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THE EQUALITY $(A \cap B)^n = A^n \cap B^n$ FOR IDEALS

ROBERT GILMER AND ANNE GRAMS

1. Introduction. Let D be an integral domain with identity, and let R be a commutative ring. If n is a positive integer, R will be said to *have property* (n), $(n)'$, or $(n)''$ according as

property (n): For any $x, y \in R$, $(x, y)^n = (x^n, y^n)$.
property $(n)'$: For any $x \in R$ and any ideal A of R such that $x^n \in A^n$, it follows that $x \in A$.
property $(n)''$: For any ideals A, B of R, $(A \cap B)^n = A^n \cap B^n$.

J. Ohm introduced property (n) in [**7**] in connection with the question: If $n \geqq 2$ and if D has property (n), must D be a Prüfer domain? (The integral domain D with identity is a Prüfer domain if each nonzero finitely generated ideal of D is invertible; equivalently, D_P is a valuation ring for each proper prime ideal P of D.) Prior to Ohm's paper, it was known that if D has property (2) and if D is integrally closed, then D is Prüfer. In [**7**, Theorem 1.4], Ohm showed that D is Prüfer if D has property (2) and if 2 is a unit of D. Example 4.6 of [**7**] is a domain which has property (n) for all n, but which is not integrally closed, and hence not Prüfer.

In [**3**], Gilmer extended Ohm's investigation of property (n), and in the process he defined property $(n)'$. He showed that property $(n)'$ implies property (n) and that an integrally closed domain having property (n) for any $n > 1$, is Prüfer. Two examples in [**3**] show that a domain with property $(n)'$, for each positive n need not be Prüfer.

In this paper, we investigate the question:

Does property $(n)''$ imply that D is a Prüfer domain?

We show that property $(n)''$ implies both properties (n) and $(n)'$, and thus if D is integrally closed, it implies that D is a Prüfer domain. Example 3.4 shows, however, that property $(n)''$, for all n, is not strong enough to imply that the domain is integrally closed. Finally, we show in Example 3.7 that property $(n)''$ is not equivalent to either property (n) or $(n)'$. Our notation and terminology will be that of [**2**].

2. Property $(n)''$ and Prüfer domains. Throughout this section, D is an integral domain with identity, and n is an integer greater than 1.

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LEMMA 2.1. *Property* $(n)'' \Rightarrow$ *property* $(n)' \Rightarrow$ *property* (n).

Proof. Theorem 5.3 in [**3**] shows that property $(n)'$ implies property (n). To see that property $(n)''$ implies property $(n)'$, suppose that A is an ideal of D, $x \in D - \{0\}$, and $x^n \in A^n$. Then $(x)^n = (x)^n \cap A^n = [(x) \cap A]^n \subseteq (x)^{n-1}A$. Hence $(x) \subseteq A$ since $(x)^{n-1}$ is a cancellation ideal.

THEOREM 2.2. *Let D be an integrally closed domain. The following conditions are equivalent in D:*

(a) *D is a Prüfer domain.*

(b) *Property $(n)''$ holds in D.*

(c) *Property $(n)''$ for finitely generated ideals holds in D.*

Proof. (a) $\Rightarrow$ (b). It is clear that any valuation ring has property $(n)''$. Hence if D is Prüfer, and if $\{M_\lambda\}$ is the set of maximal ideals of D, then for any ideals A and B of D,

$$(A \cap B)^n = \bigcap_\lambda (A \cap B)^n D_{M_\lambda} = \bigcap_\lambda [(A \cap B)D_{M_\lambda}]^n = \bigcap_\lambda (AD_{M_\lambda} \cap BD_{M_\lambda})^n$$
$$= \bigcap_\lambda [(AD_{M_\lambda})^n \cap (BD_{M_\lambda})^n]$$
$$= [\bigcap_\lambda A^n D_{M_\lambda}] \cap [\bigcap_\lambda B^n D_{M_\lambda}] = A^n \cap B^n.$$

(b) $\Leftrightarrow$ (c). This is true in any commutative ring.

(b) $\Rightarrow$ (a). This follows from Lemma 2.1 above and from Theorem (20.3) in [**2**].

Remark 2.3. It is apparent from our proof of the implication (a) $\Rightarrow$ (b) that $[\bigcap_{i=1}^k A_i]^n = \bigcap_{i=1}^k A_i^n$ for any finite family $\{A_i\}_{i=1}^k$ of ideals of a Prüfer domain. The analogous equality for an arbitrary family of ideals of a Prüfer domain is not valid. For example, if $V = \mathbf{Q}[[X]]$ and if $M = XV$, where $\mathbf{Q}$ is the field of rational numbers, then if $\mathbf{Z}$ denotes the ring of integers, the domain $D = \mathbf{Z} + M$ is Prüfer [**2**, p. 561]; but if $A_i = p_i D$, where $p_1 < p_2 < \ldots$ is the sequence of positive prime integers, then for any positive integer n,

$$[\bigcap_{i=1}^{\infty} A_i]^n = M^n \subset \bigcap_{i=1}^{\infty} A_i^n = M.$$

At this point we detect a breakdown in the duality between the operations of addition and intersection on the set of ideals of a Prüfer domain, for it is true that

$$(\textstyle\sum_\lambda A_\lambda)^n = \sum_\lambda A_\lambda^n$$

for any family $\{A_\lambda\}$ of ideals of a Prüfer domain, and for any positive integer n.

Remark 2.4. A careful analysis of the proof of Lemma 2.1 and of Theorem 4.3 of [**3**] shows that the following generalization of the implication (c) $\Rightarrow$ (a) in Theorem 2.2 is valid.

(*) *If D is n-integrally closed and if $(A \cap B)^n = A^n \cap B^n$ for each pair A, B of ideals of D with a basis of two elements, then D is a Prüfer domain.* (If n is a positive integer and if J is an integral domain with identity with quotient field K, then J is said to be *n-integrally closed* [**3**] if J contains each element θ in K such that θ is a root of a monic polynomial $f(X) \in J[X]$ of degree n.)

Result (*) is of some interest because of its connection with one of the more important open questions concerning Prüfer domains, namely: Does every finitely generated ideal of a Prüfer domain have a basis of two elements? [**5**].

Remark 2.5. The concept of a Prüfer domain has been extended to commutative rings with zero divisors, thereby obtaining Prüfer rings. M. Griffin's paper [**6**] contains much of what is known about Prüfer rings. Using the results of [**6**], it is straightforward to prove the following generalization of Theorem 2.2:

Let R be an integrally closed ring with few zero divisors, and let n be an integer greater than one. The following conditions are equivalent in R:

(a) *R is a Prüfer ring.*

(b) *If A and B are regular ideals of R, then $(A \cap B)^n = A^n \cap B^n$.*

(c) *If A and B are finitely generated regular ideals of R, then $(A \cap B)^n = A^n \cap B^n$.*

The question arises as to the relationship between conditions (a), (b), and (c) if the hypothesis "R has few zero divisors" is dropped. In partial answer to this question, we can prove (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). An examination of our proof of Lemma 2.1, together with results of [**3**], show that (c) implies the following condition (c)′:

(c)′: If $\{r_i\}_{i=1}^n$ is a finite set of regular elements of R, then $(r_1, \ldots, r_n)$ is invertible.

An example in [**4**] shows that an integrally closed ring in which (c)′ holds need not be a Prüfer ring, but we have no example to show that (c) does not imply (a).

3. Examples.

In this section, we present a class of domains with property $(n)''$ for every positive n, but which are not integrally closed.

Let V be a valuation ring of the form $K + M$, where K is a field and M is the maximal ideal of V, and let v be a valuation associated with V. Let k be a proper subfield of K, and set $D = k + M$. In order to present Example 3.4, we shall need a characterization of the finitely generated ideals of D and of the powers of such ideals.

LEMMA 3.1 (Gilmer [**3**]). *If $x \in D - \{0\}$, xD contains each element y of V such that $v(y) > v(x)$. If A is a finitely generated ideal of D, say $A = (a_1, \ldots, a_n)$, and if $t = \min\{v(a_i) | 1 \leqq i \leqq n\}$, then for any element a of A such that $v(a) = t$, A has a basis $\{a, k_2a, \ldots, k_ma\}$ for some $k_2, \ldots, k_m \in K - k$. Moreover,*

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$A = Wa + C$, where W is the k-subspace of K spanned by $\{1, k_2, \ldots, k_m\}$ and $C = \{y \in V | v(y) > t\}$.

LEMMA 3.2. *Let $A = Wa + C$ be a finitely generated ideal of D as given in Lemma 3.1. Then $A^n = W^n a^n + C_1$, where $C_1 = \{y \in V | v(y) > nv(a)\}$. If B is a finitely generated ideal of D of the form $W_1 b + C$, where $v(b) = v(a)$ and W_1 is a finite-dimensional k-subspace of K, then*

$$(A \cap B)^n = (Wa \cap W_1 b)^n + C_1.$$

Proof. Any element $x \in A^n$ is a finite sum of elements $a_1 a_2 \ldots a_n$, $a_i \in A$. Writing $a_i = w_i a + c_i$, $w_i \in W$, $c_i \in C$, we obtain $a_1 a_2 \ldots a_n = w_1 \ldots w_n a^n + w_1 \ldots w_{n-1} a^{n-1} c_n + \ldots + c_1 c_2 \ldots c_n$. Since each term in this expression except the first has v-value greater than $nv(a)$, $a_1 \ldots a_n \in W^n a^n + C_1$. Hence $x \in W^n a^n + C_1$.

Conversely, $Wa \subseteq A$ implies $W^n a^n \subseteq A^n$. Also if $y \in C_1$, then $z(y) > nv(a) = v(a^n)$ implies that $y \in a^n D \subseteq A^n$. It then follows that $W^n a^n + C_1 \subseteq A^n$, and so equality holds.

The proof that $(A \cap B)^n = (Wa \cap W_1 b)^n + C_1$ follows similarly.

If R is a subring of the commutative ring S, then R is said to have *property* (n) *with respect to* S if for each $\xi \in S$, there exist $a_i, b_i \in R$ such that $\xi^i = a_i \xi^n + b_i$, $i = 1, \ldots, n - 1$. We are interested in the case where R and S are fields.

THEOREM 3.3. *Let V be a valuation ring of the form $K + M$, K a field, M the maximal ideal of V, and let v be a valuation associated with V. Suppose that k is a proper subfield of K such that k has property (n) with respect to K for some positive integer n. Then $D = k + M$ has property $(n)''$.*

Proof. We remark that Ohm [**7**] observed that A has property (n), and Gilmer [**3**] showed that A has property $(n)'$.

Let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_m)$ be finitely generated ideals of D, let $t_1 = \min \{v(a_i) | 1 \leqq i \leqq n\}$ and $t_2 = \min \{v(b_i) | 1 \leqq i \leqq m\}$. If $t_1 > t_2 = v(b_j)$, then by Lemma 3.1, each $a_i \in b_j D$, so that $A \subseteq B$, and the result is clear. Thus we may assume that $t_1 = t_2$, $A = W_1 a + C$ and $B = W_2 b + C$, $a \in A$, $b \in B$, $v(a) = v(b) = t_1$, $C = \{y \in V | v(y) > t_1\}$, and W_1, W_2 are finite-dimensional k-subspaces of K.

In [**1**], J. W. Brewer showed that for k to have property (n) with respect to K, it is necessary that $[K : k] = 2$. It follows that we have the following three cases to consider:

1. $A = Ka + C$, $B = Kb + C$.
2. $A = ka + C$, $B = Kb + C$.
3. $A = ka + C$, $B = kb + C$.

Since $v(a) = v(b)$, there exists $\gamma \in K - \{0\}$ and $m \in M$ such that $b/a = \gamma + m$. Since $am \in C$, there is no loss of generality in assuming that $m = 0$

(that is, γa is in B and is an element of B of minimal value). Using this relationship and Lemma 3.2, the three cases become

1. $A = Ka + C = K\gamma a + C = Kb + C = B$.
2. $A = ka + C \subseteq Ka + C = K\gamma a + C = B$.
3. $(A \cap B)^n = (ka \cap k\gamma a)^n + C_1 = (k \cap k\gamma)^n a^n + C_1$.
   $A^n \cap B^n = [k^n a^n \cap k^n(\gamma a)^n] + C_1 = (k \cap k\gamma^n)a^n + C_1$.

Now the containment $(k \cap k\gamma)^n \subseteq k \cap k\gamma^n$ always holds, and since k, $k\gamma$ and $k\gamma^n$ are one-dimensional k-subspaces of K, it follows that, for $1 \leqq i \leqq n$, $k \cap k\gamma^i$ is either k or (0), depending upon whether γ^i is, or is not, in k. Since k has property (n) with respect to K, it follows from Lemma 5.5 of [**3**] that $\gamma \in k$ if and only if $\gamma^n \in k$. Thus $(k \cap k\gamma)^n = (k \cap k\gamma^n)$, and hence $(A \cap B)^n = A^n \cap B^n$.

Example 3.4. In [**7**], Ohm constructed fields k, K, with k a proper subfield of K, such that k has property (n) with respect to K for each positive integer n. If M is the maximal ideal of the valuation ring $K[[X]]$, then the domain $D = k + M$ has property $(n)''$ for each n. Since $K[[X]]$ in the integral closure of D, D is not Prüfer.

Our next example shows that property $(n)''$ is indeed stronger than properties (n) and $(n)'$. Suppose that V_1 and V_2 are rank one discrete valuation rings having a common quotient field L, that K is a common subfield of V_1 and V_2, and that $V_i = K + M_i$, where M_i is the maximal ideal of V_i. We are interested in the domain $D = K + (M_1 \cap M_2)$. If v_i is a valuation associated with V_i, then by the approximation theorem for independent valuations, there exist $a, b \in L$ such that $v_1(a) = v_2(b) = 1$, $v_1(b) = v_2(a) = 0$. Using this notation, we have

LEMMA 3.5. *The domain D is local. In particular, if A is a nonprincipal ideal of D, then there exist positive integers n, m such that $A = (a^n b^m, a^{n+1}b^m)$. Moreover,*

$$(a^n b^m, a^{n+1}b^m) = \{d \in D | v_1(d) \geqq n, v_2(d) \geqq m\}.$$

Proof. We show first that if $t \geqq n$, $s \geqq m$, and if $v_1(x) = v_2(x) = 0$, then there exist $\xi_1, \xi_2 \in K$ and $z \in M_1 \cap M_2$ such that

$$(\#) \qquad a^t b^s x = (\xi_1 + z)a^n b^m + \xi_2 a^{n+1} b^m.$$

Suppose that $a^{t-n}b^{s-m}\, x \equiv \mu_i(M_i)$, and that $a \equiv \eta(M_2)$, where $\mu_i, \eta \in K$, $\eta \neq 0$. Then $\xi_1 = \mu_1$, $\xi_2 = \eta^{-1}(\mu_2 - \mu_1)$ is the unique solution in K of the system of equations

$$\mu_1 = X,$$
$$\mu_2 = X + \eta Y.$$

It follows that $a^{t-n}\, b^{s-m}\, x - \xi_1 - a\xi_2 = z \in M_1 \cap M_2$. Hence (#) follows, and from this we have $(a^n b^m, a^{n+1}b^m) = \{d \in D | v_1(d) \geqq n, v_2(d) \geqq m\}$.

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Now let A be a nonprincipal ideal of D. Let $n = \min \{v_1(x)|x \in A\}$, let $m = \min \{v_2(x)|x \in A \text{ and } v_1(x) = n\}$, and let y be an element in A such that $v_1(y) = n$, $v_2(y) = m$. Write $y = a^n b^m u$, where $v_1(u) = v_2(u) = 0$. We choose $x \in A - (a^n b^m u)D$. It is clear that $v_1(x) \geqq n$. We show that $v_2(x) \geqq m$. If $v_1(x) = n$, it is clear that $v_2(x) \geqq m$; and if $v_1(x) > n$, then $x + a^n b^m u \in A$, $v_1(x + a^n b^m u) = n$, and hence $v_2(x + a^n b^m u) \geqq m$, so that $v_2(x) \geqq m$. It then follows from (#) that there exist $\xi_1, \xi_2 \in K$ and $z \in M_1 \cap M_2$ such that $xu^{-1} = (\xi_1 + z)a^n b^m + \xi_2 a^{n+1} b^m$. Therefore, $x = (\xi_1 + z)a^n b^m u + \xi_2 u a^{n+1} b^m = (\xi_1 + z)a^n b^m u + \xi_2 a^{n+1} b^m (\mu + hb)$, where $u = \mu(M_2)$ and where $h \in V_2$. It then follows that $x = (\xi_1 + z + \xi_2 hab)a^n b^m + \mu\xi_2 a^{n+1} b^m$, where $\xi_2 hab \in M_1 \cap M_2$. Since $x \notin a^n b^m uD$, $\mu\xi_2 \neq 0$ and thus $a^{n+1} b^m \in (x, a^n b^m u) \subseteq A$. Hence $(a^n b^m, a^{n+1} b^m) \subseteq A \subseteq (a^n b^m, a^{n+1} b^m)$, and equality follows.

THEOREM 3.6. *Let n be an integer greater than one. The domain $D = K + (M_1 \cap M_2)$ does not have property $(n)''$. D has property $(n)'$ if and only if the mapping $x \to x^n$ of K into K is one-to-one.*

Proof. Let $A = (ab)D$ and $B = (a^2 b)D$. Then $(A \cap B)^n = (a^{3n} b^{2n}, a^{3n+1} b^{2n})$ while $A^n \cap B^n = (a^{2n+1} b^{n+1}, a^{2n+2} b^{n+1})$, so that $(A \cap B)^n \subset A^n \cap B^n$.

Theorem 7.1 of [**3**] shows that D has property (n) if and only if the mapping $x \to x^n$ of K is one-to-one. Hence it suffices to show that if D has property (n), then it also has property $(n)'$.

Now if J is any domain with property (n), then J has property $(n)'$ with respect to principal ideals. That is, if $x^n \in (y)^n$, then $x \in (y)$, for since J has property (n), $xy^{n-1} \in (x^n, y^n) = (y^n)$. Thus we need only consider the nonprincipal ideals of D.

If A is a nonprincipal ideal of D, then it follows from Lemma 3.5 that $A = a^n b^m (V_1 \cap V_2)$ is an ideal of $V_1 \cap V_2$, and hence A is an intersection of valuation ideals. Lemma 5.1 of [**3**] shows that if the ideal A of the domain D is an intersection of valuation ideals of D, and if $x \in D$ is such that $x^n \in A^n$, then $x \in A$. Thus the proof is complete.

Example 3.7. The prime field π_2 with two elements has the property that $x \to x^n$ is one-to-one for any positive integer n. Let $V_1 = (\pi_2[X])_{(X)} = \pi_2 + M_1$, and let $V_2 = (\pi_2[X])_{(X+1)} = \pi_2 + M_2$. Then if $D = \pi_2 + (M_1 \cap M_2)$, we obtain an example of a domain having property $(n)'$ for each positive integer, but having property $(n)''$ for no $n > 1$.

Remark 3.8. For a positive integer $n > 1$, there are essentially two different methods of obtaining domains with property (n) which are not integrally closed. One is the $k + M$ construction of our Theorem 3.3. As we have previously remarked, Gilmer in [**3**] showed that domains constructed in this way have property $(n)'$. The second way of obtaining non-integrally closed domains with property (n) is the $K + (M_1 \cap M_2)$-construction of our Theorem 3.6. As we have shown in the proof of Theorem 3.6, property (n) in such a domain is equivalent to property $(n)'$. It follows that no example has been pointed out in the literature to show that (n) does not imply $(n)'$. In fact, it is conceivable that the properties $(n)'$, (n), and $(n)^*$ of [**3**] are equivalent.

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Florida State University,
Tallahassee, Florida