SOME STUDIES ON SEMI-LOCAL RINGS

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Introduction. The concept of semi-local rings was introduced by C. Chevalley [1][0], which the writer has generalized in a recent paper [7] by removing the chain condition. The present paper aims mainly at the study of completions of semi-local rings. First in § 1 we investigate semi-local rings which are subdirect sums of semi-local rings, and we see in § 2 that a Noetherian semi-local ring R is complete if (and only if) $R/\mathfrak{p}$ is complete for every minimal prime divisor $\mathfrak{p}$ of zero ideal, together with some other properties. Further we consider in § 3 subrings of the completion of a semi-local ring. § 4 gives some supplementary remarks to [7], Chapter II, Proposition 8.

TERMS. A ring means a commutative ring with identity and under the term "subring" we mean a subring having the same identity. Semi-local rings or local rings are those in the sense of Nagata [7] (or [6]). So, (semi-)local rings in the sense of Chevally [1] (or Cohen [2]) are called Noetherian (semi-)local rings.

1. Subdirect sums of semi-local rings.

LEMMA 1.1. Let R and R^* be a subdirect sum and the direct sum of rings $R_1, R_2, \ldots, R_n$ respectively, and suppose that R is quasi-semi-local.[1] We denote by φ_i the natural homomorphism of R onto R_i, by $\mathfrak{n}_i$ the kernel of φ_i and by $\mathfrak{m}, \mathfrak{m}^*, \mathfrak{m}_i$ the J-radicals[2] of R, R^*, R_i respectively ($i=1, 2, \ldots, n$). Then we have (1) $\mathfrak{m}^*=\mathfrak{m}R^*$, (2) $\mathfrak{m}^*\cap R=\mathfrak{m}$, (3) $\mathfrak{m}^*=\mathfrak{m}_1+\mathfrak{m}_2+\ldots+\mathfrak{m}_n$, (4) $\varphi_i(\mathfrak{m}^k)=\mathfrak{m}_i^k$ ($k=1, 2, \ldots$), (5) $(\mathfrak{n}_1+\mathfrak{n}_2)((\mathfrak{m}^*)^k\cap R)\subseteqq(\mathfrak{n}_1+\mathfrak{n}_2)\mathfrak{m}^k$ ($k=1, 2, \ldots$) provided $n=2$.

Proof. (1), (2) and (3) are almost evident.[3] To prove (4), let $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ be the totality of maximal ideals of R. Then it follows $\mathfrak{m}^k=\mathfrak{p}_1^k\cap\ldots\cap\mathfrak{p}_h^k$ $=\mathfrak{p}_1^h\ldots\mathfrak{p}_h^k$ and $\varphi_i^{-1}(\mathfrak{m}_i^k)=(\mathfrak{p}_1^k+\mathfrak{n}_i)\cap\ldots\cap(\mathfrak{p}_h^k+\mathfrak{n}_i)=(\mathfrak{p}_1^k+\mathfrak{n}_i)\ldots(\mathfrak{p}_h^k+\mathfrak{n}_i)$. This proves (4). Finally, assume that $n=2$, and consider an element b_1 of

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0) Numbers in brackets refer to the bibliography at the end.

1) A quasi-semi-local ring is a ring which has only a finite number of maximal ideals; cf. [9].

2) J-radical (Jacobson radical) of a ring is the intersection of all maximal ideals in the ring.

3) Cf. [9, Lemma 2].

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$(\mathfrak{m}^*)^k \cap R$. Then we can choose an element b_2 from $\mathfrak{m}^k$ so that $\varphi_1(b_1)=\varphi_1(b_2)$, i.e., $b=b_1-b_2 \in (\mathfrak{m}^k+\mathfrak{n}_2)\cap\mathfrak{n}_1$ (by virtue of (3) just above). Then we have $(\mathfrak{n}_1+\mathfrak{n}_2)b = \mathfrak{n}_1 b \subseteqq \mathfrak{n}_1((\mathfrak{m}^k+\mathfrak{n}_2)\cap\mathfrak{n}_1) \subseteqq \mathfrak{n}_1\mathfrak{m}^k$, which proves (5).

Next we cite lemmas due to Chevalley:

LEMMA 1.2. Let R be a complete Noetherian semi-local ring. If every $\mathfrak{a}_n$ is an open ideal ($n=1, 2, \ldots$) and if $\bigcap_{n=1}^{\infty}\mathfrak{a}_n=(0)$, then $\{\mathfrak{a}_n;\ n=1, 2, \ldots\}$ is a system of neighbourhoods of zero. [1, § II, Lemma 7]

LEMMA 1.3. Let R be a Noetherian semi-local ring with J-radical $\mathfrak{m}$ and let c be an element of R which is not a zero divisor. Then $\{\mathfrak{m}^n : cR;\ n=1, 2, \ldots\}$ forms a system of neighbourhoods of zero. [1, § II, Lemma 9]

Now we prove

THEOREM 1. *Let a Noetherian semi-local ring R be a subdirect sum of two rings R_1 and R_2. Let $\mathfrak{n}_i$ be the kernel of natural homomorphism φ_i of R onto R_i ($i=1, 2$). If $\mathfrak{n}_1+\mathfrak{n}_2$ contains a non-zero-divisor c, then R is a subspace of the direct sum R^* of R_1 and R_2. (R^* is clearly a Noetherian semi-local ring.)*

Proof. Let $\mathfrak{m}$ and $\mathfrak{m}^*$ be the J-radicals of R and R^* respectively. Then we have $\mathfrak{m}^k \subseteqq (\mathfrak{m}^*)^k \cap R$, since $\mathfrak{m}=\mathfrak{m}^*\cap R$ by Lemma 1.1. On the other hand, it follows from Lemma 1.1 also that $c((\mathfrak{m}^*)^k\cap R)\subseteqq \mathfrak{m}^k$, i.e., $(\mathfrak{m}^*)^k\cap R \subseteqq \mathfrak{m}^k : cR$. These prove our assertion by virtue of Lemma 1.3.

COROLLARY. Let R be a Noetherian semi-local ring. If the intersection of ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are zero and if $\mathfrak{q}_i : \mathfrak{q}_j = \mathfrak{q}_i$ for every pair $i \neq j$, then R is a subspace of the direct sum of rings $R/\mathfrak{q}_1, \ldots, R/\mathfrak{q}_n$; in fact, these assumptions for $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ are satisfied if $\mathfrak{q}_1\cap\ldots\cap\mathfrak{q}_n$ is a shortest representation of zero ideal as an intersection of primary ideals and if zero ideal has no imbedded prime divisor.

On the other hand, we have

THEOREM 2. *Let a Noetherian semi-local ring R be a subdirect sum of (Noetherian semi-local) rings $R_1, \ldots, R_n$. We denote by $\mathfrak{n}_i$ the kernel of natural homomorphism φ_i of R onto R_i for each i. Let $\bar{R}$ be the completion of R. Then R is a subspace of the direct sum R^* of $R_1, \ldots, R_n$ if and only if $\bigcap_{i=1}^{n}\mathfrak{n}_i\bar{R}=(0)$.*

Proof. We denote by $\bar{R}^*$ the completion of R^* and by $\mathfrak{m}$, $\bar{\mathfrak{m}}$, $\mathfrak{m}^*$, $\bar{\mathfrak{m}}^*$ the J-radicals of R, $\bar{R}$, R^*, $\bar{R}^*$ respectively.

If R is a subspace of R^*, it is evident that $\bigcap_{i=1}^{n}\mathfrak{n}_i\bar{R}=(0)$. Conversely, assume that $\bigcap_{i=1}^{n}\mathfrak{n}_i\bar{R}=(0)$. Then $\bar{R}$ is a subdirect sum of completions $\bar{R}_i$ of R_i ($i=1, 2, \ldots, n$)

by the natural way.[4] Whence $\{(\bar{\mathfrak{m}}^*)^k \cap \bar{R};\ k=1, 2, \ldots\}$ forms a system of neighbourhoods of zero in $\bar{R}$ by virtue of Lemma 1.2, that is, for any positive integer k there exists a positive integer $n(k)$ such that $(\bar{\mathfrak{m}}^*)^{n(k)} \cap \bar{R} \subseteqq \bar{\mathfrak{m}}^k$. Whence $(\mathfrak{m}^*)^{n(k)} \cap R \subseteqq \mathfrak{m}^k$, which shows that R is a subspace of R^*.

COROLLARY 1. If a Noetherian semi-local ring R is complete and if $\mathfrak{n}_1, \ldots, \mathfrak{n}_n$ are ideals in R such that $\bigcap_{i=1}^{n} \mathfrak{n}_i = (0)$, then R is a subspace of the direct sum of $R/\mathfrak{n}_1, \ldots, R/\mathfrak{n}_n$.

COROLLARY 2. Let R be a Noetherian semi-local ring, and let there be ideals $\mathfrak{q}_i$ $(i=1, 2, \ldots, n)$ in R such that $\mathfrak{q}_i : \mathfrak{q}_j = \mathfrak{q}_i$ for every pair $i \neq j$. Then we have $(\bigcap_{i=1}^{n} \mathfrak{q}_i)\bar{R} = \bigcap_{i=1}^{n} \mathfrak{q}_i \bar{R}$, where $\bar{R}$ denotes the completion of R.

Proof. This is an immediate consequence of our Theorem 2 and Corollary to Theorem 1.

THEOREM 3. *Let a semi-local ring R be a subdirect sum of semi-local rings $R_1, \ldots, R_n$. If R is a subspace of the direct sum R^* of $R_1, \ldots, R_n$, then R is a closed subspace of R^*. In particular, if moreover R^* is complete, i.e., if every R_i is complete, then so is R too.*[5]

Proof. Let $(a_i = a_{i1} + \ldots + a_{in})$ $(a_i \in R,\ a_{ik} \in R_k)$ $(i=1, 2, \ldots)$ be a convergent sequence in R with limit $c = c_1 + \ldots + c_n$ $(c_k \in R_k)$ in R^*. Suppose that $c \notin R$. Let c_i' be, for each i, an element of R which is mapped on c_i by the natural homomorphism φ_i of R onto R_i. Then we would have $\bigcap_{i=1}^{n} c_i' + \mathfrak{n}_i = \theta$[6], where $\mathfrak{n}_i$ denotes the kernel of φ_i. Since every semi-local ring is a normal space and since each $\mathfrak{n}_i$ is closed in R, there exists, for each i, an open set U_i in R such that $U_j \supseteqq c_i' + \mathfrak{n}_i$ and $\bigcap_{i=1}^{n} U_i = \theta$. This contradicts to our assumption that c is the limit of the sequence (a_i) in R, and we have $c \in R$.

2. Completeness of a semi-local ring.

LEMMA 2.1. Let R be a semi-local ring and $\mathfrak{a}$ a closed ideal in R. Then R is complete if both $R/\mathfrak{a}$ and $\mathfrak{a}$ are complete.

Proof. Let $\bar{R}$ be the completion of R. Since $\mathfrak{a}$ is complete, it follows that $\mathfrak{a}\bar{R} = \mathfrak{a}$ and $\mathfrak{a}$ is closed in $\bar{R}$. Further, since $R/\mathfrak{a}$ is complete, it follows $\bar{R}/\mathfrak{a}\bar{R} = \bar{R}/\mathfrak{a} = R/\mathfrak{a}$,[7] and this proves our assertion.

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4) Cf. [1, II, Proposition 13] or [7, Chapter II, Proposition 1].

5) If R is complete, then R^* is complete without the assumption that R is a subspace of R^*.

6) θ denotes the empty set.

7) Cf. l.c. note 4).

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LEMMA 2.2. Let R be a Noetherian semi-local ring. Let c be an element of R such that $c^2=0$. Then R is complete whenever R/cR is complete.

Proof. By virtue of preceding lemma, we have only to prove that cR is complete. Let (ca_n) $(n=1, 2, \ldots)$ $(a_n \in R)$ be a convergent sequence in R such that $c(a_n - a_{n+1}) \in \mathfrak{m}^n$ $(n=1, 2, \ldots)$, where $\mathfrak{m}$ denotes the J-radical of R. Set $\mathfrak{q}_n = \mathfrak{m}^n : cR$ $(n=1, 2, \ldots)$ and $\mathfrak{d} = (0) : cR$. Then we have $\bigcap_{n=1}^{\infty} \mathfrak{q}_n = \mathfrak{d}$ because $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$. Since $R/\mathfrak{d}$ is complete, $\{\mathfrak{q}_n/\mathfrak{d}\,;\ n=1, 2, \ldots\}$ forms a system of neighbourhoods of zero in $R/\mathfrak{d}$, by virtue of Lemma 1.2, this shows that (a_n) is a convergent sequence in $R/\mathfrak{d}$. Let a be its limit, then ca is the limit of (ca_n). This proves our assertion.

THEOREM 4. *Let R be a Noetherian semi-local ring with p-radical*[8] $\mathfrak{n}$. *Then R is complete whenever $R/\mathfrak{n}$ is complete.*

This is an immediate consequence of Lemma 2.2.

THEOREM 5. *Let R be a Noetherian semi-local ring. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ be the totality of minimal prime divisors of zero ideal in R. Then R is complete whenever every $R/\mathfrak{p}_i$ is complete.*

This follows immediately from Corollary to Theorem 1 and Theorems 3, 4.

3. Subrings of the completion of a semi-local ring.

Let R be a ring and $\mathfrak{m}$ its ideal. Suppose that $\bigcap_{n=1}^{\infty} \mathfrak{m}^n = (0)$. Then R is called an $\mathfrak{m}$-adic ring if R is topologized by taking $\{\mathfrak{m}^n\,;\ n=1, 2, \ldots\}$ as a system of neighbourhoods of zero.

THEOREM 6. *Let R be a semi-local ring and let $\bar{R}$ be its completion. Let $\mathfrak{m}$ and $\bar{\mathfrak{m}}$ be the J-radicals of R and $\bar{R}$ respectively. If R' is a subring of $\bar{R}$ containing R and if we set $\mathfrak{m}' = \bar{\mathfrak{m}} \cap R'$, then we have $R \cap \mathfrak{m}'^k = \mathfrak{m}^k$ $(k=1, 2, \ldots)$. Consequently, R is a subspace of $\mathfrak{m}'$-adic ring R'.*

Proof. Since $\bar{R}$ is an $\bar{\mathfrak{m}}$-adic ring, R' becomes an $\mathfrak{m}'$-adic ring. Since clearly $\mathfrak{m} = \mathfrak{m}' \cap R$, we have $\mathfrak{m}^k \subseteq \mathfrak{m}'^k \cap R$. On the other hand, it follows from $\bar{\mathfrak{m}}^k \cap R = \mathfrak{m}^k$ that $\mathfrak{m}^k = (\bar{\mathfrak{m}}^k \cap R') \cap R \supseteq \mathfrak{m}'^k \cap R$, because $\mathfrak{m}' = \bar{\mathfrak{m}} \cap R$. These prove our assertion.

THEOREM 7. *Let R be a semi-local ring and $\bar{R}$ its completion. If a subring R' of $\bar{R}$ containing R is finite with respect to R, then R' is a semi-local ring and R is a subspace of R', but* (*the semi-local ring*) *R' is not a subspace of $\bar{R}$ unless*

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8) The p-radical of a ring R is the intersection of all prime ideals in R; cf. [8]. If R is Noetherian, it is the largest nilpotent ideal.

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R' coincides with R.

Proof. Let $\mathfrak{m}$, $\mathfrak{m}'$ and $\overline{\mathfrak{m}}$ be the J-radicals of R, R' and $\overline{R}$ respectively. Put further $\mathfrak{m}''=\overline{\mathfrak{m}}\cap R'$. Then it follows from Theorem 5 that $(\mathfrak{m}'')^k\cap R=\mathfrak{m}^k$ ($k=1, 2, \ldots$), while we have clearly that $\mathfrak{m}R'\subseteqq\mathfrak{m}'\subseteqq\mathfrak{m}''$, which shows that R is a subspace of R'. Since $R/\mathfrak{m}=\overline{R}/\overline{\mathfrak{m}}$, we have $R/\mathfrak{m}=R'/\mathfrak{m}''$. Suppose now that R' is a subspace of R, then $\mathfrak{m}R'=\mathfrak{m}\overline{R}\cap R'=\mathfrak{m}''$, because $\mathfrak{m}R'$ is (open whence) closed in R'. We have therefore $R+\mathfrak{m}R'=R'$, which implies $R=R'$ by virtue of [6, Appendix, Corollary to Proposition 4].

Remark. As was shown in the above proof, we have also that $\mathfrak{m}R'\neq\mathfrak{m}''$ if $R'\neq R$.

Corollary 1. Let R and $\overline{R}$ be the same as in Theorem 6. Then $\overline{R}$ is not finite with respect to R whenever $R\neq\overline{R}$.

Corollary 2. Let R be a Noetherian semi-local ring and let R' be a semi-local ring in which R is contained as a subring as well as a subspace. Then R is closed in R' whenever R' is finite with respect to R.

We prove, by the way, some properties of $\mathfrak{m}$-adic rings.

Proposition 3.1. If an $\mathfrak{m}$-adic ring R is a subspace as well as a subring of an $\mathfrak{m}'$-adic ring R' and if both $\mathfrak{m}$ and $\mathfrak{m}'$ are semi-prime ideals[9] in R and R' respectively, then we have $\mathfrak{m}'\cap R=\mathfrak{m}$.

Proof. Since $\mathfrak{m}'\cap R$ is an open semi-prime ideal in R, we have $\mathfrak{m}'\cap R\supseteqq\mathfrak{m}$. On the other hand, since we can find a natural number k so that $\mathfrak{m}\supseteqq(\mathfrak{m}')^k\cap R\supseteqq(\mathfrak{m}'\cap R)^k$ and since $\mathfrak{m}$ is a semi-prime ideal, we have $\mathfrak{m}\supseteqq\mathfrak{m}'\cap R$.

Proposition 3.2. Let R be an $\mathfrak{m}$-adic ring, and suppose that $\mathfrak{m}$ is a finite intersection of maximal ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_h$ of R. Let S be the complementary set of $\bigcup_{i=1}^{h}\mathfrak{p}_i$ with respect to R. Then the ring R_S of quotients of S with respect to R in the sense of H. Grell [4] is definable and is a semi-local ring. Further R is a dense subset of R_S.[10]

Proof. S is clearly multiplicatively closed. S contains no zero divisor, because $\mathfrak{m}\cap S=\theta$, every $\mathfrak{m}^n$ ($n=1, 2, \ldots$) is an intersection of primary ideals and $\bigcap_{n=1}^{\infty}\mathfrak{m}^n=(0)$.[11] Therefore R_S is definable. Further, since $\mathfrak{m}^nR_S\cap R=\mathfrak{m}^n$, R_S is a semi-

[9] A semi-prime ideal in a ring R is an ideal which is an intersection of prime ideals in R; cf. [8].

[10] Cf. [11, § 7].

[11] Cf. [7, Chapter I, Lemma 3].

local ring and R is a subspace of R_S. Now we prove that R is dense in R_S. That $((b/a)+\mathfrak{m}^n R_S)\cap R \neq \emptyset$ ($b \in R$, $a \in S$) is equivalent to that $b - ac_n \in \mathfrak{m}^n$ for a suitable $c_n \in R$. Since $a \in S$, a is unit in $R/\mathfrak{m}^n$, and this shows the existence of such c_n (for each n). This completes our proof.

Remark. Set $S' = \{a(\in R);\ a-1 \in \mathfrak{m}\}$. Then R_S coincides with the ring of quotients of S' with respect to R, because every element of S is unit in $R/\mathfrak{m}$.[10]

4. Supplementary remarks to [7, Chapter II, Proposition 8].

First we prove

PROPOSITION 4.1.[12] Let R be a ring in which every maximal ideal is principal. Then the following five conditions for R are equivalent to each other:

(1) R is a direct sum of a finite number of principal ideal rings each of which is einartig.[13]

(2) R is a principal ideal ring.

(3) R is Noetherian.

(4) R is a subdirect sum of a finite number of einartig rings.

(5) Zero ideal of R is an intersection of a finite number of primary ideals $\mathfrak{q}_1, \ldots \mathfrak{q}_s$ such that $\bigcap_{n=1}^{\infty}\mathfrak{p}^n \subseteqq \mathfrak{q}_i$ for any maximal ideal $\mathfrak{p}$ containing $\mathfrak{q}_i$ (for each i).

Before proving this, we state some lemmas:

LEMMA 4.1. If a ring R is a subdirect sum of a finite number of Noetherian rings, then R is Noetherian, too.

Proof, Let R be a subdirect sum of Noetherian rings $R_1, \ldots, R_n$. Let $\mathfrak{a}$ be an ideal in R. Let $\mathfrak{a}_1$ be the natural image of $\mathfrak{a}$ in R_1. Then there exists a finite basis $(a_1', \ldots, a_r')$ for $\mathfrak{a}_1$ in R_1. Let a_i be, for each i, an element of $\mathfrak{a}$ whose R_1-component is a_i'. Then clearly $\mathfrak{a} = (a_1, \ldots, a_r) + \mathfrak{a} \cap (R_2 + \cdots + R_n)$. Thus we can prove our assertion by induction on n.

LEMMA 4.2. Every local ring with principal maximal ideal is an einartig principal ideal ring. [7, Chapter II, Proposition 8.]

LEMMA 4.3. An einartig ring R is a principal ideal ring whenever every maximal ideal is principal.

For, R is Noetherian by virtue of [3, Theorem 2].

Proof of Proposition 4.1. It is clear that (2), (3), (4) and (5) follows from (1) and that (3) follows from (2). (3) follows from (4) by virtue of Lemmas

[12] As for the equivalence of (1), (2) and (3), cf. [5, Theorem 9].

[13] A ring R is said to be einartig if every proper prime ideal is maximal.

4.1 and 4.3. To prove that (1) follows from (3), let $\mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$ be a shortest representation of zero ideal in R as an intersection of primary ideals. Let $\mathfrak{p}$ be a maximal ideal in R, then the ring of quotients[14] of $\mathfrak{p}$ with respect to R is a local ring, whence an einartig principal ideal ring. This shows that $\mathfrak{p}$ contains only one $\mathfrak{q}_i$ and $R/\mathfrak{q}_i$ is einartig.[15] It follows from this that R is the direct sum of $R/\mathfrak{q}_1, \ldots, R/\mathfrak{q}_n$ each of which is, by virtue of Lemma 4.3, an einartig principal ideal ring. That (4) follows from (5) is easy if we observe the following

LEMMA 4.4. If a principal ideal aR in a ring R contains properly a prime ideal $\mathfrak{p}$, then $\mathfrak{p} \subseteqq \bigcap_{n=1}^{\infty} a^n R$.

Proof. Since $aR \supset \mathfrak{p}$, we have $\mathfrak{p} = a\mathfrak{p}'$ for an ideal $\mathfrak{p}'$ in R. Since $a \notin \mathfrak{p}'$, we have $\mathfrak{p} = \mathfrak{p}'$. This shows that $\mathfrak{p} = a^n \mathfrak{p}$, which proves our assertion.

Next we construct a semi-local ring R which is not Noetherian, but every maximal ideal in R is principal:

EXAMPLE. Let K be a field and let x, y and z be indeterminates. Let R_1 be the subring of $K(x, y)$ generated by $K[x, y]$ and y/x. Then $R_1 \neq K(x, y)$ and xR_1 is a maximal ideal in R_1. Let R_2 be the ring of quotients of xR with respect to R_1. Let S be the intersection of complementary sets of $xR_2[z]$ and $zR_2[z]$ with respect to $R_2[z]$. Then the ring R of quotients of S with respect to $R_2[z]$ is a required ring.

In fact, R has only two maximal ideals xR and zR, while R is semi-local because $\bigcap_{n=1}^{\infty} z^n R = (0)$.

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