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SOME COMPARISON THEOREMS FOR CONJUGATE AND σ-POINTS

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Introduction. Section 1 of this paper is concerned with the effect on conjugate and σ-points of various perturbations of $q(x)$ for differential equations of the form

$$z'' + q(x)z = 0.$$

An integral inequality is developed in Section 2 that involves corresponding focal and conjugate points of such a differential equation.

1. On perturbations. In this section of the paper we shall consider solutions $z(x)$ and $y(x)$, respectively, of differential equations

$$(1) \qquad z'' + q(x)z = 0,$$

$$(2) \qquad y'' + p(x)y = 0,$$

where $q(x)$ and $p(x)$ are positive functions, continuous on an interval $[0, c]$, except possibly at a finite number of points of the interval $(0, c)$ at each of which both left- and right-hand limits of $p(x)$ and $q(x)$ exist. The points of discontinuity of $p(x)$ and $q(x)$ are not necessarily the same points. Unless otherwise noted, a solution will always mean a nonnull solution.

We shall suppose that $x = c$ is the first conjugate point of $x = 0$ with respect to equation (1); that is, there exists a solution $z(x)$ of (1), positive on $(0, c)$, such that

$$z(0) = z(c) = 0, \quad z'(0) = 1.$$

The σ-point of $z(x)$ is the (unique) point σ on $(0, c)$ at which $z'(\sigma) = 0$, and similarly for $y(x)$.

The paper is concerned with comparison theorems for the first conjugate point and for the σ-point of $x = 0$ with respect to equation (2), when $p(x)$ is a perturbation of $q(x)$.

Use will be made of the following fundamental lemma (see [**4**]).

LEMMA 1. *If*

$$(3) \qquad \int_0^c [p(x) - q(x)]z^2(x)dx \geqq 0 \quad [p(x) \not\equiv q(x)],$$

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a solution $y(x)$ of (2) *that vanishes at $x = 0$ will have a zero on the open interval* $(0, c)$.

Let $x = \sigma$ be the σ-point of $z(x)$ and let $[\alpha, \beta]$ with midpoint m be any subinterval of $[0, \sigma]$. On $[\alpha, \beta]$ replace $q(x)$ by a function $p(x)$ with the property that $q(x) - p(x) > 0$ on $[\alpha, m)$ while $p(x) - q(x) > 0$ on $(m, \beta]$. Finally, let $p(x) \equiv q(x)$ outside $[\alpha, \beta]$.

We have then the following result.

THEOREM 1. *If*

$$(4) \qquad p(m + \epsilon) - q(m + \epsilon) \geqq q(m - \epsilon) - p(m - \epsilon)$$

for each $\epsilon > 0$ such that $m + \epsilon < \beta$, the conjugate point of $y(x)$ precedes $x = c$.

In other words, a solution $y(x)$ that vanishes at $x = 0$ must vanish again on $(0, c)$.

To prove the theorem, draw the graph of the function $z^2(x)$, note that $z^2(x)$ is an increasing function on $[0, \sigma]$ and that (3) holds.

Clearly there is a dual theorem when the subinterval $[\alpha, \beta] \subset [\sigma, c]$. One replaces $q(x)$ on $[\alpha, \beta]$ by a function $p(x)$ with the property that $p(x) - q(x) > 0$ on $[\alpha, m)$ and $q(x) - p(x) > 0$ on $(m, \beta]$, where m is the midpoint of $[\alpha, \beta]$. Let $p(x) \equiv q(x)$ outside $[\alpha, \beta]$. We then have the following result.

THEOREM 2. *If*

$$(5) \qquad p(m - \epsilon) - q(m - \epsilon) \geqq q(m + \epsilon) - p(m + \epsilon)$$

for each $\epsilon > 0$ such that $m + \epsilon < \beta$, the conjugate point of $y(x)$ precedes $x = c$.

Next, let $z(x)$ be a solution of the system

$$(6) \qquad z'' + q(x)z = 0, \quad z(0) = 0, \quad z'(0) = 1,$$

let $y(x)$ be a solution of the system

$$(7) \qquad y'' + p(x)y = 0, \quad y(0) = 0, \quad y'(0) = 1,$$

and let σ be the σ-point of $z(x)$.

LEMMA 2. *If*

$$(8) \qquad \int_0^\sigma [p(x) - q(x)]z^2(x)dx \geqq 0 \quad [p(x) \not\equiv q(x)],$$

the σ-point of $y(x)$ precedes that of $z(x)$.

A special case of this result was proved in [**4**]. To prove the lemma, we employ the Picone formula

$$(9) \qquad \int_0^\sigma (p - q)z^2 dx + \int_0^\sigma \left(\frac{yz' - zy'}{y}\right)^2 dx = \left[\frac{z}{y}(yz' - zy')\right]_0^\sigma .$$

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Suppose first that $y(x) > 0$ on $(0, \sigma]$. The right-hand member of (9) is then zero. We shall have a contradiction unless the first two integrals in (9) are both zero. But the second integral vanishing implies that $y(x) \equiv kz(x)$, where k is a constant; that is, both $y(x)$ and $z(x)$ are solutions of (1) and of (2). Accordingly, $[p(x) - q(x)]y(x) \equiv 0$. Then $y(x)$ must be identically zero on a subinterval (by hypothesis, known to exist) on which $p(x) \not\equiv q(x)$. It would follow that $y(x) \equiv 0$ on $[0, \sigma]$.

If the lemma is false, $y(\sigma)$ must then be zero. The right-hand member of (9) is again zero, since $\lim z/y$ exists at both $x = 0$ and $x = \sigma$. The second integral in (9) exists, and the above argument may be repeated.

The proof of the lemma is complete.

THEOREM 3. *Under the hypotheses of either Theorem* 1 *or Theorem* 2, *the σ-point of $y(x)$ precedes that of $z(x)$.*

The arguments in support of Theorems 1 and 2, modified in an obvious way, are valid for Theorem 3.

COROLLARY 1. *Let $[\alpha_1, \beta_1]$ and $[\alpha_2, \beta_2]$ $(\alpha_1 < \alpha_2)$ be equal (possibly overlapping) subintervals of $[0, \sigma]$ and consider the two differential equations*

$$\text{(i)} \qquad y_1'' + p_1(x)y_1 = 0,$$

$$\text{(ii)} \qquad y_2'' + p_2(x)y_2 = 0,$$

where $p_1(x) = q(x) + \delta (\delta > 0)$ on $[\alpha_1, \beta_1]$ and $p_1(x) = q(x)$ outside this subinterval of $[0, c]$, while $p_2(x) = q(x) + \delta$ on $[\alpha_2, \beta_2]$ and $p_2(x) = q(x)$ outside this subinterval. Then, the conjugate point and the σ-point of $x = 0$ with respect to (ii) *precede, respectively, the corresponding points with respect to* (i).

This conclusion also holds in the dual situation—when the subintervals lie on $[\sigma, c]$ and $\alpha_2 < \alpha_1$.

It is clear that the constant δ in the corollary can be replaced by a positive function $\delta(x)$ *mutatis mutandis.*

Critique. Note that m need not be the midpoint of $[\alpha, \beta]$ in the above theorems. It may be any point of (α, β) less than or equal to the midpoint in the case of Theorem 1 and in the first part of Theorem 3. Similarly, it may be any point greater than or equal to the midpoint of $[\alpha, \beta]$ in Theorem 2 and in the second part of Theorem 3. Then conditions (4) and (5) are required to hold only where applicable—that is, on a subinterval of $[\alpha, \beta]$ of which m is the midpoint. Outside such a subinterval, on the remainder of $[\alpha, \beta]$, the left-hand members of inequalities (4) and (5) will then be required simply to be nonnegative.

The following corollary is useful in computing estimates of the σ-point of $z(x)$ for some functions $q(x)$ (we return to this idea later in the paper).

COROLLARY 2. *Suppose that $q(x)$ is an increasing convex function, and let $x = \sigma$ be the σ-point of the corresponding solution $z(x)$ of* (6). *Let the interval*

$[0, \sigma]$ be divided into n subintervals on each of which $q(x)$ is replaced by its value at the midpoint of the subinterval and let $p(x)$ be the resulting step-function on $[0, \sigma]$, while $p(x) \equiv q(x)$ on $(\sigma, c]$. The σ-point of the solution $y(x)$ of (7) will follow the σ-point of $z(x)$, and there will be no conjugate point of $x = 0$ corresponding to the equation $y'' + p(x)y = 0$ on the interval $(0, c]$.

If $q(x)$ is decreasing and concave and $q(x)$ is replaced on $[0, \sigma]$ by the above step-function, the σ-point of $y(x)$ will precede the σ-point of $z(x)$, and, likewise, the conjugate point of $y(x)$ will precede that of $z(x)$.

We continue with generalizations of earlier results of the present writer [**1**, Theorem 6], the proofs of which are less immediate.

Suppose now that in addition to previous assumptions $q(x)$ is nondecreasing on $[0, c]$, and let m be any point of $(0, c)$. We employ the following lemma.

LEMMA 3. *Suppose $q(x)$ is nondecreasing on $[0, c]$, let m be any point of $(0, c)$ such that $2m \leqq c$, and let $z(x)$ be the solution defined by (6). Then, $z(m + \epsilon) > z(m - \epsilon)$ for each ϵ $(0 < \epsilon < m)$.*

To prove the lemma; construct the auxiliary differential equation

$$z_1'' + q_1(x)z = 0, \tag{10}$$

where $q_1(x) = q(2m - x)(m \leqq x \leqq 2m)$. Note that $q_1(x)$ is the reflection in the line $x = m$ of $q(x)$ and that $q_1(x)$ is defined only on the interval $[m, 2m]$. Let $z_1(x)$ be the solution of (10) defined by the conditions

$$z_1(2m) = 0, \quad z_1'(2m) = -1.$$

Then $z_1(x)$ will be well defined on $[m, 2m]$ and will be the reflection on that interval in the line $x = m$ of $z(x)$ on the interval $[0, m]$, and $z_1(m) = z(m) > 0$, $z_1'(m) = -z'(m)$. Observe that [**4**] $m \leqq c/2 \leqq \sigma$; consequently,

$$z_1'(m) = -z'(m) \leqq 0. \tag{11}$$

The conclusion of the lemma is obvious, unless $m + \epsilon > \sigma$. So assume that this inequality holds. It follows from (11) that $z_1(x) < z(x)$ on a small interval $(m, m + \epsilon_1)(\epsilon_1 > 0)$. Suppose that at some point

$$x = x_1(m < x_1 < 2m), \quad z(x_1) = z_1(x_1).$$

At such a point, $z_1'(x_1) > z'(x_1)$. But the "wronskian"

$$w = z_1 z' - z_1' z$$

has the property that

$$w'(x) = [q_1(x) - q(x)]z(x)z_1(x) < 0$$

for x on $(x_1, 2m]$; that is, $w(x)$ is nonincreasing on this interval. We have, however,

$$w(x_1) = z(x_1)[z'(x_1) - z_1'(x_1)] < 0,$$
$$w(2m) = -z_1'(2m)z(2m) > 0,$$

a contradiction. Accordingly, $z_1(x) < z(x)(m < x \leqq 2m)$, and the conclusion of the lemma follows.

THEOREM 4. *Let $q(x)$ be nondecreasing on $(0, c)$ and let m be any point of $(0, c)$ such that $2m \leqq c$. If on $[0, 2m]$ $q(x)$ is replaced by a function $p(x)$ with the property that*

$$p(m + \epsilon) - q(m + \epsilon) \geqq q(m - \epsilon) - p(m - \epsilon) \geqq 0 \quad (0 < \epsilon \leqq m),$$

with $p(x) \equiv q(x)$ on $(2m, c]$, the conjugate point of $x = 0$ with respect to the differential equation $y'' + p(x)y = 0$ precedes that of $z(x)$, unless $p(x) \equiv q(x)$.

An appeal to Lemmas 1 and 3 yields the proof of the theorem.

The dual of Theorem 4 is the following.

THEOREM 5. *Let $q(x)$ be a nonincreasing function on $(0, c)$ and let m be any point of $(0, c)$ such that $2m \geqq c$. If, on the interval $[2m - c, c]$, $q(x)$ is replaced by a function $p(x)$ with the property that*

$$(12) \quad p(m - \epsilon) - q(m - \epsilon) \geqq q(m + \epsilon) - p(m + \epsilon) \geqq 0 \quad (0 < \epsilon \leqq c - m)$$

with $p(x) \equiv q(x)$ on $[0, 2m - c)$, the conjugate point of $x = 0$ with respect to the differential equation $y'' + p(x)y = 0$ precedes that of $z(x)$, unless $p(x) \equiv q(x)$.

Theorems 4 and 5 generalize Theorems 1.1 and 1.2 of [**4**].

An instructive example. Let $q(x) = 1$. Then $z(x) = \sin x$, and $c = \pi$, $\sigma = \pi/2$. Set

$$p(x) = \begin{cases} \lambda^2 & (0 \leqq x < \pi/2) \\ \mu^2 & (\pi/2 < x \leqq \pi), \end{cases}$$

where λ and μ are numbers on the interval $(0, 2)$. A solution of the system

$$(13) \quad y'' + p(x)y = 0, \quad y(0) = 0$$

is, then,

$$(14) \quad y = \begin{cases} \sin \lambda x & (0 \leqq x \leqq \pi/2) \\ \sin (\lambda\pi/2) \cos \mu(x - \pi/2) + (\lambda/\mu) \cos (\lambda\pi/2) \sin \mu(x - \pi/2) & (\pi/2 < x \leqq \pi). \end{cases}$$

A little trigonometry reveals that for (13)

$$c = \frac{\pi}{2} + \frac{\pi}{2\beta}\left[\pi - \text{arc tan}\left(\frac{\beta}{\alpha}\tan\alpha\right)\right] \quad (\alpha = \lambda\pi/2, \beta = \mu\pi/2),$$

$$\sigma = \frac{\pi}{2\lambda} \quad (\lambda > 1),$$

$$\sigma = \frac{\pi}{2}\left[1 + \frac{1}{\beta}\text{arc tan}\left(\frac{\alpha}{\beta}\frac{1}{\tan\alpha}\right)\right] \quad (\lambda < 1).$$

When, for example, $\lambda = 3/2$, $\mu = 1/2$, we have $c = 2.2143$, $\sigma = \pi/3$. Note that $c < \pi$ and $\sigma < c/2$. When $\lambda = 1/2$, $\mu = 3/2$, we have $c = 2.8325$, $\sigma = 1.7853$. In this case, $c < \pi$, while $\sigma > c/2$ (cf. [**4**]).

In the limiting case $\lambda = 0$, if we take $\mu = \sqrt{2}$, say, we have $c = 2.9806 < \pi$, and $\sigma = 2.3824 > c/2$. This is, of course, equivalent to defining $p(x) = 0$ on $[0, \pi/2)$ and $p(x) = 2$ on $[\pi/2, \pi]$. When $\lambda = \sqrt{2}$, $\mu = 0$, then $\sigma = \pi/2\sqrt{2} = 1.1107$, and c does not exist.

Finally, it is of interest to determine μ^2, when λ^2 is an arbitrary number < 1 and $c = \pi$. This leads at once to the equation

$$\frac{\tan \beta}{\beta} + \frac{\tan \alpha}{\alpha} = 0.$$

A little computation leads to the following paired values of λ^2 and μ^2:

λ^2 :	0.000	0.0625	0.2500	0.5625	1.000
μ^2 :	1.664	1.638	1.538	1.369	1.000

Approximating σ. A method of obtaining both lower and upper bounds of σ-points was developed in [**3**]. Corollary 2 to Theorem 3 above permits sharper results in two situations. Consider the differential equation

$$y'' + q(x)y = 0, \tag{15}$$

where $q(x)$ is positive, nondecreasing, and convex on $[0, \sigma]$, and where $x = \sigma$ is the σ-point of a solution of (15) that vanishes at $x = 0$. Divide the interval $[0, \sigma]$ into n subintervals $[0, h]$, $[h, 2h]$, $\cdots$ $[(n-1)h, nh]$ of common length h. If $q(x)$ is then replaced by the step-function $p(x)$, where

$$p(x) = q\left(\frac{2i-1}{2}h\right) = c_i^{\,2}$$

on the ith subinterval ($i = 1, 2, \cdots, n$), the σ-point of $x = 0$ with respect to

$$y'' + p(x)y = 0 \tag{16}$$

will, by Corollary 2 to Theorem 3, follow that of (15)—that is, will provide an upper bound for σ of (15).

The above observations yield a method of obtaining such a bound (see [**3, 6**]). One solves the equations

$$\begin{aligned} &\tan z_2 = \frac{c_2}{c_1}\tan c_1 h, \\ (17)\quad &\tan z_i = \frac{c_i}{c_{i-1}}\tan\,(c_{i-1}h + z_{i-1}) \quad (i = 3, 4, \cdots, n), \\ &h = \frac{1}{c_n}\left(\frac{\pi}{2} - z_n\right) \end{aligned}$$

for h using a modified version of successive approximations (see [**6**]).

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Similarly, if $q(x)$ is positive, nonincreasing, and concave on $[0, \sigma]$, equations (17) will provide a lower bound for the σ-point associated with (15).

For example, the σ-point of $x = 0$ associated with (17), when $q(x) = 7 - x^2$, is known to be

$$\frac{\sqrt{9 - \sqrt{57}}}{2} = 0.6021.$$

Taking $n = 5$ in (17) one obtains the lower bound 0.6020^-—a very good lower bound considering the small value of n employed.

2. Some integral inequalities. In another paper [**4**] it was shown that if $p(x)$ is positive and increasing on $[a, c]$ and $x = c$ is the first conjugate point of $x = a$, then

$$(18) \quad \int_a^\sigma p(x)dx < \int_\sigma^c p(x)dx,$$

where σ is the σ-point of a solution vanishing at $x = a$. The inequality is reversed when $p(x)$ is a decreasing function. It is also true when $p(x)$ is increasing that $\sigma < f$, where $x = f$ is the focal point of the line $x = a$. If we write

$$(19) \quad p(x) = 1/h^2(x)$$

and assume that $p(x)$ is of class C'', an inequality stronger than (18) may be available, as the following result shows.

THEOREM 6. *Suppose that $p(x) > 0$ is an increasing function of class C'', that $h''(x) < 0$, and that $y_1(x)$ is a solution of the differential equation*

$$(20) \quad y'' + p(x)y = 0$$

such that $y_1(a) = y_1(c) = 0$, $y_1(x) \neq 0$ on (a, c). Then

$$(21) \quad c/2 < \sigma < f$$

and

$$(22) \quad \int_a^f p(x)dx < \int_f^c p(x)dx.$$

The result (21) is known [**4**]. To prove (22), let $y(x)$ be a solution of (20) and set

$$z(x) = y'(x).$$

Then $z(x)$ is a solution of the differential equation

$$(23) \quad (z'/p)' + z = 0.$$

If in this equation we substitute

$$(24) \quad t = \int_a^x p(x)dx,$$

we have

$$\ddot{w} + \frac{1}{p_1(t)} w = 0,$$

where $w(t) = z(x) = y'(x)$, and $p_1(t) = p(x)$, subject to (24). Further, we have the following identities:

$$\dot{w}(t) = \frac{z'(x)}{p(x)} = \frac{y''(x)}{p(x)} = -y(x).$$

Suppose now that $y(x)$ is a solution of (20) such that $y(a) = 1$, $y'(a) = 0$, $y(f) = 0$, $y(x) > 0$ on (a, f). Then $x = f$ is the focal point of the line $x = a$. We observe that when $x = a$, $t = 0$, and write

$$t_1 = \int_a^f p(x)dx, \quad t_2 = \int_a^g p(x)dx,$$

where $x = g$ is the first zero of $y'(x)$ following $x = a$. It follows that $w(0) = 0 = w(t_2)$.

Because $p(x)$ is an increasing function, $1/p_1(t)$ decreases, as t increases. It follows [**1**] that $2t_1 < t_2$; that is,

$$\int_a^f p(x)dx < \int_f^g p(x)dx. \tag{25}$$

But (see [**5**]) because $h''(x) < 0$, $g < c$, and (22) follows.

A companion result is the following.

THEOREM 7. *If in Theorem* 6, $p(x)$ *is a decreasing function with* $h''(x) > 0$, *then* $f < \sigma < c/2$ *and*

$$\int_a^f p(x)dx > \int_f^c p(x)dx.$$

The proof is analogous to that of Theorem 6.

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