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An application of combinatorial techniques to a topological problem

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The following statement is proved: Let X be a set having at most continuously many elements and $f : X \to X$ a mapping such that each iteration f^n $(n = 1, 2, \ldots)$ has a unique fixed point. Then for every number $c \in (0, 1)$ there exists a metric ρ on X such that the metric space (X, ρ) is separable and the mapping f is a contraction with the Lipschitz constant c .

1. Introduction

In recent two decades different mathematicians asked the following question: Given an abstract set X and a mapping $f : X \to X$, does there exist a non-trivial topology on X which would render f continuous and would satisfy at the same time some prescribed conditions (compactness, separability, metrizability, Hausdorff property, and so forth)? de Groot and de Vries [3] proved that if X has at most continuously many elements then for every $f : X \to X$ there exists a non-discrete separable metric topology on X rendering f continuous. Bessaga [2] obtained the following result (a converse to the Banach fixed point theorem).

THEOREM 1 (Bessaga). *Let X be a set and $f : X \to X$ such that all the iterates f^n have a unique fixed point. Assuming the weak (countable) form of the axiom of choice, then for any $c \in (0, 1)$ there exists a complete metric on X rendering f a c-contraction.*

The purpose of this note is to show that in case X has at most continuously many elements then the separability of the metric in the above

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theorem can be claimed. In the construction of this metric we will use the following combinatorial theorem of Ramsey (see, for example, [1]).

THEOREM 2 (Ramsey). *If the set of all unordered pairs* $\{n, m\}$ *of natural numbers* N *is decomposed in finite number of sets, say* $R_1, R_2, \ldots, R_k$, *that is,*

$$\left\{A \;\middle|\; |A| = 2 \text{ and } A \subset N\right\} = R_1 \cup R_2 \cup \ldots \cup R_k$$

then there exists an infinite subset $M \subset N$ *and an index* $i \in \{1, 2, \ldots, k\}$ *such that all pairs* $\{n, m\} \subset M$ *belong to* R_i .

Finally we will need the following result of Meyers [4].

THEOREM 3 (Meyers). *If* X *is a metrizable topological space and* $f : X \to X$ *a continuous mapping satisfying:*

(i) f *has a unique fixed point* a , *that is,* $f(a) = a$;

(ii) *for every* $x \in X$ *the sequence of iterates* $x, f(x), f^2(x), \ldots$ *converges to* a ;

(iii) *there exists a neighbourhood* U_a *of* a *such that for any neighbourhood* V_a *of* a *there exists* n_0 *such that* $n \geq n_0$ *implies* $f^n(U_a) \subset V_a$;

then for every $c \in (0, 1)$ *there exists a metric on* X *which is compatible with the topology of* X *and with respect to which* f *is a* c*-contraction.*

2. Proof of the theorem

Let X be an abstract set with at most continuously many elements and let $f : X \to X$ satisfy the conditions of Theorem 1. Choosing $c = \frac{1}{2}$ we denote by ρ the corresponding metric on X existing by this theorem. If a is the fixed point of f we define the sets A_n (n integer) by:

$$A_n = \{x \mid x \in X \text{ and } 2^{n-1} < \rho(a, x) \leq 2^n\} .$$

Thus we obtain a disjoint partition of X in the form $X = \{a\} \cup \bigcup_{-\infty}^{+\infty} A_n$ satisfying the condition that the image $f(A_n)$ of A_n under f is contained in $\{a\} \cup \bigcup_{-\infty}^{n-1} A_k$. Once this result is achieved, we disregard the metric ρ (since it is not separable in general) and proceed in the following way.

We consider the subset $\{0\} \cup \bigcup_{-\infty}^{+\infty} C_n$ of the euclidean plane where 0 is the origin and C_n is the circle with centre in 0 and of radius 2^n. Since each set A_n has at most continuously many elements one can identify A_n with a certain subset $B_n \subset C_n$ of C_n. Doing this for every n and identifying a with the origin 0, our set X can be thought of as the set $\{0\} \cup \bigcup_{-\infty}^{+\infty} B_n$. Denoting by d_2 the euclidean metric we thus obtain a separable metric space (X, d_2) and it follows from the definition that each subset $\{0\} \cup \bigcup_{-\infty}^{n} B_k$ is totally bounded and invariant under f.

We now define a new metric d_2^* on X with respect to which f will be continuous as follows:

$$d_2^*(x, y) = \sup_{n \geq 0} d_2\left(f^n(x), f^n(y)\right) ,$$

for $x, y \in X$ and where $f^0(x)$ stands for x. It is clear that d_2^* is a metric and that f is continuous with respect to d_2^*, since from the definition it follows immediately that f is non-expanding:

$$d_2^*\left(f(x), f(y)\right) \leq d_2^*(x, y) .$$

Since the circles C_n shrink to 0 it follows that for each pair $x, y \in X$ there is a number $n = n(x, y)$ such that

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$d_2^*(x, y) = d_2\left(f^n(x), f^n(y)\right)$. In order to show that the sets $\{0\} \cup \bigcup_{-\infty}^{n} B_k$ are totally bounded also with respect to the metric d_2^* we need the following.

LEMMA. *Let* (Y, d) *be a totally bounded metric space and let* $f : Y \to Y$ *(not necessarily continuous) be such that the diameters* δ_n *of the iterated images* $f^n(Y)$ *converge to zero as* $n \to \infty$. *Then the metric* d^* *on* Y *defined by*

$$d^*(x, y) = \sup_{n \geq 0} d\left(f^n(x), f^n(y)\right)$$

is also totally bounded.

Proof. First we observe that due to $\delta_n \to 0$ there is an integer $n = n(x, y)$ for each pair of points $x, y \in Y$ such that $d^*(x, y) = d\left(f^n(x), f^n(y)\right)$. Now if d^* were not totally bounded there would be a number $\varepsilon > 0$ and a sequence $\{x_k\} \subset Y$ such that

$$d^*\left(x_k, x_l\right) \geq \varepsilon \quad \text{for all} \quad k \neq l .$$

But this would mean that there is a function $n(k, l)$ on the set of all unordered paiı $\{k, l\}$ of natural numbers such that $d\left(f^{n(k,l)}\left(x_k\right), f^{n(k,l)}\left(x_l\right)\right) \geq \varepsilon$ for all pairs $\{k, l\} \subset N$. Again due to the shrinkage $\delta_n \to 0$ it is obvious that the function $n(k, l)$ must be bounded and so its range consists of finite numbers of values, say $n_1, n_2, \ldots, n_r$. But Theorem 2 would then imply that for some $i \in \{1, 2, \ldots, r\}$ the inequality $d\left(f^{n_i}\left(x_k\right), f^{n_i}\left(x_l\right)\right) \geq \varepsilon$ would hold for some infinite subset of indices which would contradict the assumption that d is totally bounded. This proves that d^* must be totally bounded as well.

Observing that the restriction of $f : X \to X$ to the invariant subset $X_n = \{0\} \cup \bigcup_{-\infty}^{n} B_k$ satisfies the hypothesis of our lemma we arrive at the following conclusion.

As a countable union of totally bounded sets, (X, d_2^*) is a separable metric space and $f : X \to X$ a continuous mapping. Since $d_2^* \geq d_2$ it follows that the topology generated by d_2^* is in general finer than the Euclidean generated by d_2. Since each set X_n is d_2-open, it is also d_2^*-open and observing that for each $x \in X$ we have $d_2^*(0, x) = d_2(0, x)$ it follows that each open neighbourhood of 0 with respect to d_2^* contains some set X_n. Since $f(X_n) \subset X_{n-1}$ this implies that the conditions of Theorem 3 are satisfied for the topology generated by d_2^* and our theorem follows from Theorem 3.

REMARK. It is so far not known if the space (X, d_2^*) can be assumed *topologically* complete. In this case the result of Meyers [4] would furnish at the same time a separable and complete metric. So it appears that the gain of separability was paid for by the loss of completeness.

References

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