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ENSURING COMMUTATIVITY OF FINITE GROUPS

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To Laci Kovács on his 65th birthday

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Abstract

Comments are made on the following question. Let m, n be positive integers and $\mathscr{G}$ a finite group. Suppose that for all choices of a subset of cardinality m and of a subset of cardinality n in $\mathscr{G}$ some member of the first commutes with some member of the second. Under what conditions on m, n is the group abelian?

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This note arose out of a discussion of a paper presented at AGRAM 2000 at the University of Western Australia by Howard Bell. 'Some setwise commutativity conditions for rings': since then Professor Bell has with Professor Abraham Klein found some interesting results related to the results below [1]. The question raised at AGRAM 2000 was:

Let $\mathscr{G}$ be a finite group of order g and assume that however a set M of m elements and a set N of n elements of the group is chosen, at least one element of M commutes with at least one element of N (call this condition Comm). What relations between g, m, n guarantee that $\mathscr{G}$ is abelian?

Clearly if one of M, N contains an element of the centre of $\mathscr{G}$ or if M and N overlap, condition Comm is satisfied. Thus if $m + n = g$, or even only $m + n = g - z + 1$, where z is the order of the centre of $\mathscr{G}$, Comm is satisfied without $\mathscr{G}$ having to be abelian. An example is every non-abelian group, the smallest being the S_3 of order 6: if M is chosen to consist of one or two elements of order 2, the two elements of order 3 together with the remaining elements or element of order 2 can be taken to form N, showing that $m = 1, n = 5$ or $m = 2, n = 4$ are needed to ensure a group of order 6 is abelian. If we choose $m = 1$ [which is the most interesting case, anyway] and $n = 5$, Comm ensures the group is abelian, whatever g.

There are of course values of g such that all groups of that order are abelian. There is a recent characterisation of such 'abelian' numbers in Pakianathan and Shankar [2]: for such orders g we can choose $m = n = 1$. For the 'nilpotent' numbers of [2] that are not 'abelian' (because they are not cube-free), $m = 1, n = 5$ is again best possible as exemplified by the quaternion group or the dihedral group of order 8. In this case we can do a little better: while in general $m = 2, n = 4$ forces a group to be abelian, whatever its order, the case of the groups of order 8 is exceptional in that we need $m = 2, n = 5$ to force the group to be abelian. More generally, if $g = p^3$ for p a prime, $m = p, n = g - p^2 + 1$ will ensure commutativity. It is not very difficult to compute optimal values for m and n for other values of g to ensure commutativity, but *sapienti sat.*

References

[1] H. E. Bell and A. A. Klein, 'Combinatorial commutativity and finiteness for rings, II', Preprint.

[2] J. Pakianathan and K. Shankar, 'Nilpotent numbers', *Amer. Math. Monthly* **107** (2000), 631–634.

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