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JACOBIAN IDEALS AND THE NEWTON NON-DEGENERACY CONDITION

CARLES BIVIÀ-AUSINA

Departament de Matemàtica Aplicada, Universitat Politècnica de València, Escola Politècnica Superior d'Alcoi, Plaça Ferrándiz i Carbonell 2, 03801 Alcoi (Alacant), Spain (carbivia@mat.upv.es)

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Abstract In this paper we extract some conclusions about Newton non-degenerate ideals and the computation of Łojasiewicz exponents relative to this kind of ideal. This motivates us to study the Newton non-degeneracy condition on the Jacobian ideal of a given analytic function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. In particular, we establish a connection between Newton non-degenerate functions and functions whose Jacobian ideal is Newton non-degenerate.

Keywords: reduced orders; Łojasiewicz exponents; Newton polyhedra of ideals

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1. Introduction

The computation of the integral closure of an ideal is a central problem in commutative algebra. Moreover, the notion of integral dependence has a transcendental or analytical interpretation in terms of Łojasiewicz exponents. This point of view allows us to study some geometrical incidence relations in singularity theory, such as the Whitney conditions (see [**28**]). Let $\mathcal{O}_n$ denote the ring of analytic function germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ and let us denote the integral closure of an ideal I of $\mathcal{O}_n$ by $\bar{I}$. There is a class of ideals in $\mathcal{O}_n$ whose integral closures are easily computable; these are called *Newton non-degenerate ideals*. The definition of Newton non-degenerate ideal is formulated in terms of the Newton polyhedron of I, which is denoted by $\Gamma_+(I)$ (see Definition 3.2). It was proved by Saia [**26**] that an ideal $I \subseteq \mathcal{O}_n$ of finite colength is Newton non-degenerate if and only if the integral closure of I is generated by the monomials $x_1^{k_n} \cdots x_n^{k_n}$ such that $(k_1, \dots, k_n)$ belongs to $\Gamma_+(I)$ (see also [**2**] for an algebraic approach to this result). We recall that Newton non-degenerate functions in the sense of Kouchnirenko [**15**] are those functions such that the ideal of $\mathcal{O}_n$ generated by $x_i \partial f/\partial x_i$, $i = 1, \dots, n$, is Newton non-degenerate.

We are interested in computing the Łojasiewicz exponent $\alpha_0(f)$ of an analytic function germ $f \in \mathcal{O}_n$. This is defined as the infimum of those real numbers $\alpha > 0$ such that there

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exists a positive constant $C > 0$ and an open neighbourhood U of $0 \in \mathbb{C}^n$ such that

$$|x|^\alpha \leqslant C \sup_i \left| \frac{\partial f}{\partial x_i}(x) \right|, \quad \text{for all } x \in U.$$

There are many works dealing with the computation of the number $\alpha_0(f)$ (see [**1**, **12**, **17**, **22**]). Let $J(f)$ denote the Jacobian ideal of a given $f \in \mathcal{O}_n$, that is, the ideal of $\mathcal{O}_n$ generated by the partial derivatives $\partial f/\partial x_1, \dots, \partial f/\partial x_n$. When the Jacobian ideal $J(f)$ is Newton non-degenerate we prove that the number $\alpha_0(f)$ can be determined easily (see Corollary 3.6). On the other hand, if f is a Newton non-degenerate function with an isolated singularity at the origin, there exists a sharp upper bound for $\alpha_0(f)$ given by Fukui [**12**]. Therefore, the main objective of this paper is to study the Newton non-degeneracy condition on a given Jacobian ideal and its relation with Newton non-degenerate functions. Although both conditions are independent in general, we obtain a characterization of the Newton polyhedra $\Gamma_+ \subseteq \mathbb{R}^n_+$ satisfying the condition that every Newton non-degenerate function $f \in \mathcal{O}_n$ with $\Gamma_+(f) = \Gamma_+$ is such that the Jacobian ideal $J(f)$ is Newton non-degenerate. We also show that it is possible to find Newton polyhedra $\Gamma_+ \subseteq \mathbb{R}^n_+$ such that no function $f \in \mathcal{O}_n$ with $\Gamma_+(f) = \Gamma_+$ has Newton non-degenerate Jacobian ideal. This part is developed in § 5 of the paper.

The importance of the Łojasiewicz exponents $\alpha_0(f)$ in singularity theory lies also in the following fact. Let $r_0(f)$ denote the degree of C^0-sufficiency in $\mathcal{O}_n$ of a complex germ $f \in \mathcal{O}_n$. Then it is known (see [**4**, **7**]) that

$$r_0(f) = [\alpha_0(f)] + 1, \tag{1.1}$$

where $[a]$ stands for the integer part of a given $a \in \mathbb{R}$. We recall that $r_0(f)$ is defined as the minimum of those $r > 1$ such that, for all $g \in \mathcal{O}_n$ with the same r-jet as f, there exists a germ of homeomorphism $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g = f \circ \phi$.

It is known that some topological invariants of a given Newton non-degenerate function f are codified by means of $\Gamma_+(f)$, as can be seen in [**10**] or [**30**]. Therefore, in view of these results and our interest in studying the Newton non-degeneracy condition on Jacobian ideals, it is convenient to have some rapid method to test if a given function or ideal is Newton non-degenerate. This is the objective of § 4. We take advantage of the techniques of this section to show an effective method to test if a given point $k \in \mathbb{Z}^n_+$ belongs to a fixed Newton polyhedron Γ_+.

In § 6 we obtain an effective formula to compute the sequence of mixed multiplicities $\{e_i(I, J) : i = 1, \dots, n\}$ for a pair of Newton non-degenerate ideals of finite codimension in $\mathcal{O}_n$. In particular, this allows us to compute the sequence $\{e_i(m_n, J(f)) : i = 1, \dots, n\}$ when the ideal $J(f)$ is Newton non-degenerate. As a consequence we establish a result on the μ^*-constancy of analytic families $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ such that the ideal $J(f_0)$ is Newton non-degenerate.

2. Reduced orders and Łojasiewicz exponents

In this section we give the definitions of reduced order and of Łojasiewicz exponent and we recall some known fundamental results.

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Definition 2.1. Let $I \subseteq \mathcal{O}_n$ be an ideal and let $h \in I$, $h \neq 0$. We define $\nu_I(h)$ to be the largest power of I which contains the element h. Then, if $h \in \sqrt{I}$, $h \neq 0$, the *reduced order of* h *with respect to* I is

$$\bar{\nu}_I(h) = \lim_{r \to \infty} \frac{\nu_I(h^r)}{r}.$$

Moreover, we set $\bar{\nu}_I(0) = +\infty$ and $\bar{\nu}_I(h) = 0$, if $h \notin \sqrt{I}$. The definition of reduced order was originally given by Samuel in [**27**].

It is proved in the works of Nagata [**21**] and Rees [**23**] that if $\bar{\nu}_I(h) < \infty$, then $\bar{\nu}_I(h) \in \mathbb{Q}$. Moreover, an element $h \in \mathcal{O}_n$ is integral over an ideal I if and only if $\bar{\nu}_I(h) \geqslant 1$ (see [**21**] or [**28**]). We recall that h is said to be *integral* over I when h satisfies an equation of the form

$$h^r + a_1 h^{r-1} + \cdots + a_{r-1} h + a_r = 0,$$

where $a_i \in I^i$, $i = 1, \ldots, r$, for some $r \geqslant 1$. The set of elements that are integral over I is another ideal of $\mathcal{O}_n$ containing I that is denoted by $\bar{I}$, and is called the *integral closure of* I.

By the properties of $\bar{\nu}_I$ (see [**19, 28**]), given an integer $r \geqslant 1$, the integral closure of I^r is equal to the set of those $h \in \mathcal{O}_n$ such that $\bar{\nu}_I(h) \geqslant r$. Therefore, given a positive rational number θ and an ideal $I \subseteq \mathcal{O}_n$, the ideal $\overline{I^\theta}$ is defined in [**28**] as

$$\overline{I^\theta} = \{h \in \mathcal{O}_n : \bar{\nu}_I(h) \geqslant \theta\}. \tag{2.1}$$

Therefore, if p, q are positive integers, an element $h \in \mathcal{O}_n$ belongs to $\overline{I^{p/q}}$ if and only if $h^q \in \overline{I^p}$. As we shall see, the reduced-order filtration $\bar{\nu}_I$ can be reformulated in analytical terms.

If $S \subseteq \mathcal{O}_n$, we denote by $V(S)$ the set germ at $0 \in \mathbb{C}^n$ of the set of common zeros of S. Moreover, we denote by $|x|$ the Euclidean norm of a point $x \in \mathbb{R}^n$.

Definition 2.2. Let I be an ideal of $\mathcal{O}_n$ generated by the germs $g_1, \ldots, g_s$ and let $h \in \mathcal{O}_n$. We define $\mathcal{L}(h, I)$ as the set of those $\alpha \in \mathbb{R}_+$ such that there exists some open neighbourhood U of 0 in $\mathbb{C}^n$ and a constant $C > 0$ such that

$$|h(x)|^\alpha \leqslant C \sup_i |g_i(x)|, \quad \text{for all } x \in U. \tag{2.2}$$

It is proved in [**18**, p. 136] that when $V(I) \subseteq V(h)$, the set $\mathcal{L}(h, I)$ is non-empty. Then, in this case, we define the *Łojasiewicz exponent of* h *with respect to* I as

$$\ell(h, I) = \inf \mathcal{L}(h, I).$$

Moreover, the number $\ell(h, I)$ belongs to $\mathcal{L}(h, I)$ (that is, the number $\ell(h, I)$ is actually a minimum), as can be seen in [**16**].

We observe that the definition of $\mathcal{L}(h, I)$ does not depend on the system of generators of I.

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It is proved by Lejeune and Teissier [**16**] that if I is an ideal of $\mathcal{O}_n$, then the following equality holds:

$$\ell(h, I) = \frac{1}{\bar{\nu}_I(h)}, \tag{2.3}$$

for any $h \in \sqrt{I}$. In particular, by the mentioned result of Nagata and Rees on the rationality of $\bar{\nu}_I(h)$, the Łojasiewicz exponents $\ell(h, I)$ are rational numbers.

Remark 2.3. Let us denote the ring of analytic real germs $(\mathbb{R}^n, 0) \to \mathbb{R}$ by $\mathcal{A}_n$. Then, Definition 2.2 can be reproduced analogously in the context of ideals I of $\mathcal{A}_n$, thus leading to the Łojasiewicz exponents $\ell_{\mathbb{R}}(h, I)$ for a given $h \in \mathcal{A}_n$. Therefore, following the analytic formulation of integral closure in the complex case, an element $h \in \mathcal{A}_n$ is said to be *integral* over an ideal $I \subseteq \mathcal{A}_n$ when $\ell_{\mathbb{R}}(h, I) \leqslant 1$. The set of integral elements over I forms an ideal of $\mathcal{A}_n$ that we also call the *integral closure of* I and we denote by $\bar{I}$. Fekak [**11**] has given an algebraic formulation of this notion of real integral closure in terms of order functions. If I is an ideal of $\mathcal{A}_n$ and $h \in \mathcal{A}_n$, the numbers $\ell_{\mathbb{R}}(h, I)$ are also rational, by virtue of [**5**].

We denote by m_n the maximal ideal of $\mathcal{O}_n$. By an abuse of notation, if there is no risk of confusion, we also denote the maximal ideal of $\mathcal{A}_n$ by m_n. Let us fix, from now on, a system of coordinates $x_1, \dots, x_n$ in $\mathbb{C}^n$.

Definition 2.4. We say that an ideal $I \subseteq \mathcal{O}_n$ has *finite colength* when $\dim_{\mathbb{C}} \mathcal{O}_n/I < \infty$. Since this is equivalent to saying that $V(I) = \{0\}$, we can consider the number $\ell(m_n, I) = \max\{\ell(x_i, I) : i = 1, \dots, n\}$. Therefore, the number $\ell(m_n, I)$ is equal to the infimum of those $\alpha > 0$ such that there exists a constant $C > 0$ and an open neighbourhood U of 0 in $\mathbb{C}^n$ such that

$$|x|^\alpha \leqslant C \sup_i |g_i(x)|, \quad \text{for all } x \in U.$$

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ with an isolated singularity at the origin. We define the *Łojasiewicz exponent of* f as $\alpha_0(f) = \ell(m_n, J(f))$, where $J(f)$ denotes the ideal of $\mathcal{O}_n$ generated by the partial derivatives

$$\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}.$$

3. The Newton non-degeneracy condition

As mentioned in the introduction, there is a characterization of the class of ideals $I \subseteq \mathcal{O}_n$ of finite colength such that $\bar{I}$ is generated by monomials. In order to show the motivation of our problem, here we expose some basic facts about this characterization and its implications for the computation of Łojasiewicz exponents.

We denote by $\mathbb{R}_+$ the set of non-negative real numbers. Therefore, let $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$ and let $\mathbb{Q}_+ = \mathbb{Q} \cap \mathbb{R}_+$.

Definition 3.1. We say that a subset $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a *Newton polyhedron* when there exists some $A \subseteq \mathbb{Z}^n_+$ such that Γ_+ is equal to the convex hull in $\mathbb{R}^n_+$ of the set $\{k + v :$

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$k \in A,\ v \in \mathbb{R}^n_+\}$. In this case, we say that Γ_+ is the *Newton polyhedron determined by* A and we write $\Gamma_+ = \Gamma_+(A)$.

Let us fix a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$. Let A_0 be the minimal subset of $\mathbb{Z}^n_+$, with respect to inclusion, between those $A \subseteq \mathbb{Z}^n_+$ such that $\Gamma_+(A) = \Gamma_+$. We say that $k \in \mathbb{Z}^n_+$ is a *vertex* of Γ_+ when $k \in A_0$. Thus, we call A_0 the *set of vertices* of Γ_+.

If $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$, we define $\ell(v, \Gamma_+) = \min\{\langle v, k\rangle : k \in \Gamma_+\}$ and the subset $\Delta(v, \Gamma_+) = \{k \in \Gamma_+ : \langle v, k\rangle = \ell(v, \Gamma_+)\}$. The sets of the form $\Delta(v, \Gamma_+)$, for some $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ are called *faces of* Γ_+. If Δ is a face of Γ_+ and $\Delta = \Delta(v, \Gamma_+)$, where $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$, then we say that Δ is the face of Γ_+ *supported* by v. We observe that a face Δ of Γ_+ is compact if and only if it is supported by a vector $v \in (\mathbb{R} \smallsetminus \{0\})^n$. We denote by Γ the union of compact faces of Γ_+. Then we will denote by Γ_- the union of all segments joining the origin and some point of Γ.

We say that a vector $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$ is *primitive* when the non-zero coordinates of v are mutually prime integer numbers. Then any face of Γ_+ of dimension $n-1$ is supported by a unique primitive vector. Let us denote by $\mathcal{F}(\Gamma_+)$ the set of primitive vectors of $\mathbb{R}^n_+$ supporting some compact face of Γ_+ of maximal dimension.

If $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$, we denote the monomial $x_1^{k_1} \cdots x_n^{k_n}$ by x^k. Given a function germ $f \in \mathcal{O}_n$, if $f = \sum_k a_k x^k$ is the Taylor expansion of f, then the *support* of f is defined as $\operatorname{supp}(f) = \{k \in \mathbb{Z}^n_+ : a_k \neq 0\}$. We define the *Newton polyhedron of* f as $\Gamma_+(f) = \Gamma_+(\operatorname{supp}(f))$. If I is an ideal of $\mathcal{O}_n$, the *support of* I, denoted by $\operatorname{supp}(I)$, is defined as the union of the supports of the elements of I. We denote by $\Gamma_+(I)$ the Newton polyhedron determined by $\operatorname{supp}(I)$. It is a simple exercise to check that if $g_1, \dots, g_s$ is a system of generators of I, then $\Gamma_+(I)$ is equal to the convex hull of $\Gamma_+(g_1) \cup \cdots \cup \Gamma_+(g_s)$. It is known that I and $\bar{I}$ have the same Newton polyhedron (see [**2**] or [**3**]).

Definition 3.2. If $g = \sum_k a_k x^k \in \mathcal{O}_n$ and A is a subset of $\mathbb{R}^n_+$, we denote by g_A the series given by the sum of those $a_k x^k$ such that $k \in A$. If $\operatorname{supp}(g) \cap A = \emptyset$, then we fix $g_A = 0$.

Let $I \subseteq \mathcal{O}_n$ be an ideal. Suppose that I is generated by the germs $g_1, \dots, g_s$. We say that I is *Newton non-degenerate* when, for each compact face Δ of $\Gamma_+(I)$, we have that

$$\{x \in \mathbb{C}^n : (g_1)_\Delta(x) = \cdots = (g_s)_\Delta(x) = 0\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}. \tag{3.1}$$

It is easy to check that the above definition does not depend on the chosen system of generators of I.

Definition 3.3 (see [15, 31]). Let $f \in \mathcal{O}_n$. Then we denote by $I(f)$ the ideal of $\mathcal{O}_n$ generated by the germs

$$x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n}.$$

The function f is said to be *Newton non-degenerate* when the ideal $I(f)$ is Newton non-degenerate.

As mentioned in the introduction, several topological invariants of Newton non-degenerate functions are expressed in terms of their Newton polyhedra.

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Theorem 3.4 (see [26]). *Let $I \subseteq \mathcal{O}_n$ be an ideal of finite colength. Then I is Newton non-degenerate if and only if the integral closure $\bar{I}$ is equal to the ideal generated by the monomials x^k such that $k \in \Gamma_+(I)$.*

A version of the above result for ideals $I(f)$, $f \in \mathcal{O}_n$, was initially proved by Yoshinaga [**31**], although the notion of integral closure is not explicitly mentioned in [**31**]. We now give some details about the construction of the Newton filtration attached to a Newton polyhedron (see also [**15**] or [**30**]). This will help us in determining the reduced-order filtration $\bar{\nu}_I$ when I is a Newton non-degenerate ideal.

Let us fix a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$ intersecting each coordinate axis. Given a face Δ of Γ_+, the *cone* over Δ, denoted by $C(\Delta)$, is the closure of the union of all half lines starting at the origin and passing through a point of Δ. It is clear that the union of all cones $C(\Delta)$, where Δ varies in the set of faces of Γ_+, is equal to $\mathbb{R}^n_+$.

Let $\{v^1, \dots, v^r\}$ be the family of vectors of $\mathcal{F}(\Gamma_+)$ such that $\ell(v^i, \Gamma_+) \neq 0$, for all $i = 1, \dots, r$. Then we denote by M_Γ the least common multiple of the integers $\ell(v^1, \Gamma_+), \dots, \ell(v^r, \Gamma_+)$. Thus we define

$$\Phi_\Gamma(k) = \min\left\{ \frac{M_\Gamma}{\ell(v^i, \Gamma_+)} \langle k, v^i \rangle : i = 1, \dots, r \right\}, \quad \text{for all } k \in \mathbb{R}^n_+.$$

We observe that Φ_Γ is constant (and equal to M_Γ) on Γ, the map Φ_Γ is linear on each cone $C(\Delta)$, where Δ is any compact face of Γ_+, and it is clear that $\Phi_\Gamma(\mathbb{Z}^n_+) \subseteq \mathbb{Z}^n_+$.

Now we define the ideals

$$\mathcal{R}_q = \{h \in \mathcal{O}_n : \Phi_\Gamma(\operatorname{supp}(h)) \subseteq [q, \infty[\,\} \cup \{0\}, \quad \text{for all } q \in \mathbb{Z}^n_+. \tag{3.2}$$

We also define the map $\nu_\Gamma : \mathcal{O}_n \to \mathbb{R}_+ \cup \{+\infty\}$ given by $\nu_\Gamma(h) = \max\{q : h \in \mathcal{R}_q\}$, if $h \neq 0$, and $\nu_\Gamma(0) = +\infty$. We refer to ν_Γ as the *Newton filtration* induced by Γ_+ (see also [**3**]).

Proposition 3.5. *Let $I \subseteq \mathcal{O}_n$ be an ideal of finite codimension, let $\Gamma_+ = \Gamma_+(I)$ and let $M = M_\Gamma$. Then $M\bar{\nu}_I \leqslant \nu_\Gamma$ and equality holds if and only if I is a Newton non-degenerate ideal.*

Proof. Let $\{\mathcal{R}_q\}_{q \geqslant 0}$ denote the ideals associated with the Newton filtration $\bar{\nu}_I$ (see (3.2)). The ideal I is contained in $\mathcal{R}_M$. Therefore, given an integer $p > 0$, we have

$$\overline{I^p} \subseteq \overline{\mathcal{R}^p_M} = \mathcal{R}_{Mp}. \tag{3.3}$$

Let $p/q \in \mathbb{Q}_+$ and let $h \in \mathcal{O}_n$. Then $h \in \overline{I^{p/q}}$ if and only if $h^q \in \overline{I^p}$. But $\overline{I^p} \subseteq \mathcal{R}_{Mp}$, by (3.3). Then the element h^q belongs to $\mathcal{R}_{Mp}$. But this is equivalent to saying that $h \in \mathcal{R}_{Mp/q}$. Therefore, we obtain the inclusion

$$\overline{I^{p/q}} \subseteq \mathcal{R}_{Mp/q}, \quad \text{for any } p/q \in \mathbb{Q}_+. \tag{3.4}$$

If I were Newton non-degenerate, then $\bar{I} = \mathcal{R}_M$ and the inclusion of (3.3) becomes an equality, for any positive integer p. Then, by a similar argument, we find that the inclusion (3.4) must also be an equality, if I is Newton non-degenerate. The converse is a consequence of Theorem 3.4. Now the result comes from the definition of $\bar{\nu}_I$ and ν_Γ. □

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Let $I \subseteq \mathcal{O}_n$ be an ideal of finite colength. Then the Newton polyhedron of I intersects each coordinate axis. Let P_i be the intersection of $\Gamma_+(I)$ with the x_i-axis, $i = 1, \dots, n$. A version of the following result for the case $n = 2$ can be found in [**14**, p. 9].

Corollary 3.6. *Under the above conditions, we have*

$$\ell(m_n, I) \geqslant \max\{|P_1|, \dots, |P_n|\}, \tag{3.5}$$

and equality holds if I is Newton non-degenerate.

Proof. From relation (2.3) and Proposition 3.5, we deduce that

$$\begin{aligned} \ell(m_n, I) &= \max\{\ell(x_i, I) : i = 1, \dots, n\} = \max\{(\bar{\nu}_I(x_i))^{-1} : i = 1, \dots, n\} \\ &\geqslant \max\{M/\nu_\Gamma(x_i) : i = 1, \dots, n\} = \max\{|P_1|, \dots, |P_n|\}. \end{aligned} \tag{3.6}$$

If I is Newton non-degenerate, the inequality in (3.6) becomes an equality, by virtue of Proposition 3.5. □

Therefore, it is natural to study the Newton non-degeneracy condition on Jacobian ideals, since the computation of the Łojasiewicz exponent $\ell(m_n, J(f))$ is quite direct in this case, as we have deduced in Corollary 3.6.

If I is an ideal of $\mathcal{A}_n$, the Newton polyhedron of I is defined analogously. The Newton non-degeneracy condition is also extended to ideals of $\mathcal{A}_n$ by replacing $\mathbb{C}$ by $\mathbb{R}$ in (3.1). Moreover, the proof of Theorem 3.4 is based on the analytic formulation of the integral closure of an ideal in $\mathcal{O}_n$. Therefore, Theorem 3.4 is also valid in the context of elliptic ideals of $\mathcal{A}_n$. We recall that an ideal I of $\mathcal{A}_n$ is said to be *elliptic* when the germ at $0 \in \mathbb{R}^n$ of the set of real zeros of I is equal to $\{0\}$.

Given an ideal I of $\mathcal{A}_n$ and $p/q \in \mathbb{Q}_+$, we define

$$\overline{I^{p/q}} = \{h \in \mathcal{A}_n : h^q \in \overline{I^p}\}. \tag{3.7}$$

Considering this definition and the real version of Theorem 3.4, the statement and the proof of Proposition 3.5 can be extended naturally to ideals of $\mathcal{A}_n$. Thus if I is an elliptic ideal of $\mathcal{A}_n$, we also have the inequality

$$\ell_{\mathbb{R}}(m_n, I) \geqslant \{|P_1|, \dots, |P_n|\}, \tag{3.8}$$

where P_i denotes the point where $\Gamma_+(I)$ meets the x_i-axis, $i = 1, \dots, n$, as in Corollary 3.6.

Remark 3.7. Let $f \in \mathcal{A}_n$ and suppose that the Taylor expansion of f around the origin defines a complex germ $f_{\mathbb{C}} \in \mathcal{O}_n$. If $f_{\mathbb{C}}$ has an isolated singularity at the origin and the ideal $J(f_{\mathbb{C}})$ is Newton non-degenerate, then we observe that the Łojasiewicz exponents $\ell_{\mathbb{R}}(m_n, J(f_{\mathbb{R}}))$ and $\ell(m_n, J(f_{\mathbb{C}}))$ are equal, by virtue of (3.8) and Corollary 3.6. In general, equality between these numbers does not hold, as the reader may check in [**5**].

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4. A method to test the Newton non-degeneracy condition

The task of deciding whether an ideal of finite colength $I \subseteq \mathcal{O}_n$ is Newton non-degenerate or not using Definition 3.2 involves the task of listing the set of compact faces of a Newton polyhedron in $\mathbb{R}^n_+$. This is not a straightforward task at all, even when $n = 3$. Here we present an alternative way to check when an ideal $I \subseteq \mathcal{O}_n$ is Newton non-degenerate by means of computing multiplicities. In particular, by Definition 3.3, the resulting test is useful in detecting Newton non-degenerate functions $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$.

We denote the n-dimensional volume of a compact subset $K \subseteq \mathbb{R}^n$ by $\mathrm{V}_n(K)$. However, by an abuse of notation, if Γ_+ is a Newton polyhedron in $\mathbb{R}^n_+$, we denote the n-dimensional volume of Γ_- by $\mathrm{V}_n(\Gamma_+)$.

If I is an ideal of finite colength of $\mathcal{O}_n$, we will denote the number $\mathrm{V}_n(\Gamma_+(I))$ by $v(I)$. Moreover, the multiplicity of I will be denoted by $e(I)$ (see [**20**, § 14]).

Theorem 4.1 (see [2, 3]). *Let I be an ideal of $\mathcal{O}_n$ of finite colength. Then $e(I) \geqslant n!v(I)$ and equality holds if and only if I is Newton non-degenerate.*

Therefore, in view of the above result, we can say that Newton non-degenerate ideals of finite colength of $\mathcal{O}_n$ are those ideals of $\mathcal{O}_n$ having *minimal* multiplicity.

If $\Gamma_+ \subseteq \mathbb{R}^n$ is a Newton polyhedron, we denote by ρ_Γ the polynomial obtained as the sum of those monomials x^k such that $k \in \Gamma$.

Lemma 4.2 (see [2]). *If Γ_+ is a Newton polyhedron in $\mathbb{R}^n$, then ρ_Γ is a Newton non-degenerate function.*

Corollary 4.3. *Let $I \subseteq \mathcal{O}_n$ be an ideal of finite colength and let $\Gamma = \Gamma(I)$. Then I is Newton non-degenerate if and only if*

$$e(I) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(\rho_\Gamma)}.$$

Proof. The function ρ_Γ is Newton non-degenerate, by Lemma 4.2. Then the colength of the ideal $I(\rho_\Gamma)$ is equal to $n!\mathrm{V}_n(\Gamma_+(\rho_\Gamma))$. But $\Gamma_+(I) = \Gamma_+(\rho_\Gamma)$. Hence the number $n!\mathrm{V}_n(\Gamma_+(I))$ is equal to the colength of $I(\rho_\Gamma)$ and the result follows from Theorem 4.1. □

We recall that if $I \subseteq \mathcal{O}_n$ is an ideal of finite colength generated by n elements, then $e(I) = \dim_{\mathbb{C}} \mathcal{O}_n/I$. Hence, as a consequence of the above lemma, if $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ is an analytic map germ such that the ideal $I(f)$ has finite colength and if $\Gamma_+ = \Gamma_+(f)$, then f is Newton non-degenerate if and only if

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(f)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(\rho_\Gamma)}. \tag{4.1}$$

If $f = \sum_k a_k x^k$ is the Taylor expansion of a function germ $f \in \mathcal{O}_n$, we denote by $p(f)$ the polynomial obtained as the sum of those terms $a_k x^k$ such that $k \in \mathrm{supp}(f) \cap \Gamma(f)$. We will refer to $p(f)$ as the *principal part of* f. We observe that the definition of a Newton non-degenerate function $f \in \mathcal{O}_n$ depends only on $p(f)$. Therefore, if f is a polynomial

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function such that $I(f)$ has finite colength, we have that f is Newton non-degenerate if and only if

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(f)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(\rho_f)}, \tag{4.2}$$

where ρ_f denotes the sum of the monomials x^k such that $k \in \operatorname{supp}(f)$. We remark that the dimensions appearing on both sides of (4.2) are easily computable via a computer algebra system such as SINGULAR [**13**].

Example 4.4. Let $f : (\mathbb{C}^4, 0) \to (\mathbb{C}, 0)$ be given by $f(x, y, z, t) = x^9 - y^9 + z^9 - t^9 + (x^2t - y^3z)^2 + xyzt$. Using the program SINGULAR, we can check that

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_4}{I(f)} = 2268.$$

On the other hand, the colength of the ideal $I(\rho_f)$ in $\mathcal{O}_4$ is also equal to 2268. Therefore, the function f is Newton non-degenerate, by Corollary 4.3.

The test given in (4.1) works when $I(f)$ has finite colength; in particular, the Newton polyhedron $\Gamma_+(f)$ must intersect each coordinate axis. We can extend this test to arbitrary functions $f \in \mathcal{O}_n$ with an isolated singularity at the origin using a result of Kouchnirenko. Let us give some preliminary definitions in order to state this result.

We say that a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$ is *q-installed*, where $q \in \{0, 1, \dots, n\}$, when Γ_+ intersects the axes corresponding to the variables $x_1, \dots, x_q$ and Γ_+ does not intersect the remaining coordinate axes. For an arbitrary Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$, we define

$$d(\Gamma_+) = \max\{|k| : k \in \Gamma\}. \tag{4.3}$$

Lemma 4.5 (see [15]). *Let $f \in \mathcal{O}_n$ have an isolated singularity at the origin. Suppose that $\Gamma_+(f)$ is q-installed and let $r_0 = d(\Gamma_+(f))^n + 1$. Then f is Newton non-degenerate if and only if $f + x_{q+1}^{r_0} + \dots + x_n^{r_0}$ is Newton non-degenerate.*

Under the conditions of the previous lemma, the Newton polyhedron of the function $g = f + x_{q+1}^{r_0} + \dots + x_n^{r_0}$ intersects each coordinate axis. Then it makes sense to ask if equality (4.1) is satisfied for the function g.

It is worth noting here the following observation. If we fix a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$, then we can deduce an effective criterion to determine whether a point $k = (k_1, \dots, k_n) \in \mathbb{Z}^n_+$ belongs to Γ_+ or not. This method helps in determining equisingular deformations of a given function $f \in \mathcal{O}_n$, by virtue of Theorem 6.3 and other results on equisingularity theory of deformations (see [**8**]).

Corollary 4.6. *Let $\Gamma_+ \subseteq \mathbb{R}^n_+$ be a Newton polyhedron intersecting each coordinate axis, let $k_0 \in \mathbb{Z}^n_+$ and let $\rho = \rho_\Gamma + x^{k_0}$. Then the point k_0 belongs to Γ_+ if and only if*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(\rho_\Gamma)} = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(\rho)}.$$

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Proof. Let I denote the ideal generated by all monomials x^k such that $k \in \Gamma_+$. Then we observe that $k_0 \in \Gamma_+$ if and only if $x^{k_0} \in \bar{I}$. Moreover, the monomial x^{k_0} is integral over I if and only if $e(I) = e(x^{k_0}, I)$, by Rees's theorem (see [**24**] or [**28**]). The ideals I and $\langle I, x^{k_0} \rangle$ are generated by monomials; in particular, both ideals are Newton non-degenerate. Then the result follows from Corollary 4.3. □

There is an alternative method to check if a point $k \in \mathbb{Z}^n_+$ belongs to a Newton polyhedron Γ_+ in the paper of Delfino *et al*. [**9**]. This method is based on integer programming techniques.

5. Newton non-degenerate Jacobian ideals

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic map germ. In this section we study the relation between Newton non-degenerate functions (see Definition 3.3) and functions whose Jacobian ideal $J(f)$ is Newton non-degenerate.

Let $L \subseteq \{1, \dots, n\}$. Then we denote the set $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0, \text{ for all } i \in L\}$ by $\mathbb{R}^n_L$ and the number of elements of L by $|L|$. If $L \subseteq \{1, \dots, n\}$ and $f = \sum_k a_k x^k \in \mathcal{O}_n$, we write f_L to denote the sum of those $a_k x^k$ such that $k \in \operatorname{supp}(f) \cap \mathbb{R}^n_L$.

Suppose that $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a Newton polyhedron intersecting each coordinate axis. Then in [**15**] the *Newton number* of Γ_+ is defined as

$$\nu(\Gamma_+) = n!\mathrm{V}_n - (n-1)!\mathrm{V}_{n-1} + \cdots + (-1)^{n-1} 1!\mathrm{V}_1 + (-1)^n,$$

where V_i denotes the sum of i-dimensional volumes of the intersection of Γ_- with coordinate planes of dimension i, for $i \in \{1, \dots, n\}$.

The *Newton number* of a q-installed Newton polyhedron Γ_+ is defined as

$$\nu(\Gamma_+) = \sup_{r \in \mathbb{N}} \nu(\Gamma_+(\rho_\Gamma + x^r_{q+1} + \cdots + x^r_n)).$$

It is proved in [**15**, p. 19] that if Γ_+ is a Newton polyhedron as above, then

$$\nu(\Gamma_+) = \nu(\Gamma_+(\rho_\Gamma + x^{r_0}_{q+1} + \cdots + x^{r_0}_n)),$$

where $r_0 = (d(\Gamma_+))^n + 1$ (see the definition given in (4.3)).

If $f \in \mathcal{O}_n$ has an isolated singularity at the origin, the *Milnor number* of f is defined as

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f)}.$$

Here we state a known result of Kouchnirenko on the computation of the Milnor number of a Newton non-degenerate function.

Theorem 5.1 (see [15]). *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic map germ with an isolated singularity at the origin. Then $\mu(f) \geqslant \nu(\Gamma_+(f))$, and equality holds if f is Newton non-degenerate.*

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Theorem 5.2. *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ with an isolated singularity at the origin such that*

$$\frac{\partial f_L}{\partial x_i} = \left(\frac{\partial f}{\partial x_i}\right)_L, \tag{5.1}$$

for all $L \subseteq \{1, \dots, n\}$ and all $i = 1, \dots, n$. Suppose that the ideal $I(f)$ has finite colength in $\mathcal{O}_n$. If $J(f)$ is Newton non-degenerate, then $I(f)$ is also Newton non-degenerate.

Proof. We observe that, since the ideal $I(f)$ has finite colength, the Newton polyhedron $\Gamma_+ = \Gamma_+(f)$ intersects each coordinate axis.

By [**15**, Lemma 3.2], we know that

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(f)} = \sum_{L \subseteq \{1,\dots,n\}} \mu(f_L), \tag{5.2}$$

where we set $\mu(f_\emptyset) = \mu(f)$ and $\mu(f_{\{1,\dots,n\}}) = 0$.

Moreover, since the ideal $J(f)$ is Newton non-degenerate, the ideal generated by the elements $(\partial f/\partial x_i)_L$, $i = 1, \dots, n$, is also Newton non-degenerate (as an ideal of $\mathcal{O}_{n-|L|}$), for all $L \subseteq \{1, \dots, n\}$. Then, as a consequence of condition (5.1), we deduce that $J(f_L)$ is Newton non-degenerate, for all $L \subseteq \{1, \dots, n\}$. Thus, applying Theorem 4.1 and relation (5.2), we have

$$n!\mathrm{V}_n(\Gamma_+) \leqslant \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(f)} = \sum_{L \subseteq \{1,\dots,n\}} (n - |L|)!\mathrm{V}_{n-|L|}(\Gamma_+(J(f_L))). \tag{5.3}$$

Let us denote by g the polynomial ρ_Γ. Considering condition (5.1), we observe that

$$\operatorname{supp}\left(\frac{\partial f_L}{\partial x_i}\right) = \operatorname{supp}\left(\frac{\partial f}{\partial x_i}\right)_L \supseteq \operatorname{supp}\left(\frac{\partial g}{\partial x_i}\right)_L \supseteq \operatorname{supp}\left(\frac{\partial g_L}{\partial x_i}\right).$$

Then we have the inclusion

$$\Gamma_+(J(g_L)) \subseteq \Gamma_+(J(f_L)), \tag{5.4}$$

for all $L \subseteq \{1, \dots, n\}$.

Therefore, since the function g is Newton non-degenerate, we have

$$\begin{aligned} n!\mathrm{V}_n(\Gamma_+) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(g)} = \sum_{L \subseteq \{1,\dots,n\}} \mu(g_L) &\geqslant \sum_{L \subseteq \{1,\dots,n\}} (n - |L|)!\mathrm{V}_{n-|L|}(J(g_L)) \\ &\geqslant \sum_{L \subseteq \{1,\dots,n\}} (n - |L|)!\mathrm{V}_{n-|L|}(J(f_L)), \end{aligned}$$

where the last inequality is a consequence of (5.4). Then joining the above inequalities with relation (5.3), we deduce that $n!\mathrm{V}_n(\Gamma_+)$ is equal to the colength of $I(f)$. But this implies that $I(f)$ is Newton non-degenerate, by Theorem 4.1. □

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We give an example showing that condition (5.1) cannot be removed from Theorem 5.2.

Example 5.3. Let us consider the map $f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by $f(x, y, z) = yz^3 + x^{10} + z^{10} + x^2z^4(x-z)^2 + y^5$. We observe that f is not Newton non-degenerate, since f does not satisfy condition (4.2). Let us see that the Jacobian ideal of f is Newton non-degenerate. The Milnor number of f is $\mu(f) = 99$. Let S be the union of the supports of $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ and let g_S denote the sum of those monomials x^k such that $k \in S$. Then we observe that

$$g_S(x, y, z) = x^9 + xz^6 + x^3z^4 + x^2z^5 + z^3 + y^4 + yz^2 + z^9 + x^4z^3.$$

and $\Gamma_+(g_S) = \Gamma_+(J(f))$. Moreover, by Theorem 4.2, the map g_S is Newton non-degenerate. Therefore,

$$3!\mathrm{V}_3(J(f)) = 3!\mathrm{V}_3(\Gamma_+(g_S)) = \dim_{\mathbb{C}} \frac{\mathcal{O}_3}{I(g_S)} = 99.$$

Then we have obtained the equality $\mu(f) = 3!\mathrm{V}_3(J(f))$, which implies that $J(f)$ is Newton non-degenerate, by Theorem 4.1. Hence, we conclude that the Łojasiewicz exponent $\alpha_0(f)$ is the least possible, that is, it is equal to 9, by virtue of Corollary 3.6.

Given a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$, we define

$$\mathcal{O}(\Gamma_+) = \{f \in \mathcal{O}_n : \Gamma_+(f) = \Gamma_+ \text{ and } V(J(f)) = \{0\}\}.$$

Then, when $\mathcal{O}(\Gamma_+) \neq \emptyset$, we can consider the number

$$\delta(\Gamma_+) = \min\{n!\mathrm{V}_n(\Gamma_+(J(f))) : f \in \mathcal{O}(\Gamma_+)\}.$$

By Theorem 5.1, we have that

$$\nu(\Gamma_+) = \min\{\mu(f) : f \in \mathcal{O}(\Gamma_+)\}.$$

Therefore, the inequality $\nu(\Gamma_+) \geqslant \delta(\Gamma_+)$ holds, by Theorem 4.1, and these numbers are different in general. We observe that the minimum $\delta(\Gamma_+)$ is attained at the function f defined as the sum of all monomials x^k such that $k \in \Gamma_+$ and $\{k - e_i : k_i > 0\}$ is contained in Γ_-, where $\{e_i\}_{i=1}^n$ denotes the canonical basis in $\mathbb{R}^n$.

Given a Newton polyhedron Γ_+ in $\mathbb{R}^n_+$, we define

$$\mathcal{K}(\Gamma_+) = \{f \in \mathcal{O}(\Gamma_+) : f \text{ is Newton non-degenerate}\},$$
$$\mathcal{B}(\Gamma_+) = \{f \in \mathcal{O}(\Gamma_+) : J(f) \text{ is a Newton non-degenerate ideal}\}.$$

Proposition 5.4. *Let $\Gamma_+ \subseteq \mathbb{R}^n_+$ be a Newton polyhedron. Then $\mathcal{K}(\Gamma_+) \subseteq \mathcal{B}(\Gamma_+)$ if and only if $\nu(\Gamma_+) = \delta(\Gamma_+)$.*

Proof. Suppose that $\nu(\Gamma_+) = \delta(\Gamma_+)$ and let $f \in \mathcal{K}(\Gamma_+)$. Then we obtain the following inequalities:

$$\nu(\Gamma_+) = \mu(f) \geqslant n!\mathrm{V}_n(\Gamma_+(J(f))) \geqslant \delta(\Gamma_+).$$

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Since we are assuming that $\nu(\Gamma_+) = \delta(\Gamma_+)$, the above relation implies that $\mu(f)$ is equal to $n!\mathrm{V}_n(\Gamma_+(J(f)))$ and then the ideal $J(f)$ is Newton non-degenerate, by Theorem 4.1.

Suppose that $\mathcal{K}(\Gamma_+) \subseteq \mathcal{B}(\Gamma_+)$ and let $f \in \mathcal{O}(\Gamma_+)$ be such that

$$\delta(\Gamma_+) = n!\mathrm{V}_n(\Gamma_+(J(f))).$$

The Newton non-degeneracy condition for functions is generic in the space of polynomial functions g such that $\mathrm{supp}(g) \subseteq \Gamma_+$ (see [**15**, Theorem 6.1]). Thus, modifying the coefficients of f, we can suppose that f is Newton non-degenerate. In particular, the ideal $J(f)$ is Newton non-degenerate, since we are assuming $\mathcal{K}(\Gamma_+) \subseteq \mathcal{B}(\Gamma_+)$. Therefore,

$$\nu(\Gamma_+) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f)} = n!\mathrm{V}_n(\Gamma_+(J(f))) = \delta(\Gamma_+).$$

□

It is worth remarking here that the class of functions $f \in \mathcal{O}_n$ with isolated singularity at the origin and such that $\nu(\Gamma_+(f)) = \delta(\Gamma_+(f))$ includes the class of homogeneous functions of $\mathcal{O}_n$ with isolated singularity at the origin.

We denote by $\mathrm{V}\Gamma_+$ the set of vertexes of a Newton polyhedron Γ_+. Then we will denote by $\rho_{\mathrm{V}\Gamma_+}$ the polynomial obtained as the sum of those monomials x^k such that $k \in \mathrm{V}\Gamma_+$. It is clear that $\mathcal{K}(\Gamma_+) \neq \emptyset$ for any Newton polyhedron Γ_+ (see also Lemma 4.2). On the other hand, the family $\mathcal{B}(\Gamma_+)$ is not always non-empty, as the following proposition shows.

Proposition 5.5. *Let Γ_+ be a Newton polyhedron in $\mathbb{R}^n_+$. Then $\mathcal{B}(\Gamma_+) \neq \emptyset$ if and only if the ideal $J(\rho_{\mathrm{V}\Gamma_+})$ is Newton non-degenerate.*

Proof. The *if* part is obvious. Let $f \in \mathcal{O}(\Gamma_+)$ be such that the ideal $J(f)$ is Newton non-degenerate. Then we observe that $\mathrm{supp}(\rho_{\mathrm{V}\Gamma_+})$ is contained in $\mathrm{supp}(f)$, which implies that $\mathrm{supp}(J(\rho_{\mathrm{V}\Gamma_+})) \subseteq \mathrm{supp}(J(f))$. In particular, we have the inequality

$$\mathrm{V}_n(\Gamma_+(J(f))) \leqslant \mathrm{V}_n(\Gamma_+(J(\rho_{\mathrm{V}\Gamma_+}))). \tag{5.5}$$

Since the ideal $J(f)$ is Newton non-degenerate and the function $\rho_{\mathrm{V}\Gamma_+}$ is always Newton non-degenerate, we deduce that

$$\mathrm{V}_n(\Gamma_+(J(\rho_{\mathrm{V}\Gamma_+}))) \geqslant \mathrm{V}_n(\Gamma_+(J(f))) = \mu(f) \geqslant \nu(\Gamma_+) = \mu(\rho_{\mathrm{V}\Gamma_+}) \geqslant \mathrm{V}_n(\Gamma_+(J(\rho_{\mathrm{V}\Gamma_+}))).$$

Then the ideal $J(\rho_{\mathrm{V}\Gamma_+})$ is Newton non-degenerate, by Theorem 4.1. □

Example 5.6. Let $f = x^5 + x^3y^2 + xy^5 + y^{10}$ and let $\Gamma_+ = \Gamma_+(f)$. It is easy to check that $\mathrm{supp}(f)$ is the set of vertices of Γ_+; then $f = \rho_{\mathrm{V}\Gamma_+}$. The ideal $J(f)$ is not Newton non-degenerate, as can be checked by applying Corollary 4.3. Hence there is no function $g \in \mathcal{O}_n$ such that $\Gamma_+(g) = \Gamma_+$ and $J(g)$ is Newton non-degenerate, by the previous result.

In view of the preceding results and examples, we can say that Newton polyhedra such that $\mathcal{B}(\Gamma_+) \neq \emptyset$ are somehow special. It would be desirable to characterize this kind of Newton polyhedron from a different point of view.

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6. Mixed multiplicities

We recall that if I, J are two ideals of $\mathcal{O}_n$ of finite colength, then $e_i(I, J)$ denotes the mixed multiplicity $e(I, \dots, I, J, \dots, J)$, where I is repeated i times and J is repeated $n - i$ times, $i \in \{0, 1, \dots, n\}$. Therefore, we have $e_0(I, J) = e(J)$ and $e_n(I, J) = e(I)$. We refer to the works of Teissier [**28**] and Rees [**25**] for the definition of, properties of and nice results concerning the notion of mixed multiplicity of a set of n ideals of finite colength in $\mathcal{O}_n$.

The following result of Rees [**25**] will help us in finding an expression for the set of mixed multiplicities $\{e_i(m_n, J(f)) : i = 0, 1, \dots, n\}$ in terms of $\Gamma_+(J(f))$ when $J(f)$ is a Newton non-degenerate ideal.

Theorem 6.1 (see [25]). *Let $I_1, \dots, I_n$ be ideals in $\mathcal{O}_n$ of finite colength. Then*

$$e(I_1, \dots, I_n) = \frac{1}{n!} \sum_{\substack{J \subseteq \{1, \dots, n\} \\ J \neq \emptyset}} (-1)^{n-|J|} e\biggl(\prod_{j \in J} I_j\biggr),$$

where $|J|$ denotes the number of elements of a subset $J \subseteq \{1, \dots, n\}$.

We recall that if $g \in \mathcal{O}_n$, then we denote by $I(g)$ the ideal of $\mathcal{O}_n$ generated by the germs

$$x_1 \frac{\partial g}{\partial x_1}, \dots, x_n \frac{\partial g}{\partial x_n}.$$

Corollary 6.2. *Let I, J be ideals of $\mathcal{O}_n$ of finite colength. If r, s are non-negative integer numbers, we denote by $g_{r,s}$ the sum of the monomials x^k whose support belongs to $\Gamma(I^r J^s)$. If I and J are Newton non-degenerate and $i \in \{0, 1, \dots, n\}$, then*

$$e_i(I, J) = \frac{1}{n!} \sum_{s=1}^{n} (-1)^{n-s} \biggl(\sum_{r=\max\{0, i-(n-s)\}}^{\min\{i, s\}} \binom{i}{r} \binom{n-i}{s-r} \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(g_{r,s-r})} \biggr).$$

Proof. If I and J are Newton non-degenerate ideals, then $\bar{I}$ and $\bar{J}$ are monomial ideals. However, the integral closure of IJ is equal to the integral closure of the product $\bar{I} \cdot \bar{J}$. Then the ideal IJ also has monomial integral closure, and, consequently, IJ is also a Newton non-degenerate ideal, by Theorem 3.4. Therefore, the ideals $I^r J^s$ are Newton non-degenerate, for all $r, s \in \mathbb{Z}_+^n$. Then the result follows as an application of Theorem 6.1, Corollary 4.3 and a simple combinatorial computation. □

Consider an analytic deformation $F : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ of a function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, where we write $F(t, x) = f_t(x)$. Suppose that $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ has an isolated singularity at the origin for all t. Then the deformation is said to be *μ^*-constant*, when $e_i(m_n, J(f_t))$ does not depend on t, for all $i = 0, 1, \dots, n$.

Theorem 6.3. *Let $f_t : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic deformation of a function germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$. Suppose that f_t has an isolated singularity at the origin for all t and suppose that*

$$\Gamma_+(J(f_t)) \subseteq \Gamma_+(J(f)), \tag{6.1}$$

for all t. If $J(f)$ is a Newton non-degenerate ideal, then the family f_t is μ^*-constant.

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Proof. The ideal $J(f)$ is Newton non-degenerate. Then, by Theorem 4.1 we have

$$n!v(J(f)) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f)} \geqslant \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f_t)} \geqslant n!v(J(f_t)), \tag{6.2}$$

for all t small enough, where the first inequality comes from the upper semicontinuity of the colength function [**29**, p. 39]. However, we have that $n!v(J(f_t)) \geqslant n!v(J(f))$ from inclusion (6.1). Then, joining this fact with (6.2), we deduce the equality $\Gamma_+(J(f)) = \Gamma_+(J(f_t))$. Hence each ideal $J(f_t)$ is Newton non-degenerate, by Theorem 4.1, and the mixed multiplicities $e_i(m_n, J(f_t))$ only depend on the Newton polyhedron of $J(f_t)$, by virtue of Corollary 6.2. Therefore, the family f_t is μ^*-constant. □

Under the conditions of the above result we conclude that the deformation f_t is Whitney equisingular along $\{0\} \times \mathbb{C}^n$ (see [**28**]).

In particular cases, we can apply Corollary 4.6 in order to study when condition (6.1) is satisfied. As we see in the next example, which consists of the example of Briançon and Speder [**6**], condition (6.1) cannot be removed from the previous theorem.

Example 6.4. Let us consider the analytic deformation $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ given by $f_t(x, y, z) = z^5 + xy^7 + x^{15} + ty^6 z$. We observe that $J(f_0)$ is Newton non-degenerate, as a consequence of Corollary 4.3. Moreover, a simple computation shows that condition (6.1) does not hold in this case. It is proved in [**6**] that the deformation f_t is not μ^*-constant. However, by the previous proposition, if we consider a deformation of the form $g_t = z^5 + xy^7 + x^{15} + tx^{k_1} y^{k_2} z^{k_3}$, such that the supports of the partial derivatives of $x^{k_1} y^{k_2} z^{k_3}$ belong to $\Gamma_+(J(f)) = \Gamma_+(y^7, x^{14}, xy^6, z^4)$, where $k_1, k_2, k_3 > 0$, then the deformation g_t is μ^*-constant.

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