Glasgow Math. J. **51A** (2009) 129–135. © 2009 Glasgow Mathematical Journal Trust.
doi:10.1017/S0017089508004849. Printed in the United Kingdom

NOTE ON THE TWO-COMPONENT ANALOGUE OF TWO-DIMENSIONAL LONG WAVE – SHORT WAVE RESONANCE INTERACTION SYSTEM

KEN-ICHI MARUNO
Department of Mathematics, The University of Texas-Pan American, Edinburg, TX 78541, USA
e-mail: kmaruno@utpa.edu

YASUHIRO OHTA
Department of Mathematics, Kobe University, Rokko, Kobe 657-8501, Japan

and MASAYUKI OIKAWA
Research Institute for Applied Mechanics, Kyushu University, Kasuga, Fukuoka, 816-8580, Japan

Abstract. An integrable two-component analogue of the two-dimensional long wave – short wave resonance interaction (2c-2d-LSRI) system is studied. Wronskian solutions of 2c-2d-LSRI system are presented. A reduced case, which describes resonant interaction between an interfacial wave and two surface wave packets in a two-layer fluid, is also discussed.

2002 *Mathematics Subject Classification.* 35Q51, 35Q55, 37K40

1. Introduction. In these past decades, vector soliton equations have received so much attention in mathematical physics and non-linear physics [**1**, **2**, **8**, **13**]. Recently, we derived the following system in a two-layer fluid using reductive perturbation method, which was motivated by a paper by Onorato et al. [**9**, **12**]:

$$\mathrm{i}\big(S_t^{(1)} + S_y^{(1)}\big) - S_{xx}^{(1)} + LS^{(1)} = 0, \quad \mathrm{i}\big(S_t^{(2)} - S_y^{(2)}\big) - S_{xx}^{(2)} + LS^{(2)} = 0,$$
$$L_t = 2\big(|S^{(1)}|^2 + |S^{(2)}|^2\big)_x. \tag{1}$$

This system is an extension of the two-dimensional long wave – short wave resonance interaction system [**14**, **10**] and describes the two-dimensional resonant interaction between an interfacial gravity wave and two surface gravity packets propagating in directions symmetric about the propagation direction of the interfacial wave in a two-layer fluid.

In this paper, we will study this system and its integrable modification,

$$\begin{aligned}\mathrm{i}\big(S_t^{(1)} + S_y^{(1)}\big) - S_{xx}^{(1)} + LS^{(1)} &= 2\mathrm{i}S^{(2)*}Q,\\ \mathrm{i}\big(S_t^{(2)} - S_y^{(2)}\big) - S_{xx}^{(2)} + LS^{(2)} &= 2\mathrm{i}S^{(1)*}Q,\\ L_t = 2\big(|S^{(1)}|^2 + |S^{(2)}|^2\big)_{x,} \qquad Q_x &= S^{(1)}S^{(2)}.\end{aligned} \tag{2}$$

where * means complex conjugate. In our recent paper [**11**], we studied

$$\mathrm{i}\big(S_t^{(1)} + S_y^{(1)}\big) - S_{xx}^{(1)} + LS^{(1)} = 0, \quad \mathrm{i}\big(S_t^{(2)} + S_y^{(2)}\big) - S_{xx}^{(2)} + LS^{(2)} = 0,$$
$$L_t = 2\big(|S^{(1)}|^2 + |S^{(2)}|^2\big)_x. \tag{3}$$

Note that this system is different from system (1) only in the sign of y-derivative term $S_y^{(2)}$.

2. Bilinear forms and Wronskian solutions. Consider a two-component analogue of two-dimensional long wave – short wave resonance interaction (2c-2d-LSRI) system (2). Using the dependent-variable transformation $L = -(2\log F)_{xx}$, $S^{(1)} = G/F$, $S^{(2)} = H/F$, $Q = -K^*/F$, we obtain

$$\begin{aligned} &\big(D_x^2 - \mathrm{i}(D_t + D_y)\big)G\cdot F = 2\mathrm{i}H^*K^*, \qquad D_xD_tF\cdot F = -2(GG^* + HH^*),\\ &\big(D_x^2 - \mathrm{i}(D_t - D_y)\big)H\cdot F = 2\mathrm{i}G^*K^*, \qquad D_xK\cdot F = -G^*H^*. \end{aligned} \tag{4}$$

These bilinear forms have the three-component Wronskian solution [**3**, **4**, **7**].

Consider the following three-component Wronskian:

$$\tau_{NML} = |\,\boldsymbol{\varphi} \quad \boldsymbol{\psi} \quad \boldsymbol{\chi}\,|,$$

where $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$ and $\boldsymbol{\chi}$ are $(N+M+L)\times N$, $(N+M+L)\times M$ and $(N+M+L)\times L$ matrices, respectively: $\boldsymbol{\varphi} = (\partial_{x_1}^{j-1}\varphi_i)_{1\le i\le N+M+L}^{1\le j\le N}$, $\boldsymbol{\psi} = (\partial_{x_1}^{j-1}\psi_i)_{1\le i\le N+M+L}^{1\le j\le M}$ and $\boldsymbol{\chi} = (\partial_{x_1}^{j-1}\chi_i)_{1\le i\le N+M+L}^{1\le j\le L}$, and φ_i is an arbitrary function of x_1 and x_2 satisfying $\partial_{x_2}\varphi_i = \partial_{x_1}^2\varphi_i$, and ψ_i and χ_i are arbitrary functions of y_1 and z_1, respectively. The above Wronskian satisfies

$$\begin{aligned} &\big(D_{x_1}^2 - D_{x_2}\big)\tau_{N+1,M-1,L}\cdot\tau_{NML} = 0, \quad \big(D_{x_1}^2 - D_{x_2}\big)\tau_{N+1,M,L-1}\cdot\tau_{NML} = 0,\\ &D_{x_1}D_{y_1}\tau_{NML}\cdot\tau_{NML} = 2\tau_{N+1,M-1,L}\tau_{N-1,M+1,L},\\ &D_{x_1}D_{z_1}\tau_{NML}\cdot\tau_{NML} = 2\tau_{N+1,M,L-1}\tau_{N-1,M,L+1},\\ &D_{x_1}\tau_{N,M+1,L-1}\cdot\tau_{NML} = -\tau_{N-1,M+1,L}\tau_{N+1,M,L-1},\\ &D_{y_1}\tau_{N-1,M,L+1}\cdot\tau_{NML} = -\tau_{N,M-1,L+1}\tau_{N-1,M+1,L},\\ &D_{z_1}\tau_{N+1,M-1,L}\cdot\tau_{NML} = -\tau_{N+1,M,L-1}\tau_{N,M-1,L+1}. \end{aligned}$$

Setting

$$\begin{aligned} f = \tau_{NML},\quad &g = \tau_{N+1,M-1,L}, \quad h = \tau_{N-1,M,L+1}, \quad k = \tau_{N,M+1,L-1},\\ &\bar{g} = \tau_{N-1,M+1,L}, \quad \bar{h} = \tau_{N+1,M,L-1}, \quad \bar{k} = \tau_{N,M-1,L+1}, \end{aligned}$$

we have the following bilinear forms:

$$\begin{aligned} &\big(D_{x_1}^2 - D_{x_2}\big)g\cdot f = 0, \qquad \big(D_{x_1}^2 + D_{x_2}\big)\bar{g}\cdot f = 0, \qquad D_{x_1}D_{y_1}f\cdot f = 2g\bar{g},\\ &\big(D_{x_1}^2 + D_{x_2}\big)h\cdot f = 0, \qquad \big(D_{x_1}^2 - D_{x_2}\big)\bar{h}\cdot f = 0, \qquad D_{x_1}D_{z_1}f\cdot f = 2h\bar{h},\\ &D_{x_1}k\cdot f = -\bar{g}\bar{h}, \qquad D_{y_1}h\cdot f = -\bar{g}\bar{k}, \qquad D_{z_1}g\cdot f = -\bar{h}\bar{k},\\ &D_{x_1}\bar{k}\cdot f = gh, \qquad D_{y_1}\bar{h}\cdot f = gk, \qquad D_{z_1}\bar{g}\cdot f = hk. \end{aligned}$$

By the change of independent variables $x_1 = x$, $x_2 = -\mathrm{i}y$, $y_1 = y - t$, $z_1 = -y - t$ $(x, y, t : \text{real})$, we have $\partial_x = \partial_{x_1}$, $\partial_y = -\mathrm{i}\partial_{x_2} + \partial_{y_1} - \partial_{z_1}$, $\partial_t = -\partial_{y_1} - \partial_{z_1}$. Thus, we obtain

$$\begin{aligned} &(D_x^2 - \mathrm{i}(D_t + D_y))g\cdot f = -2\mathrm{i}\bar{h}\bar{k}, \qquad (D_x^2 + \mathrm{i}(D_t + D_y))\bar{g}\cdot f = -2\mathrm{i}hk,\\ &(D_x^2 - \mathrm{i}(D_t - D_y))h\cdot f = -2\mathrm{i}\bar{g}\bar{k}, \qquad (D_x^2 + \mathrm{i}(D_t - D_y))\bar{h}\cdot f = -2\mathrm{i}gk,\\ &D_xD_tf\cdot f = -2(g\bar{g} + h\bar{h}), \qquad D_xk\cdot f = -\bar{g}\bar{h}, \qquad D_x\bar{k}\cdot f = gh\,. \end{aligned}$$

https://doi.org/10.1017/S0017089508004849 Published online by Cambridge University Press

Consider solutions satisfying the following condition:

$$\bar{g}\mathcal{G} = (g\mathcal{G})^*, \quad \bar{h}\mathcal{G} = (h\mathcal{G})^*, \quad \bar{k}\mathcal{G} = -(k\mathcal{G})^*, \quad f\mathcal{G} : \text{real}, \tag{5}$$

where $\mathcal{G}$ is a gauge factor. Then, for $F = f\mathcal{G}$, $G = g\mathcal{G}$, $H = h\mathcal{G}$ and $K = k\mathcal{G}$, we will obtain the bilinear equations of the 2c-2d-LSRI system (4). Thus, the 2c-2d-LSRI system has a three-component Wronskian solution.

To satisfy the condition (5), we consider the following constrained case: $N = M + L$, $\psi_i = 0$ for $2M + 1 \le i \le 2M + 2L$, $\chi_i = 0$ for $1 \le i \le 2M$ and

$$\varphi_i = e^{\xi_i}, \quad \varphi_{M+i} = e^{-\xi_i^*}, \quad \xi_i = p_i x_1 + p_i^2 x_2,$$
$$\psi_i = a_i e^{\eta_i}, \quad \psi_{M+i} = a_{M+i} e^{-\eta_i^*}, \quad \eta_i = q_i y_1 + \eta_{i0},$$

for $i = 1, 2, \ldots, M$, and

$$\varphi_{2M+i} = e^{\theta_i}, \quad \varphi_{2M+L+i} = e^{-\theta_i^*}, \quad \theta_i = s_i x_1 + s_i^2 x_2,$$
$$\chi_{2M+i} = b_i e^{\zeta_i}, \quad \chi_{2M+L+i} = b_{L+i} e^{-\zeta_i^*}, \quad \zeta_i = r_i z_1 + \zeta_{i0},$$

for $i = 1, 2, \ldots, L$, where p_i, s_i, q_i and r_i are wave numbers and η_{i0} and ζ_{i0} are phase constants. The parameters a_i and b_i must be determined from the condition of complex conjugacy. By using the standard technique [**6**], a_i and b_i are determined as

$$a_i = \prod_{\substack{k=1\\k\neq i}}^{M} \frac{p_k - p_i}{q_k - q_i} \prod_{k=1}^{M} \frac{p_k^* + p_i}{q_k^* + q_i}, \quad a_{M+i} = \prod_{k=1}^{L} (s_k + p_i^*)(s_k^* - p_i^*), \quad 1 \le i \le M,$$
$$b_i = \prod_{\substack{k=1\\k\neq i}}^{L} \frac{s_k - s_i}{r_k - r_i} \prod_{k=1}^{L} \frac{s_k^* + s_i}{r_k^* + r_i}, \quad b_{L+i} = \prod_{k=1}^{M} (p_k + s_i^*)(p_k^* - s_i^*), \quad 1 \le i \le L,$$

and condition (5) is satisfied for the gauge factor,

$$\mathcal{G} = \prod_{1\le i<j\le M} (p_j^* - p_i^*)(q_i - q_j) \prod_{1\le i<j\le L} (s_j^* - s_i^*)(r_i - r_j) \prod_{i=1}^{M} \prod_{j=1}^{L} (p_i - s_j)$$
$$\times e^{\sum_{i=1}^{M}(\xi_i^* - \eta_i) + \sum_{j=1}^{L}(\theta_j^* - \zeta_j)}.$$

This solution represents the $(M + L)$-soliton, i.e., M solitons propagate on the first component of short wave $S^{(1)}$ whose complex wave numbers are given by p_i and q_i and complex phase constants are η_{i0} and L solitons propagate on the second one $S^{(2)}$ whose complex wave numbers and phase constants are s_i, r_i and ζ_{i0}.

For instance by taking $M = L = 1$, (1+1)-soliton solution is given as

$$F = f\mathcal{G} = c\bigg(\frac{p + p^*}{q + q^*} \frac{s + s^*}{r + r^*} \frac{1}{|p + s^*|^2} - \frac{s + s^*}{r + r^*} e^{\xi + \xi^* - \eta - \eta^*} - \frac{p + p^*}{q + q^*} e^{\theta + \theta^* - \zeta - \zeta^*}$$
$$+ |p - s|^2 e^{\xi + \xi^* - \eta - \eta^* + \theta + \theta^* - \zeta - \zeta^*}\bigg),$$
$$G = g\mathcal{G} = c(p + p^*) e^{\xi - \eta} \bigg(\frac{s + s^*}{r + r^*} \frac{1}{p^* + s} - (p - s) e^{\theta + \theta^* - \zeta - \zeta^*}\bigg),$$

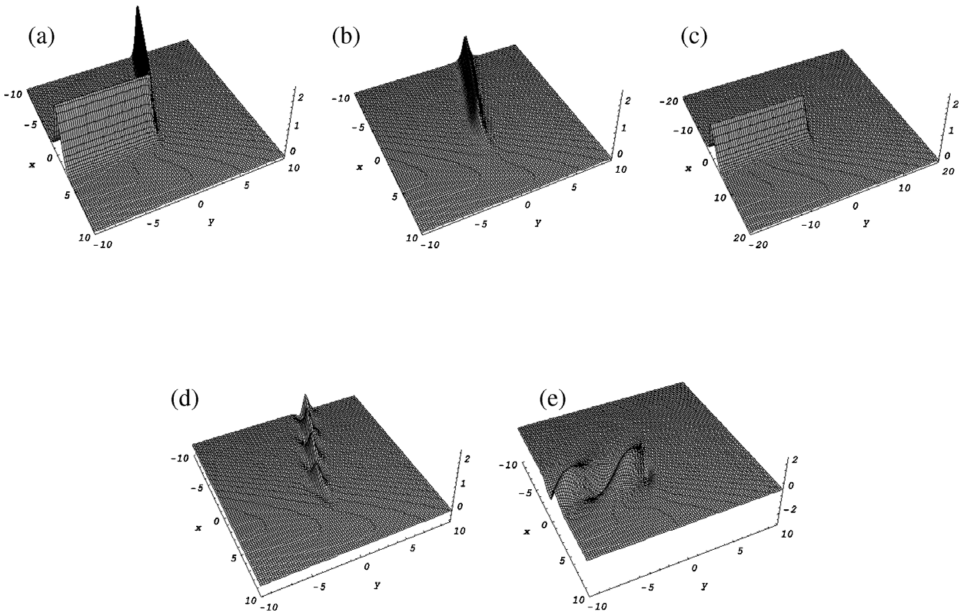

Figure 1. Single-line soliton of equations (2), which is obtained by tau-functions of (6). (a) $-L$, (b) $|S^{(1)}|$, (c) $|S^{(2)}|$, (d) $\mathrm{Re}\,[S^{(1)}]$, (e) $\mathrm{Re}\,[S^{(2)}]$. The parameters are $p = 1 + \mathrm{i},\ q = -1 + 2\mathrm{i},\ r = -2 + \mathrm{i}$.

$$H = h\mathcal{G} = -c(s+s^*)e^{\theta^*-\zeta^*}\left(\frac{p+p^*}{q+q^*}\frac{1}{p^*+s} + (p^*-s^*)e^{\xi+\xi^*-\eta-\eta^*}\right),$$
$$K = k\mathcal{G} = c\frac{(p+p^*)(s+s^*)}{p+s^*}e^{\xi^*-\eta^*+\theta-\zeta},$$

where $c = -|(p-s)(p+s^*)|^2$ and we dropped the index 1 for simplicity. In order to satisfy the regularity condition $F \neq 0$, we can take $\mathrm{Re}\,p > 0$, $\mathrm{Re}\,s > 0$, $\mathrm{Re}\,q < 0$ and $\mathrm{Re}\,r < 0$. After removing the gauge and constant factors, by choosing the same wave number in x-direction for the above two solitons, i.e. $s = p$, we obtain the single soliton solution,

$$F = \frac{1}{p+p^*} - e^{\xi+\xi^*}((q+q^*)e^{-\eta-\eta^*} + (r+r^*)e^{-\zeta-\zeta^*}),$$
$$G = (q+q^*)e^{\xi-\eta}, \quad H = -(r+r^*)e^{\xi^*-\zeta^*}, \quad K = (q+q^*)(r+r^*)e^{\xi+\xi^*-\eta^*-\zeta}, \quad (6)$$

where $\xi = px - \mathrm{i}p^2y$, $\eta = q(y-t) + \eta_0$ and $\zeta = -r(y+t) + \zeta_0$. Figure 1. shows the plots of this single soliton solution. L shows V-shape soliton, $|S^{(1)}|$ and $|S^{(2)}|$ show solitoff behaviour [**5**].

3. Solutions in the case without Q. We consider the 2c-2d-LSRI system (1) without the fourth field Q in (2). This system (1) describes waves in the two-layer fluid. Setting $L = -(2\log F)_{xx}$, $S^{(1)} = G/F$, $S^{(2)} = H/F$, we have

$$[\mathrm{i}(D_t + D_y) - D_x^2]G\cdot F = 0, \qquad [\mathrm{i}(D_t - D_y) - D_x^2]H\cdot F = 0,$$
$$-(D_tD_x - 2c)F\cdot F = 2GG^* + 2HH^*.$$

https://doi.org/10.1017/S0017089508004849 Published online by Cambridge University Press

Here we consider the case of $c = 0$.

Using the procedure of the Hirota bilinear method, we obtain the single soliton solution

$$F = 1 + A_{11}\exp(\eta_1 + \eta_1^*), \quad G = a_1\exp(\eta_1), \quad H = b_1\exp(\xi_1),$$

$$\eta_1 = p_1 x + \mathrm{i}q_1 y + \lambda_1 t + \eta_1^{(0)}, \qquad \xi_1 = p_1 x - \mathrm{i}q_1 y + \lambda_1 t + \eta_1^{(0)},$$

$$A_{11} = -\frac{a_1 a_1^* + b_1 b_1^*}{(p_1 + p_1^*)(\lambda_1 + \lambda_1^*)}, \quad \lambda_1 = -\mathrm{i}p_1^2 - \mathrm{i}q_1.$$

Here, q_1 is a real number. We can rewrite A_{11} as

$$A_{11} = -\frac{a_1 a_1^* + b_1 b_1^*}{(p_1 + p_1^*)^2(\mathrm{i}p_1^* - \mathrm{i}p_1)}.$$

Thus, we have

$$S^{(1)} = \frac{a_1\exp(\eta_1)}{1 + A_{11}\exp(\eta_1 + \eta_1^*)}, \quad S^{(2)} = \frac{b_1\exp(\xi_1)}{1 + A_{11}\exp(\eta_1 + \eta_1^*)},$$

$$L = -2\frac{\partial^2}{\partial x^2}\log(1 + A_{11}\exp(\eta_1 + \eta_1^*)).$$

Since $|S^{(1)}|^2 = GG^*/F^2$, $|S^{(2)}|^2 = HH^*/F^2$, $L = -(2\log F)_{xx}$ do not include y, all solitons propagate in the x-direction.

There is an exact solution depending on y-variable,

$$S^{(1)} = \frac{A_1\exp(px + qy + rt)}{1 + \exp(2(px + qy + rt))}\exp(\mathrm{i}(k_1 x + l_1 y + m_1 t)),$$

$$S^{(2)} = \frac{A_2\exp(px + qy + rt)}{1 + \exp(2(px + qy + rt))}\exp(\mathrm{i}(k_2 x + l_2 y + m_2 t)),$$

$$L = \frac{A\exp(2(px + qy + rt))}{(1 + \exp(2(px + qy + rt)))^2},$$

where p, q, r, k_1, l_1, m_1, k_2, l_2, m_2, A_1, A_2 and A satisfy the relations $r = (k_1 + k_2)p$, $q = (k_1 - k_2)p$, $m_1 = k_1^2 - l_1 - p^2$, $m_2 = k_2^2 + l_2 - p^2$, $A = -8p^2$, $A_1^2 + A_2^2 = -4(k_1 + k_2)p^2$, and p, q, k_1, l_1, l_2 are arbitrary parameters. In Figure 2., we see that waves in $S^{(1)}$ and $S^{(2)}$ have different modulation property, i.e. carrier waves in $S^{(1)}$ and $S^{(2)}$ has different directions of propagation. Note that the solutions of equations (2) also have this property.

It seems that equations (1) are non-integrable and do not admit general N-soliton solution. Similar system (2) has an N-soliton solution, but its physical derivation has not been done yet.

4. Concluding remarks. We have studied solutions of a new integrable 2c-2d LSRI system (2). We presented a Wronskian formula for 2c-2d LSRI system (2) with complex conjugacy condition. We have also presented solutions of the system (1) in the case of two-layer fluid, i.e. the 2c-2d LSRI system without Q. In this case, the system

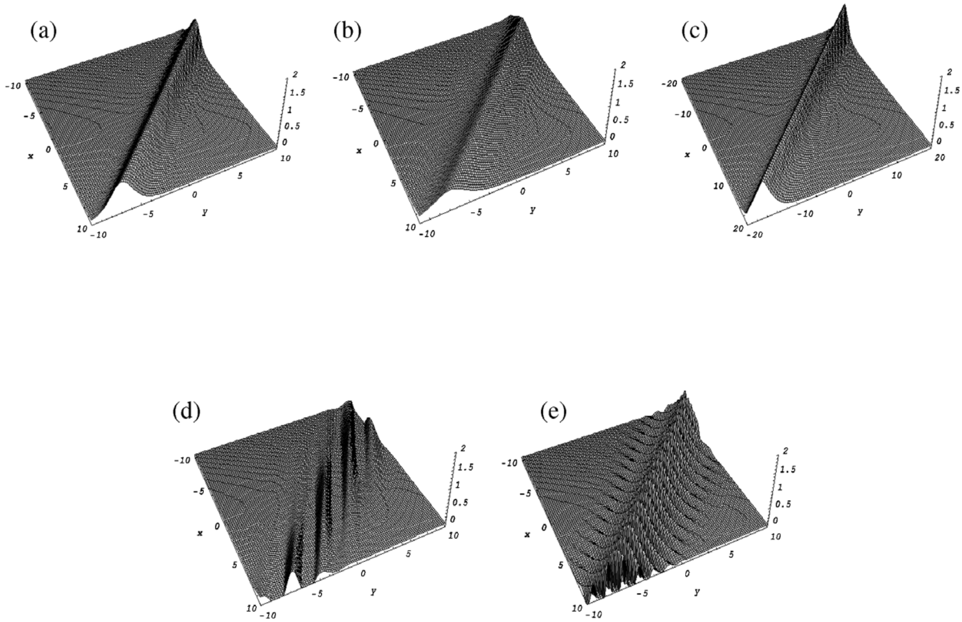

Figure 2. Line soliton of equations (1). (a) $-L$, (b) $|S^{(1)}|$, (c) $|S^{(2)}|$, (d) Re$[S^{(1)}]$, (e) Re$[S^{(2)}]$. The parameters are $k_1 = -1$, $k_2 = -2$, $A_1 = 1$, $A_2 = 2$, $l_1 = 3$, $l_2 = 4$.

(1) seems to be non-integrable, i.e. the system (1) does not have multi-soliton solutions. We have found that waves in $S^{(1)}$ and $S^{(2)}$ in both systems have different modulation property, i.e. carrier waves in $S^{(1)}$ and $S^{(2)}$ have different directions of propagation. But system (2) has much more interesting solutions such as the V-shape soliton and solitoff because of integrability.

ACKNOWLEDGEMENTS. One of the authors (Ken-ichi Maruno) wishes to acknowledge organizers for providing the financial support for the ISLAND 3 (Integrable Systems: Linear And Nonlinear Dynamics 3) conference.

REFERENCES

1. M. J. Ablowitz, B. Prinari and A. D. Trubatch, *Discrete and continuous nonlinear Schrödinger systems* (Cambridge University Press, Cambridge, UK, 2004).

2. M. J. Ablowitz, B. Prinari and A. D. Trubatch, Soliton interactions in the vector NLS equation, *Inv. Probl.* **20** (2004), 1217–1237.

3. E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations. III. Operator approach to the Kadomtsev–Petviashvili equation, *J. Phys. Soc. Jpn.* **50** (1981), 3806–3812.

4. E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation group for soliton equations: Euclidean Lie algebras and reduction of the KP hierarchy, *Publ. Res. Inst. Math. Sci.* **18** (1982), 1077–1111.

5. C. R. Gilson, Resonant behaviour in the Davey–Stewartson equation, *Phys. Lett. A* **161** (1992), 423–428.

6. J. Hietarinta and R. Hirota, Multidromion solutions to the Davey–Stewartson equation, *Phys. Lett. A* **145** (1990), 237–244.

7. R. Hirota, *The direct method in soliton theory* (Cambridge University Press, Cambridge, UK, 2004).

8. S. V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, *Sov. Phys. JETP* **38** (1974), 248–253.

9. M. Oikawa, Y. Ohta and K. Maruno, Long wave-short wave resonance interaction system, Reports of RIAM Symposium No. 18ME-S5, 2007.

10. M. Oikawa, M. Okamura and M. Funakoshi, Two-dimensional resonant interaction between long and short waves, *J. Phys. Soc. Jpn.* **58** (1989), 4416–4430.

11. Y. Ohta, K. Maruno and M. Oikawa, Two-component analogue of two-dimensional long wave-short wave resonance interaction equations: A derivation and solutions, *J. Phys. A: Math. Theor.* **40** (2007), 7659–7672.

12. M. Onorato, A. R. Osborne and M. Serio, Modulational instability in crossing sea states: A possible mechanism for the formation of freak waves, *Phys. Rev. Lett.* **96** (2006), 014503-1–014503-4.

13. R. Radhakrishnan, M. Lakshmanan and J. Hietarinta, Inelastic collision and switching of coupled bright solitons in optical fibers, *Phys. Rev. E* **56** (1997), 2213–2216.

14. N. Yajima and M. Oikawa, Formation and interaction of sonic-Langmuir solitons – inverse scattering method, *Prog. Theor. Phys.* **56** (1976), 1719–1739.