*J. Austral. Math. Soc.* (*Series A*) 44 (1988), 259–264

# THE HAUSDORFF DIMENSION OF A SET OF NORMAL NUMBERS II

A. D. POLLINGTON

(Received 14 October 1986)

Communicated by W. Moran

**Abstract**

Let $R, S$ be a partition of $2, 3, \ldots$ so that rational powers fall in the same class. Let $(\lambda_n)$ be any real sequence; we show that there exists a set $N$, of dimension 1, so that $(x + \lambda_n)$ $(n = 1, 2, \ldots)$ are normal to every base from $R$ and to no base from $S$, for every $x \in N$.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 11 K 16, 11 K 55.

*Keywords and phrases*: normal numbers, Hausdorff dimension, sum sets.

## Introduction

We call two natural numbers $r, s$ equivalent, $r \sim s$, when each is a rational power of the other. Schmidt [4] showed that given any partition of the numbers $2, 3, \ldots$ into two disjoint classes $\mathbf{R}$, $\mathbf{S}$, so that equivalent numbers fall in the same class, then there is an uncountable set, $N$, of numbers which are normal to every base from $\mathbf{R}$ and to no base from $\mathbf{S}$. In [3] we showed that the set of numbers with this property has Hausdorff dimension 1.

Pearce and Keane [2] gave a new proof of Schmidt's result. Given $r, s$, $r \not\sim s$, there are uncountably many numbers which are normal to base $r$ but not even simply normal to base $s$. Brown, Moran and Pearce [1], have recently shown, using the theory of Riesz product measures, that every real number can be expressed as the sum of four numbers none of which is normal to base $s$ but all of which are normal to every base $r \not\sim s$.

In this paper we return to the method used by Schmidt [4] and the author [3] and prove

THEOREM 1. *Given any partition of the numbers* $2, 3, \ldots$ *into two disjoint classes* $\mathbf{R}$, $\mathbf{S}$ *so that equivalent numbers fall into the same class, and any real sequence* $(\lambda_i)_{i\in N}$, *then the set of numbers* $\xi$ *for which* $\lambda_i + \xi$ $(i = 1, 2, \ldots)$ *is normal to every base from* $\mathbf{R}$ *and to no base from* $\mathbf{S}$ *has Hausdorff dimension* 1.

This extends the results of [3] to simultaneous translates. It follows immediately from Theorem 1 that every real number can be expressed as a sum of two numbers from $N$.

## Preliminaries

The proof of this result proceeds along the same lines as that given in [3]. The only changes that need to be made are in that part of the argument concerned with the non-normality with respect to the bases from $\mathbf{S}$, the construction of the sets $J_1 \supset J_2 \supset \cdots$. As before we apply our construction to bases $\geq A$, which gives us a Hausdorff dimension of $\log(A-3)/\log A$, taking unions over $A$ gives dimension 1. We assume that our sequences $\mathbf{R} = (r_1, r_2, \ldots)$, $\mathbf{S} = \{s_1, s_2, \ldots\}$ satisfy the conditions of Section 3 of [3].

We write $h(m)$ for the least number $h$, for which

(1) $$m \not\equiv 0 \pmod{2^h}, \quad \text{that is,} \quad m = 2^h \cdot k + 2^{h-1},$$

and let

(2) $$g(m) = h(k).$$

Put

(3) $$s(m) = s_{g(m)}, \qquad \lambda(m) = \lambda_{h(m)}.$$

Then as $m$ runs through the natural numbers, with the non-negative integer powers of 2 deleted, each $\lambda_j$ appears infinitely often in $\lambda(m)$, and as $m$ runs through those numbers for which $\lambda(m) = \lambda_j$ each $s_i$ appears infinitely often in the sequence $s(m)$.

## Construction of a set of nonnormal numbers

We construct sets $J_0 = [0,1] \supset J_1 \supset J_2 \supset \cdots$, each the union of closed intervals. Let $f(m) = e^{\sqrt{m}} + 2s_1 m^3$. Put

$$\langle m \rangle = \lceil f(m) \rceil, \qquad \langle m; x \rangle \lceil \langle m \rangle / \log x \rceil$$

https://doi.org/10.1017/S1446788700029840 Published online by Cambridge University Press

where $\lceil x \rceil = -[-x]$.

(4) $$b_m = \langle m+1; s(m) \rangle$$

and

(5) $$a_{m+1} = \left[\frac{b_m \log s(m)}{\log s(m+1)}\right] + 2.$$

Then

(6) $$\frac{\langle m+1 \rangle}{\log s(m+1)} + 2 \le a_{m+1} \le \frac{\langle m+1 \rangle}{\log s(m+1)} + \log\log m + 3$$

and

$$s(1)^{b_1} < s(2)^{a_2} < s(2)^{b_2} < s(3)^{a_3} < \cdots.$$

Let $J_1$ be the union of the intervals, each of length $s(1)^{-b_1}$, whose left end points are of the form

(7) $$\xi_1 = \frac{\varepsilon_1}{s(1)} + \cdots + \frac{\varepsilon_{b_1}}{s(1)^{b_1}} - \lambda(1)$$

where the $\varepsilon_i$ range over $0, 1, \ldots, s(1) - \delta(1)$ and

$$\delta(i) = \begin{cases} 2 & \text{if } s(i) \text{ is odd,} \\ 3 & \text{if } s(i) \text{ is even.} \end{cases}$$

There are $(s(1) - \delta(1))^{b_1}$ such intervals $I$ of $J_1$.

Suppose that $J_k$ has been constructed and that $I_k$ is an interval of $J_k$ of length $s(k)^{-b_k}$. By (5)

$$s(k+1)^{-a_{k+1}+2} \le s(k)^{-b_k}.$$

Thus in each interval $I_k$ there are at least

$$\left[\frac{s(k+1)^{a_{k+1}}}{s(k)^{b_k}}\right] - 2$$

intervals $I'_k$ of length $s(k+1)^{-a_{k+1}}$ with left end point, $\rho_k$, for which $\rho_k + \lambda(k+1)$ is a finite decimal of length $a_{k+1}$ in base $s(k+1)$. We construct subintervals of $I'_k$ of length $s(k+1)^{-b_{(k+1}}$ whose left end points are of the form

(8) $$\xi_{k+1} = \rho_k + \left(\frac{\varepsilon_1}{s(k+1)} + \cdots + \frac{\varepsilon_{t_{k+1}}}{s(k+1)^{t_{k+1}}}\right) s(k+1)^{-a_{k+1}}$$

where $t_k = b_k - a_k$ and the $\varepsilon_i$ can range over $0, 1, \ldots, s(k+1) - \delta(k+1)$.

In each interval $I'_k$ there are $(s(k+1) - \delta(k+1) + 1)^{t_{k+1}}$ such intervals. Let $J_{k+1}$ be the union of all such intervals taken over all $I'_k$. Then $J_{k+1}$ is the union of at least

$$\left(\left[\frac{s(k+1)^{a_{k+1}}}{s(k)^{b_k}}\right] - 2\right) (s(k+1) - \delta(k+1) + 1)^{t_{k+1}}$$

intervals of length $s(k+1)^{-b_{k+1}}$. This completes the construction of the sequence of sets $J_0 \supset J_1 \supset \cdots$.

LEMMA 1. *If* $\xi \in \bigcap_{i=1}^{\infty} J_i$ *then* $\xi + \lambda_j$ *is non-normal to each base* $s_1, s_2, \ldots,$ *for every* $j \in N$.

PROOF. Fix $g, h$ and let $\lambda = \lambda_h$, $s = s_g$. Let $q$ be so large that

$$\left(\frac{s-1}{s}\right)^q < 2^{-g-h} \tag{9}$$

For a number $M$ with $h(M) = h$, $g(M) = g$ there are at least,

$$\sum_{\substack{m \le M \\ h(m)=h \\ g(m)=g}} (t_m - 1 - q), \tag{10}$$

$q$-blocks $\varepsilon_{i+1} \cdots \varepsilon_{i+q}$ consisting of the digits $0, 1, \ldots, s-2$ in the expansion of $\xi + \lambda$, for which $i + q \le b_M$. Now $h(m) = h$ if $m = 2^h \cdot k + 2^{h-1}$ and $g(m) = g$ if $k = 2^g \cdot l + 2^{g-1}$ so $m = 2^{h+g} l + 2^{h+g-1} + 2^{h-1}$, that is,

$$m \equiv 2^{h+q-1} + 2^{h-1} \pmod{2^{g+h}}.$$

If $g(m) = g$, $h(m) = h$ and $m > 2^{h+g-1} + 2^{h-1}$, then, by (6)

$$t_m - 1 - q \ge 2^{-g-h} \sum_{j=m-2^{g+h}+1}^{m} [(\langle j+1; \delta\rangle - \langle j; s\rangle) - \log\log m - 5 - q]$$

since $t_m = b_m - a_m$ and $\langle m+1; s\rangle - \langle m; s\rangle$ is a non-decreasing function of $m$. Thus (10) is at least

$$\begin{aligned} &\sum_{\substack{m \le M \\ g(m)=g \\ h(m)=h}} \sum_{j=m-2^{g+h}+1}^{m} ((\langle j+1; s\rangle - \langle j; s\rangle) - \log\log m - 5 - q) \\ &\ge 2^{-g-h}((\langle M+1; s\rangle - \langle 1; s\rangle - M(\log\log M + 5 - q)) \\ &= 2^{-g-h} b_M (1 + O(1)). \end{aligned}$$

If $\xi + \lambda$ were normal to base $s$, the number of $q$-blocks with digits $0, 1, \ldots, s_q - 2$ and indices smaller than $b_M$ would be asymptotic to $((s-1)/s) b_M$. By (9) this is clearly not the case and Lemma 1 is proved.

## Construction of a set of normal numbers

We also have to ensure that the translate of the numbers we have constructed are also all normal to every base from $\mathbf{R}$. We do this, as in [3], by discarding certain of the intervals $J_i$ at each stage, to obtain a new sequence, $K_1 \supset K_2 \supset \cdots$, with $K_i \supset J_i$.

https://doi.org/10.1017/S1446788700029840 Published online by Cambridge University Press

Consider the intervals $I'_{m-1}$. In each such interval there are

$$(s(m) - \delta(m) + 1)^{t_m}$$

intervals of $J_m$ whose left end points we denote by $\xi_m$. Let

$$A_m(x) = \sum_{t=-m}^{m} \sum_{i=1}^{m} \left| \sum_{j=\langle m_j r_i \rangle+1}^{\langle m+1; r_i \rangle} e(r_i^j t x) \right|^2 .$$

LEMMA 2. *Let $j \in N$, then if $m \geq \delta_j$ there are at least $(s(m)-3)^{t_m}$ numbers $\xi_m \in I'_{m-1}$ for which*

$$A_m(\xi_m + \lambda_i^2) \leq cm(\langle m+1 \rangle - \langle m \rangle)^{2-\beta_m/2},$$

*for $i = 1, \ldots, j$. Here $c$ is an absolute constant and $\delta_j$ is constant depending on $j$. Here $\beta_m$ is as in [3], $\beta_m \geq \beta_1 m^{-1/4}$.*

PROOF. Let $s = s(m)$. As in the proof of Lemma 3 of [3] we have:
The number of $\xi_m \in I'_{m-1}$ for which

$$A_m(\xi_m + \lambda_j) > cm^2(\langle m+1 \rangle - \langle m \rangle)^{2-\beta_m/2}$$

is at most

$$(\langle m+1 \rangle - \langle m \rangle)^{-\beta_m/2}(s - \delta + 1)^{t_m}.$$

But $\beta_m \geq \beta_1 m^{-1/4}$ and $(\langle m+1 \rangle - \langle m \rangle) \geq e^{\sqrt{m}}/(2\sqrt{m}+1)$, and so

$$(\langle m+1 \rangle - \langle m \rangle)^{-\beta_m/2} < \frac{1}{2^{j+1}} \quad \text{for } m > \delta_j.$$

Hence there are least $(s - \delta + 1)^{t_m}/2$ numbers $\xi_m \in I'_{m-1}$ for which

$$A_m(\xi_m + \lambda_i) \leq cm^2(\langle m+1 \rangle - \langle m \rangle)^{2-\beta_m/2}, \qquad i = 1, 2, \ldots, j.$$

For $m \geq \delta_j$ $(s-3)^{t_m} < (s - \delta + 1)^{t_m}/2$, this proves the lemma.

We construct the sequence of sets $K_1 \supset K_2 \supset \cdots$ in the same way as $J_1 \supset J_2 \supset \cdots$ was constructed. But at each stage in our construction of $\{K_m\}$ we use only the $(s(m)-3)^{t_m}$ points $\xi_m$ satisfying Lemma 2. The remainder of the proof of Theorem 1 now proceeds exactly as in [3].

By a straightforward application of Weyl's criterion we have

COROLLARY 1. *Given $(a_i)$ a sequence of non-zero rational numbers, $(b_i)$ a sequence of real numbers and a partition* **R**, **S** *of the numbers $2, 3, \ldots$ so that equivalent numbers fall into the same class, then the set of numbers $\xi$ for which $a_i\xi + b_i$ is normal to every base from* **R** *and to no base from* **S** *has Hausdorff dimension* 1.

COROLLARY 2. *Given any partition of $2, 3, \ldots$ into two classes* **R**, **S**, *so that equivalent numbers fall into the same class, then every real number can be* *written as a sum of two numbers both normal to every base from* **R** *and to no base from* **S**.

## Bibliography


[1] G. Brown, W. Moran, and C. E. M. Pearce, 'Riesz products and normal numbers', *J. London Math. Soc.* (2) **32** (1985), 12–18.

[2] C. E. M. Pearce and M. S. Keane, 'On normal numbers', *J. Austral. Math. Soc.* (1) **32** (1982), 79–87.

[3] A. D. Pollington, 'The Hausdorff dimension of a set of normal numbers', *Pacific J. Math.* **95** (1980) 193–204.

[4] W. M. Schmidt, 'Über die Normalität von Zahlen zu verschiedenen Basen', *Acta Arith.* **7** (1961–62), 299–309.


Department of Mathematics
Brigham Young University
Provo, Utah 84602
U.S.A.