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SIMPLE CONDITIONS FOR MATRICES TO BE BOUNDED OPERATORS ON l_p

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ABSTRACT. The two theorems proved yield simple yet reasonably general conditions for triangular matrices to be bounded operators on l_p. The theorems are applied to Nörlund and weighted mean matrices.

1. Introduction.

Suppose throughout that

$$1 < p < \infty, \quad \frac{1}{p} + \frac{1}{q} = 1,$$

and that $A := (a_{nk})_{n,k \geq 0}$ is a triangular matrix of non-negative real numbers, that is $a_{nk} \geq 0$ for $n, k \geq 0$, and $a_{nk} = 0$ for $n > k$. Let l_p be the Banach space of all complex sequences $x = (x_n)_{n \geq 0}$ with norm

$$\|x\|_p := \Big(\sum_{n=0}^{\infty} |x_n|^p \Big)^{1/p} < \infty,$$

and let $B(l_p)$ be the Banach algebra of all bounded linear operators on l_p. Thus $A \in B(l_p)$ if and only if $Ax \in l_p$ whenever $x \in l_p$, Ax being the sequence with n-th term $(Ax)_n := \sum_{k=0}^{n} a_{nk} x_k$. Let

$$\|A\|_p := \sup_{\|x\|_p \leq 1} \|Ax\|_p,$$

so that $A \in B(l_p)$ if and only if $\|A\|_p < \infty$, in which case $\|A\|_p$ is the norm of A.

We shall prove the following two theorems:

THEOREM 1. *Suppose that*

$$M_1 := \sup_{n \geq 0} \sum_{k=0}^{n} a_{nk} < \infty, \tag{1}$$

$$M_2 := \sup_{\substack{0 \leq k \leq n/2 \\ n \geq 0}} (n+1) a_{nk} < \infty, \tag{2}$$

and

$$M_3 := \sup_{k \geq 0} \sum_{n=k}^{2k} a_{nk} < \infty. \tag{3}$$

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Then $A \in B(l_p)$ *and* $\|A\|_p \le \mu_1^{1/q}\mu_2^{1/p}$, *where*

$$\mu_1 \le 2^{1/p}M_1 + qM_2 \quad \text{and} \quad \mu_2 \le M_3 + qM_2.$$

THEOREM 2. *Suppose that (1) holds, and that*

$$a_{nk} \le M_4 a_{nj} \quad \text{for } 0 \le k \le j \le n, \tag{4}$$

where M_4 *is a positive number independent of* k, j, n. *Then* $A \in B(l_p)$ *and*

$$\max\Big(a_{00}, \frac{\lambda q}{M_4}\Big) \le \|A\|_p \le qM_1M_4^{q-1},$$

where $\lambda := \liminf na_{n0}$.

These theorems yield simple yet fairly general conditions for $A \in B(l_p)$. In Section 4 we shall illustrate their scope by applying them to Nörlund and weighted mean matrices.

2. **Lemmas.** We require the following known results:

LEMMA 1 (SEE [4, THEOREM 2]). *If*

$$\mu_1 := \sup_{n\ge 0}\sum_{k=0}^{n} a_{nk}\Big(\frac{n+1}{k+1}\Big)^{1/p} < \infty \quad \text{and} \quad \mu_2 := \sup_{k\ge 0}\sum_{n=k}^{\infty} a_{nk}\Big(\frac{k+1}{n+1}\Big)^{1/q} < \infty,$$

then $A \in B(l_p)$ *and* $\|A\|_p \le \mu_1^{1/q}\mu_2^{1/p}$.

LEMMA 2 (SEE [10, LEMMA 4] AND [8, LEMMA 1]). *If* $q > 1$ *and* $z_n \ge 0$ *for* $n = k, k+1, \ldots$, *where* k *is a non-negative integer, then*

$$\Big(\sum_{n=k}^{\infty} z_n\Big)^q \le q\sum_{n=k}^{\infty} z_n\Big(\sum_{j=n}^{\infty} z_j\Big)^{q-1}.$$

3. **Proofs of the theorems.**

PROOF OF THEOREM 1. Let $s := 1/p$, $t := 1/q$, and let μ_1, μ_2 be as in Lemma 1. Then, by (2),

$$\begin{aligned}(n+1)^s \sum_{0\le k\le n/2} \frac{a_{nk}}{(k+1)^s} &\le \Big(\sup_{0\le k\le n/2} a_{nk}\Big)(n+1)^s \sum_{0\le k\le n/2}\frac{1}{(k+1)^s}\\ &\le \Big(\sup_{0\le k\le n/2} a_{nk}\Big)\frac{(n+1)^s(n+2)^{1-s}}{(1-s)2^{1-s}} \le \frac{M_2}{1-s} = qM_2;\end{aligned}$$

and, by (1),

$$(n+1)^s \sum_{n/2<k\le n} \frac{a_{nk}}{(k+1)^s} \le \frac{(n+1)^s 2^s}{(n+2)^s} M_1 \le 2^s M_1.$$

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Hence

$$\mu_1 \le 2^s M_1 + qM_2.$$

Also, by (2),

$$(k+1)^t \sum_{n=2k+1}^{\infty} \frac{a_{nk}}{(n+1)^t} \le M_2(k+1)^t \sum_{n=2k+1}^{\infty} \frac{1}{(n+1)^{t+1}}$$
$$\le M_2(k+1)^t \int_{2k}^{\infty} \frac{dx}{(x+1)^{t+1}} = M_2 \frac{q(k+1)^t}{(2k+1)^t} \le qM_2;$$

and, by (3),

$$(k+1)^t \sum_{n=k}^{2k} \frac{a_{nk}}{(n+1)^t} \le M_3.$$

Hence

$$\mu_2 \le qM_2 + M_3.$$

The desired conclusion now follows from Lemma 1. ■

PROOF OF THEOREM 2. Our proof is modelled on the proof given by Johnson, Mohapatra and Ross of Theorem 1 in [9]. Let T be the transpose of A. We shall use the familiar result that $A \in B(l_p)$ if and only if $T \in B(l_q)$ and $\|A\|_p = \|T\|_q$. Let $y = Tx$ where $x = (x_n)$ is a real non-negative sequence in l_q. Then, by Lemma 2, (4), and Hölder's inequality,

$$\|y\|_q^q = \sum_{k=0}^{\infty} \Big(\sum_{n=k}^{\infty} a_{nk} x_n \Big)^q \le q \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{nk} x_n \Big(\sum_{j=n}^{\infty} a_{jk} x_j \Big)^{q-1}$$
$$\le qM_4^{q-1} \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{nk} x_n \Big(\sum_{j=n}^{\infty} a_{jn} x_j \Big)^{q-1} = qM_4^{q-1} \sum_{n=0}^{\infty} x_n y_n^{q-1} \sum_{k=0}^{n} a_{nk}$$
$$\le qM_1 M_4^{q-1} \sum_{n=0}^{\infty} x_n y_n^{q-1} \le qM_1 M_4^{q-1} \Big(\sum_{n=0}^{\infty} x_n^q \Big)^{1/q} \Big(\sum_{n=0}^{\infty} y_n^q \Big)^{1/p}$$
$$= qM_1 M_4^{q-1} \|x\|_q \|y\|_q^{q/p}.$$

It follows that $\|y\|_q \le qM_1 M_4^{q-1} \|x\|_q$, and hence that $\|T\|_q = \|A\|_p \le qM_1 M_4^{q-1}$.

To establish the lower estimate for $\|A\|_p$, fix $\delta \in (0,1)$ and choose a positive integer N so large that $na_{n0} > \delta\lambda$ for all $n \ge N$. Suppose $M > N$ and define $x = (x_n)$ by setting $x_n := n^{-1/p}$ for $N \le n \le M$, and $x_n := 0$ otherwise. Then, by (4),

$$\|Ax\|_p^p \ge \sum_{n=N}^{M} \Big(\sum_{k=N}^{n} a_{nk} x_k \Big)^p \ge \Big(\frac{\delta\lambda}{M_4} \Big)^p \sum_{n=N}^{M} \Big(\frac{1}{n} \sum_{k=N}^{n} k^{-1/p} \Big)^p$$
$$\ge \Big(\frac{\delta\lambda}{M_4} \Big)^p \sum_{n=N}^{M} \Big(\frac{1}{n} \int_N^n x^{-1/p}\, dx \Big)^p = \Big(\frac{\delta\lambda q}{M_4} \Big)^p \sum_{n=N}^{M} \frac{1}{n} \Big(1 - \Big(\frac{N}{n} \Big)^{1/q} \Big)^p$$
$$= \Big(\frac{\delta\lambda q}{M_4} \Big)^p \rho_M \sum_{n=N}^{M} \frac{1}{n} = \Big(\frac{\delta\lambda q}{M_4} \Big)^p \rho_M \|x\|_p^p,$$

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where $\rho_M \longrightarrow 1$ as $M \longrightarrow \infty$. It follows that $\|A\|_p \geq \frac{\delta\lambda q}{M_4}$ and hence, since δ can be arbitrarily close to 1 in $(0,1)$, that $\|A\|_p \geq \frac{\lambda q}{M_4}$. Finally, for the unit coordinate sequence $e_0 = (1,0,0,\ldots)$, we have $\|Ae_0\|_p \geq a_{00}\|e_0\|_p$, so that $\|A\|_p \geq a_{00}$. ■

4. **Examples involving Nörlund and weighted mean matrices.** Let $a := (a_n)$ be a sequence of real non-negative numbers with $a_0 > 0$, and let $A_n := a_0 + a_1 + \cdots + a_n$.

The *Nörlund matrix* N_a is defined to be the triangular matrix (a_{nk}) with $a_{nk} := \frac{a_{n-k}}{A_n}$ for $0 \leq k \leq n$, and $a_{nk} := 0$ for $k > n$.

The *weighted mean matrix* M_a is defined to be the triangular matrix (a_{nk}) with $a_{nk} := \frac{a_k}{A_n}$ for $0 \leq k \leq n$, and $a_{nk} := 0$ for $k > n$.

Observe that

$$\sum_{k=0}^{n} \frac{a_{n-k}}{A_n} = 1 \quad \text{and} \quad \sum_{n=k}^{2k} \frac{a_{n-k}}{A_n} \leq \frac{1}{A_k} \sum_{n=k}^{2k} a_{n-k} = 1,$$

so that the Nörlund matrix N_a automatically satisfies conditions (1) and (3) of Theorem 1 with $M_1 = 1$ and $M_3 \leq 1$. The weighted mean matrix M_a also satisfies (1) with $M_1 = 1$.

EXAMPLE 1. Suppose that

$$M_2' := \sup_{n \geq 0} \frac{(n+1)a_n}{A_n} < \infty. \tag{5}$$

It is immediate that, for the Nörlund matrix N_a, (2) implies (5) with $M_2' \leq M_2$. On the other hand we have, for $0 \leq k \leq n/2$,

$$\frac{(n+1)a_{n-k}}{A_n} = \frac{(n+1-k)a_{n-k}}{A_{n-k}} \cdot \frac{A_{n-k}}{A_n} \cdot \frac{n+1}{n+1-k} \leq 2\frac{(n+1-k)a_{n-k}}{A_{n-k}} \leq 2M_2',$$

so that (5) implies (2) with $M_2 \leq 2M_2'$ for the Nörlund matrix N_a.

It follows now from Theorem 1 that, subject to (5), $N_a \in B(l_p)$ and $\|N_a\|_p \leq \mu_1^{1/q}\mu_2^{1/p}$, where

$$\mu_1 \leq 2^{1/p} + 2qM_2' \quad \text{and} \quad \mu_2 \leq 1 + 2qM_2'.$$

This result was proved directly by Borwein and Cass in [3] with a slightly different and better estimate for the upper bound of the operator norm. See also [2] and [7] for related results.

EXAMPLE 2. Suppose that (a_n) *is non-increasing*. It is immediate that this implies (5) with $M_2' \leq 1$, but it also implies (4) with $M_4 = 1$ for the Nörlund matrix N_a. Hence either Theorem 1 or Theorem 2 yields that $N_a \in B(l_p)$, and Theorem 2 shows that

$$\max(1, \lambda q) \leq \|N_a\|_p \leq q,$$

where $\lambda := \liminf \frac{na_n}{A_n}$. This result was proved as Theorem 1 by Johnson, Mohapatra and Ross in [9]. Our Theorem 2 is clearly a generalization of their theorem.

EXAMPLE 3. Suppose that (a_n) *is non-decreasing*. Evidently the weighted mean matrix M_a satisfies (4) with $M_4 = 1$. It follows from Theorem 2 that $M_a \in B(l_p)$ with $\|M_a\|_p \le q$. This result was first proved by Cartlidge [6] by an entirely different method. See also [1], [5] and [7] for related and more general results.

The preceding examples involved proofs of known results. For the next example we use Theorem 2 to prove a new result which combines Examples 2 and 3. Let $a := (a_n)$, $b := (b_n)$ be sequences of real non-negative numbers with $a_0 > 0$, $b_0 > 0$, and let $c_n := a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$. The *generalized Nörlund matrix* $N_{a,b}$ is defined to be the triangular matrix (a_{nk}) with $a_{nk} := \frac{a_{n-k} b_k}{c_n}$ for $0 \le k \le n$, and $a_{nk} := 0$ for $k > n$.

EXAMPLE 4. Suppose (a_n) is non-decreasing and (b_n) is non-increasing. Then $N_{a,b} \in B(l_p)$ and $\max(1, \lambda q) \le \|N_{a,b}\|_p \le q$, where $\lambda := \liminf \frac{n a_n b_0}{c_n}$.

PROOF. Evidently the matrix $N_{a,b}$ satisfies (1) and (4) with $M_1 = M_4 = 1$. The desired conclusions follow from Theorem 2. ∎

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