^*_u(\mathcal{A})$ *(the unitary group in $W^*(\mathcal{A})$).*

(ii) $S \setminus G$ *is a two-sided ideal in S.*

PROOF: Let $s \in G$ and $s^{-1} \in G$ be such that $ss^{-1} = s^{-1}s = e$. Since S is a *-subsemigroup of $W^*(\mathcal{A})$ we have that s^* and $(s^{-1})^*$ belong to S and we have

$$\begin{aligned} ss^*\left(\left(s^{-1}\right)^* s^{-1}\right) &= s\left(s^*\left(s^{-1}\right)^*\right)s^{-1} \\ &= s\left(s^{-1}s\right)^* s^{-1} \\ &= s(e)^* s^{-1} = e \end{aligned}$$

Similarly $\left(s^{-1}\right)^* s^{-1}ss^* = e$, that is ss^* is invertible.

Since $ss^* \in S$ we have that $\|ss^*\| \leqq 1$ and $\left\|(ss^*)^{-1}\right\| \leqq 1$, and by spectral calculus we have $ss^* = e$. Similarly we get $s^*s = e$, hence s is unitary.

For (ii), let $x, y \in S$ and notice that xx^*, x^*x, yy^* and y^*y are smaller than e in the order of $W^*(\mathcal{A})$.

If $xy \in G$ then $y^*x^* \in G$ by (i) and

$$e = xyy^*x^* = y^*x^*xy$$

Now $e \geqq yy^*$ implies that

$$xx^* \geqq xyy^*x^* = e, \text{ hence } xx^* = e.$$

But as elements of $W^*(\mathcal{A})$, xx^* and x^*x have the same spectrum, hence x^*x is invertible and we get $e = x^*x$ by (i). We have shown that $xy \in G$ implies $x \in G$ and therefore $y \in G$. Thus $S \setminus G$ is a semigroup and an elementary argument now shows that $S \setminus G$ is a two sided ideal in S. ∎

Since S is compact in $\sigma(W^*(\mathcal{A}), B)$ topology we see that $G = S \cap W_u^*(\mathcal{A})$ is a locally compact topological group in the same topology.

We denote by M the proper L-ideal of $M(S)$ of measures supported on the ideal $S \setminus G$.

PROPOSITION 2. *G is dense in S.*

PROOF: If there were an element $x \in S \setminus G$ with a neighbourhood $U_x \subseteq S \setminus G$, then U_x would support a non-zero measure $\mu \in \mathcal{A} \cap M$. Thus every open set in S must intersect G, that is G is dense in S. ∎

COROLLARY. *If $\alpha \colon S \to \Gamma$ is the map induced by the inclusion $A \subseteq \overline{B}$ then $\alpha(G)$ is dense in Γ.*

Thus Γ is a compact group containing $\alpha(G)$ as a dense subgroup, hence Γ is isomorphic to the almost periodic compactification of G, and we get:

COROLLARY. *The algebra of almost periodic functionals on $L^1(G)$ is isomorphic to A.*

LEMMA 5. *G is open in S.*

PROOF: Suppose that $S \setminus G$ is not closed in S. Since G is a topological group in the topology of S, $S \setminus G$ is then dense in S. Let $\{s_\alpha\}$ be a net in $S \setminus G$ that converges to the identity e of G. Since multiplication in the W^*-algebra $W^*(\mathcal{A})$ is separately continuous in the w^*-topology we have that for μ

$$\mu * \varepsilon_{s_\alpha} \xrightarrow{w^*} \text{ hence } \mu(u_{s_\alpha}) \to \mu(u)$$

for all $u \in B$. Since $S \setminus G$ is an ideal we have that u_{s_α} is supported on $S \setminus G$, therefore $\mu(u_{s_\alpha}) = 0$, hence $\mu(u) = 0$. The contradiction shows that $S \setminus G$ is closed in S and G is open. ∎

THEOREM 2.

(i) *$\mathcal{A}$ is a * and L-subalgebra of $M(G)$.*

(ii) *$B(G) \subset B$.*

(iii) *$\mathcal{A}$ is $(M(G), C_b(G))$ dense in $M(G)$.*

(iv) *$B \subset B(G)$.*

(v) *The non-deg *-representations of $\mathcal{A}$ are in one-to-one correspondence with the continuous unitary representations of G.*

(vi) *$W^*(\mathcal{A})$ is isomorphic $W^*(G)$.*

PROOF: For a non-zero measure $\mu \in \mathcal{A}$ we have that

$$\mu = \chi_G\mu + \chi_{S\setminus G}\mu = \chi_G\mu$$

since $\chi_{S\setminus G}\mu \in \mathcal{A} \cap M = (0)$.

Since G is open in S we have that $\chi_G M(S) \subset M(G)$. Thus $\mathcal{A} \subset M(G)$. Since the involution on S and hence on G is induced by that on $\mathcal{A}$, we have that is a *-and L-subalgebra of $M(G)$.

For (ii), notice that every continuous positive definite function on G defines a bounded positive functional on $M(G)$ and hence on $\mathcal{A}$.

Let $D = \{f \in B(G) \mid (\forall \mu \in \mathcal{A})(f(\mu) = 0)\}$.
Since $\mathcal{A}$ is an L-subspace of $M(G)$ we have that D is a closed ideal in $B(G)$. It is also clear that D is $\mathcal{A}$-invariant and since $\mathcal{A}$ generates the W^*-algebra $W^*(\mathcal{A})$ and since multiplication in $W^*(\mathcal{A})$ is separately continuous we have that D is $W^*(\mathcal{A})$-invariant. Hence D is invariant under translation by elements of G since $G \subseteq W^*(\mathcal{A})$. If $D \neq (0)$ then we can find a positive definite function $g \in D$, hence $D \cap A(G)$ is a non-zero G-invariant ideal in $A(G)$. Lemma 1 now shows that $A(G) \subseteq D$. But since $A(G)$ is

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dense in $C_0(G)$ this implies that $f(\mu) = 0$ for all $f \in C_0(G)$, that is $\mathcal{A} = (0)$. We have shown that $D = (0)$, that is the restriction to $\mathcal{A}$ of a non-zero element of $B(G)$ is non-zero, that is $B(G) \subseteq B$.

For (iii), we let T denote the joint support of all elements of $\mathcal{A}$. Then T is a closed subsemigroup of G. If T is contained in a proper closed subgroup H of G, let $u \in A(G)$ be such that $u(h) = 0\,(h \in H)$ and $u \neq 0$. Then it would follow that $(\forall \mu \in \mathcal{A})(u(\mu) = 0)$, contradicting (ii). So T must be a generating subsemigroup of G. If T is proper in G let $x \in T$ with $x^{-1} \notin T$. The ideal Tx is then proper in T and will support a non-zero proper L-ideal in $\mathcal{A}$. Thus we have shown that $T = G$ and (iii) follows since $\mathcal{A}$ is $\sigma(M(G), C_b(G))$ dense in $M(G)$ if and only if $G = T$.

For (iv), let f be a positive functional on $\mathcal{A}$. Then $f \in \overline{B} = C(S)$ defines a bounded continuous function on G. Let f' denote the extension of f to $M(G)$, then f' is a positive functional on $M(G)$, hence f is a continuous positive definite function on G and $f\,|_G \neq 0$ by Proposition 2. Now since B is the span of the positive functionals on $\mathcal{A}$ we get $B \subseteq B(G)$.

For (v), let π be a non-degenerate *-representation of $\mathcal{A}$. Then $\mu \to \langle \pi(\mu)\varepsilon, \varepsilon\rangle$ defines a bounded positive functional on $\mathcal{A}$ and by (iv) $x \to \langle \pi(\mu)\varepsilon, \varepsilon\rangle$ is a non-zero continuous positive definite function on G, hence π is a continuous unitary representation on G.

Conversely suppose that π is a continuous unitary representation of G. Then π defines a *-representation of $M(G)$. Since $\mathcal{A}$ is $\sigma(M(G), C_b(G))$ dense in $M(G)$ and $\langle \pi(x)\varepsilon, \varepsilon\rangle \in B(G) \subset C_b(G)$ we have that $\langle \pi(\mu)\varepsilon, \varepsilon\rangle = 0$ for all $\mu \in \mathcal{A}$ implies $\langle \pi(e)\varepsilon, \varepsilon\rangle = 0$ hence $\varepsilon = 0$. Thus π is a non-degenerate representation of $\mathcal{A}$.

From (v) we have that the universal representation of G and $\mathcal{A}$ are the same. Since $G \subseteq W^*(\mathcal{A})$ we have that $W^*(G) \subset W^*(\mathcal{A})$. Conversely since $\mathcal{A} \subseteq M(G) \subseteq W^*(G)$ we have that $W^*(\mathcal{A}) \subseteq W^*(G)$ and (vi) follows. ∎

Corollary. $C^*(\mathcal{A}) = C^*(G)$.

Proof: By (ii) and (iv) we have that $B = B(G)$. We show that the norm in $B(G)$ is the same as the dual $C^*(\mathcal{A})$ norm. Let $f \in B(G)$ then

$$f = \sup_{\substack{\mu \in \mathcal{A} \\ \|\mu\|_* \leqq 1}} |f(\mu)| \leqq \sup_{\substack{\mu \in M(G) \\ \|\mu\|_{\sum} \leqq 1}} |f(\mu)| = \|f\|_{B(G)}$$

Notice that $\|\mu\|_* = \|\mu\|_{\sum}$ by (v) above.

The converse inequality follows from (vi).

Now $C^*(\mathcal{A}) = C^*(G)$, being the predual of a Banach space, so $B(G) = B$. ∎

Corollary. *$\mathcal{A}$ is $\| \; \|_*$-dense in $C^*(G)$.*

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Proposition 3. *If $\mathcal{A} \cap L^1(G) \neq (0)$ then $\mathcal{A} = L^1(G)$.*

Proof: $\mathcal{A} \cap L^1(G) \neq (0)$ implies that $\mathcal{A} \subseteq L^1(G)$. Let $\nu \in L^1(G)$ and $\{\mu_\alpha\} \subseteq \mathcal{A}$ be such that $\mu_\alpha \to \nu$ in the $\sigma(M(G), C_b(G))$ topology. If $f \in L^\infty(G)$ and $\mu \in \mathcal{A}$ then

$$g(x) = \int f(xy)d\mu(y)$$

is a bounded continuous function on G. Hence

$$\begin{aligned}\int g(x)d\mu_\alpha(x) &= \iint f(xy)d\mu(y)d\mu_\alpha(x)\\ &= \mu * \mu_\alpha(f)\\ &\to g(\nu) = \mu * \nu(f).\end{aligned}$$

Since μ_α and μ are elements of $\mathcal{A}$ we have that $\mu * \nu$ is a weak limit of elements of $\mathcal{A}$, hence $\mu * \nu \in \mathcal{A}$. This shows that $\mathcal{A}$ is a right L-ideal in $L^1(G)$, hence $\mathcal{A} = L^1(G)$ by Theorem 1. ∎

Recall that G is a $FIA]$ group if the group $I(G)$ of all inner automorphisms of G has compact closure in the topological group $\mathrm{Aut}(G)$ of all continuous automorphisms of G. Examples of $FIA]$ groups include compact groups and locally compact abelian groups. Mosak [4] defines the operator $\#$ on $L^1(G)$ and $C^*(G)$ for $FIA]$ groups G as the extension of

$$f^{\#}(x) = \int_{I(\overline{G})} f(\beta x)d\beta \qquad (f \in C_b(G)).$$

He obtained the useful inequalities $\|\mu^{\#}\|_* \leqq \|\mu\|_*$ $(\mu \in C^*(G))$ and $\|\mu^{\#}\|_1 \leqq \mu_1(\mu \in L^1(G))$ and showed that the image of the operator $\#$ is $ZC^*(G)$, the centre of $C^*(G)$.

Let χ denote the maximal ideal space of $ZC^*(G)$; that is χ is isomorphic to the set of extreme points of the continuous positive definite functions on G which are invariant under inner automorphisms and have norm 1. It is known that χ is discrete if G is compact. These facts will be useful in the following theorem.

Theorem 3. *If G is compact then $\mathcal{A} = L^1(G)$.*

Proof: Suppose that G is compact. Let $ZC^*(G)$ be the centre of $C^*(G)$ and $\chi = \triangle ZC^*(G)$. By the Corollary to Theorem 2 we have that $\mathcal{A}$ is $\| \ \|_*$-dense in $C^*(G)$. We apply the operator $\#$ and obtain that $\mathcal{A}^{\#}$ is dense in $ZC^*(G)$. Let 1 denote the constant function 1 on G then $1 \in \chi$ and $\{1\}$ is both open and closed in the $\sigma(X, ZC^*(G))$ topology. Since $\mathcal{A}^{\#}$ is dense in $ZC^*(G)$ we have that $1 \in \triangle\mathcal{A}^{\#}$ and 1 is both open and closed in $\triangle\mathcal{A}^{\#}$. Now the Silov idempotent theorem implies the existence of an idempotent $m \in \mathcal{A}^{\#} = Z\mathcal{A}$ such that $m(1) = 1$ and $m(\chi \setminus 1) = 0$. This idempotent must be the normalised Haar measure for G. Thus $\mathcal{A} \cap L^1(G) \neq (0)$ and the theorem follows from Proposition 3. ∎

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Remark. The above theorem shows how an extra condition on the group G gives the general theorem. Let us denote by $T]$ the class of locally compact groups G for which the only example of $\mathcal{A} \subseteq M(G)$ with our blanket condition is $L^1(G)$. In this terminology Taylor's result may be stated as all locally compact abelian groups belong to the class $T]$, and our Theorem 3 as compact groups belong to $T]$. We show in Proposition 4 that $G \in T]$ if it has an open subgroup $G_1 \in T]$. We shall need the following technical lemma.

LEMMA 6. *Suppose that M is a convolution measure algebra and N is an L-subalgebra of M with $N^\perp * N \subseteq N^\perp$. Then every proper left L-ideal in N is contained in a proper left L-ideal in M.*

PROOF: Recall that if $\mu << \mu'$ and $\mu' = \omega' + \nu'$ with $\omega' \perp \nu'$ then there are measures ω and ν such that $\omega \perp \nu$, $\mu = \omega + \nu$, $\omega << \omega'$ and $\nu << \nu$. Let I be a proper left L-ideal in N and suppose that the smallest left L-ideal containing I in M is M itself, that is each $\mu' \in M$ is absolutely continuous with a measure $\mu \in M$ of the form $\mu = \sum_{i=1}^{M} \mu_i * \ell_i$ where $\mu_i \in M$ and $\ell_i \in I$. Let $\mu' \in I^\perp = \{\nu \in M : \nu \perp I\}$ and decompose $\mu' = \omega' + \nu'$ where $\omega' \in N^\perp$ and $\nu' \in N$. Decompose each μ_i as $\omega_i + \nu_i$ with $\omega_i \in N^\perp$ and $\nu_i \in N$, then $\sum_{i=1}^{m} \omega_i * \ell_i \in N$, $\sum_{i=1}^{m} \nu_i * \ell_i \in I$ and $\mu = \sum_{i=1}^{m} \omega_i * \ell_i + \sum_{i=1}^{m} \nu_i * \ell_i$ is a decomposition of μ in $N^\perp$ and N. It follows that $\nu' \in I$, hence $\nu' = 0$ and $I^\perp \subset N^\perp$, contradicting the assumption that I is a proper L-subalgebra of N. ∎

PROPOSITION 4. *If G has an open subgroup $G_1 \in T]$, $G \in T]$.*

PROOF: Let $\mathcal{A}$ satisfy the blanket assumption and G be the group associated with $\mathcal{A}$. Let $N = \{\mu \in \mathcal{A} \mid \text{Supp}\,\mu \subseteq G_1\}$. Since G_1 is a closed subgroup of G we habe that

$$N^\perp * N \subset N^\perp.$$

It follows from Lemma 6 that N has no non-zero proper left L-ideals. Since $G_1 \subseteq T]$ we have that $N = L^1(G_1)$. Since $L^1(G_1) \subset L^1(G)$ the proposition follows. ∎

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