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A TOWER OF RIEMANN SURFACES WHICH CANNOT BE DEFINED OVER THEIR FIELD OF MODULI

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Abstract. Explicit examples of both hyperelliptic and non-hyperelliptic curves which cannot be defined over their field of moduli are known in the literature. In this paper, we construct a tower of explicit examples of such kind of curves. In that tower there are both hyperelliptic curves and non-hyperelliptic curves.

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1. Introduction. The notion of field of moduli was first introduced by Matsusaka in [**17**] for the case of polarized abelian varieties and generalized by Shimura in [**18**] for polarized abelian varieties with further structure. Later, Koizumi in [**12**] gave a more general definition of the field of moduli for general algebraic varieties (even with extra structures) which coincides with Matsusaka's and Shimura's definitions for polarized abelian varieties. In general, the field of moduli of a variety is not a field of definition for it. Both the computation of the field of moduli and to determine if it is a field of definition is a hard problem. Weil's Galois descent theorem [**20**] provides a sufficient condition for a variety X, defined over a finite Galois extension L/k, to be definable over k. The sufficient condition is given by the existence of a birational isomorphism $f_\sigma : X \to X^\sigma$, for each $\sigma \in \mathrm{Gal}(L/k)$ (defined over L) satisfying some co-cycle conditions (Weil's datum). Weil's theorem is still valid if we replace L with the complex field $\mathbb{C}$, k with the field of rationals $\mathbb{Q}$ and X with a non-singular and irreducible complex algebraic curve (that is, a closed Riemann surface) of genus at least two. If the variety has no non- trivial birational automorphisms, then the existence of a Weil's datum is clear. Unfortunately, if the variety has non-trivial automorphisms, to check the existence of a Weil's datum is not an easy task.

The first examples of explicit curves which cannot be defined over their field of moduli were provided by Earle [**4,5**] and by Shimura [**18**] around 1972; these examples are hyperelliptic curves of even genus. Other explicit examples were constructed by Huggins [**11**] for genus at least three. In [**2**] Bujalance–Turbek have provided a characterization of those hyperelliptic curves whose field of moduli is real but not a field of definition. This characterization was completed by Huggins in [**10**]. In the case of non-hyperelliptic curves, such kind of examples were obtained by the third author in [**7, 8**] and by Kontogeorgis in [**13**].

In this paper, we produce a tower of examples of curves which cannot be defined over their field of moduli. We start with the non-hyperelliptic curves as in [**7, 8**] and construct quotients of it which turn out to be non-definable over their fields of moduli. In such a tower, the lowest one is the hyperelliptic curve isomorphic to the one obtained by Earle in [**4, 5**].

THEOREM 1.1. *Let* $\theta \in (0, \pi)$, $\theta \neq \pi/2$, *and let* $r \in (1, +\infty)$, $r \notin \left\{\sqrt{1+\cos^2\theta} \pm \cos\theta\right\}$. *Set*

$$C_{r,\theta} = \left\{ \begin{array}{rl} x_1^2 + x_2^2 + x_3^2 & = 0 \\ -r^2 x_1^2 + x_2^2 + x_4^2 & = 0 \\ re^{i\theta} x_1^2 + x_2^2 + x_5^2 & = 0 \\ -re^{i\theta} x_1^2 + x_2^2 + x_6^2 & = 0 \end{array} \right\} \subset \mathbb{P}^5.$$

Then the following hold.

(1) $\mathbb{Z}_2^5 \cong H = \langle a_1, a_2, a_3, a_4, a_5 \rangle = \mathrm{Aut}(C_{r,\theta})$, *where* a_j *is multiplication by* -1 *of the* x_j-*coordinate. Furthermore* $\mathrm{Aut}^{\pm} C_{r,\theta} = \langle H, \tau \rangle$ *where* τ *is an anti-conformal automorphism of order* 4 *given by*

$$\tau([x_1 : x_2 : x_3 : x_4 : x_5 : x_6]) = [\overline{x_2} : ir\,\overline{x_1} : \overline{x_4} : ir\,\overline{x_3} : \sqrt{r}e^{i\theta/2}\,\overline{x_6} : i\sqrt{r}e^{i\theta/2}\,\overline{x_5}].$$

(2) *The conjugacy action of* τ *on the elements of* H *is described in Table 1.*

(3) *Let* N *be a subgroup of* H *with the following conditions*
  (*i*) $a_j \notin N, \forall j = 1, \ldots, 6$ *where* $a_6 = a_1a_2a_3a_4a_5$
  (*ii*) $\tau N \tau^{-1} = N$
  (*iii*) $N \cap Q_N = \emptyset$
  (*iv*) $N \cap Q_H = \emptyset$
  where $Q_K = \{(a\tau)^2 : a \in K\}$ *with* $K \leq H$. *Then,* N *acts freely on* $C_{r,\theta}$ *and* $C_{r,\theta}/N$ *has automorphism group isomorphic to* H/N. *Furthermore,* $C_{r,\theta}/N$ *cannot be defined over its field of moduli. The collection of subgroups* $N < H$, *satisfying* (i), (ii), (iii), *and* (iv) *as above, are listed in Table 2. We shall call these subgroups admissible subgroups of* H. *The lattice of these admissible subgroups is shown in Figure 1.*

The family of curves $C_{r,\theta}$ in Theorem 1.1 was obtained in [**7, 8**] to obtain genus 17 non-hyperelliptic curves not definable over the field of moduli.

2. Preliminaries.

2.1. Some preliminaries on cross ratios. A generalized circle in the Riemann sphere $\widehat{\mathbb{C}}$ is either an Euclidian circle in $\mathbb{C}$ or the union of ∞ with an Euclidian line in $\mathbb{C}$. Given four different points $a, b, c, d \in \widehat{\mathbb{C}}$, the cross-ratio is defined $[a, b, c, d] = T(d)$, where T is the unique Möbius transformation satisfying that $T(a) = \infty$, $T(b) = 0$ and

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Table 1. Conjugacy action by τ

	a	$(a\tau)^2$	$\tau^{-1}a\tau$		a	$(a\tau)^2$	$\tau^{-1}a\tau$
1	1	$a_1a_3a_5$	1	2	a_1	$a_2a_3a_5$	a_2
3	a_2	$a_2a_3a_5$	a_1	4	a_3	$a_1a_4a_5$	a_4
5	a_4	$a_1a_4a_5$	a_3	6	a_5	$a_2a_4a_5$	a_6
7	a_6	$a_2a_2a_5$	a_5	**8**	a_1a_2	$a_1a_3a_5$	a_1a_2
9	a_1a_3	$a_2a_4a_5$	a_2a_4	10	a_1a_4	$a_2a_4a_5$	a_2a_3
11	a_1a_5	$a_1a_4a_5$	a_2a_6	12	a_2a_3	$a_2a_4a_5$	a_1a_4
13	a_2a_4	$a_2a_4a_5$	a_1a_3	14	a_2a_5	$a_1a_4a_5$	a_1a_6
15	a_3a_4	$a_1a_3a_5$	a_3a_4	16	a_3a_5	$a_2a_3a_5$	a_4a_6
17	a_4a_5	$a_2a_3a_5$	a_3a_6	**18**	a_5a_6	$a_1a_3a_5$	a_5a_6
19	a_4a_6	$a_2a_3a_5$	a_3a_5	20	a_3a_6	$a_2a_3a_5$	a_4a_5
21	a_2a_5	$a_1a_4a_5$	a_1a_5	22	a_1a_6	$a_1a_4a_5$	a_2a_5
23	$a_1a_2a_3$	$a_1a_4a_5$	$a_1a_2a_4$	24	$a_1a_2a_4$	$a_1a_4a_5$	$a_1a_2a_3$
25	$a_1a_2a_5$	$a_2a_4a_5$	$a_3a_4a_5$	26	$a_1a_3a_4$	$a_2a_3a_5$	$a_2a_3a_4$
27	$a_1a_3a_5$	$a_1a_3a_5$	$a_1a_3a_5$	**28**	$a_1a_4a_5$	$a_1a_3a_5$	$a_1a_4a_5$
29	$a_2a_3a_4$	$a_2a_3a_5$	$a_1a_3a_4$	**30**	$a_2a_3a_5$	$a_1a_3a_5$	$a_2a_3a_5$
31	$a_2a_4a_5$	$a_1a_3a_5$	$a_2a_4a_5$	32	$a_3a_4a_5$	$a_2a_4a_5$	$a_1a_2a_5$

Table 2. Admissible subgroups of H

order N	N	order N	N
16	$U = \langle a_1a_2, a_2a_3, a_3a_4, a_4a_5\rangle$	8	$T_8 = \langle a_1a_2, a_3a_5, a_4a_5\rangle$
8	$T_9 = \langle a_1a_5, a_2a_5, a_3a_4\rangle$	8	$T_{10} = \langle a_1a_4, a_2a_4, a_5a_6\rangle$
4	$S_7 = \{1, a_3a_4, a_1a_2a_3, a_1a_2a_4\}$	4	$S_8 = \{1, a_1a_2, a_1a_3a_4, a_2a_3a_4\}$
4	$S_9 = \{1, a_5a_6, a_1a_2a_5, a_3a_4a_5\}$	4	$S_{10} = \{1, a_1a_2, a_3a_4, a_5a_6\}$
4	$S_{11} = \{1, a_1a_2, a_3a_5, a_4a_6\}$	4	$S_{12} = \{1, a_1a_2, a_4a_5, a_3a_6\}$
4	$S_{13} = \{1, a_3a_4, a_1a_5, a_2a_6\}$	4	$S_{14} = \{1, a_3a_4, a_2a_5, a_1a_6\}$
4	$S_{15} = \{1, a_5a_6, a_1a_3, a_2a_4\}$	4	$S_{16} = \{1, a_5a_6, a_1a_4, a_2a_3\}$
2	$R_1 = \langle a_1a_2\rangle$	2	$R_2 = \langle a_3a_4\rangle$
2	$R_3 = \langle a_5a_6\rangle$		

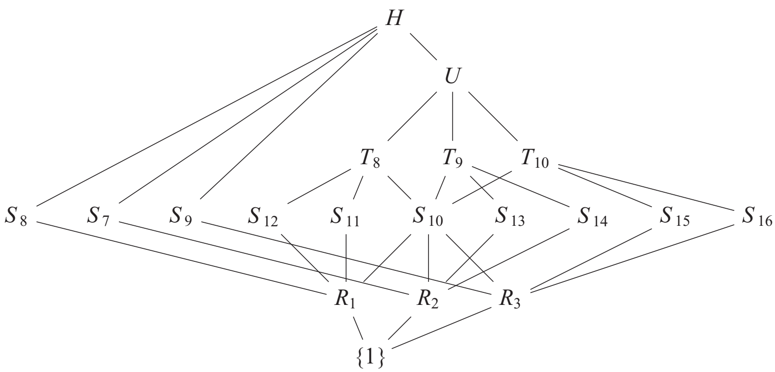

Figure 1. Lattice of admissible subgroups.

$T(c) = 1$. By the definition, $[a, b, c, d] \in \mathbb{C} - \{0, 1\}$. If S is any Möbius transformation, then $[S(a), S(b), S(c), S(d)] = [a, b, c, d]$. The points a, b, c, d belong to a common generalized circle if and only if $[a, b, c, d] \in \mathbb{R}$. In particular, Möbius transformations send generalized circles into generalized circles. Any permutation of the four points changes the value of $[a, b, c, d]$ to a value $R([a, b, c, d])$, were $R \in \mathbb{G} = \langle A(z) =$

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$1/z, B(z) = z/(z-1)\rangle \cong \mathfrak{S}_3$. In particular, if $[a, b, c, d] \in \{-1, 1/2, 2\}$ then the cross-ratio of any permutation of these four points is still in the same set. If $a \neq 0, \infty$, then $[\infty, 0, a, -a] = -1$. The only cross-ratios, obtained by permutation of ∞, 0, a and $-a$, producing the same value -1 are given by $[\infty, 0, a, -a]$, $[\infty, 0, -a, a]$, $[0, \infty, a, -a]$, $[0, \infty, -a, a]$, $[a, -a, \infty, 0]$, $[-a, a, \infty, 0]$, $[a, -a, 0, \infty]$ and $[-a, a, 0, \infty]$.

2.2. An auxiliary lemma. Let $\theta \in (0, \pi)$. If we consider the points $r_1(\theta) = \sqrt{1 + \cos(\theta)^2} - \cos(\theta)$ and $r_2(\theta) = \sqrt{1 + \cos(\theta)^2} + \cos(\theta)$, then $r_1(\theta)r_2(\theta) = 1$ and none of them is equal to ± 1. In particular, exactly one of these two points is bigger than 1; we denote it by r_θ.

LEMMA 2.1. *Let $\theta \in (0, \pi)$, $\theta \neq \pi/2$, and let $r \in (1, +\infty)$, $r \notin \left\{\sqrt{1 + \cos^2\theta} \pm \cos\theta\right\}$. If T is a Möbius transformation so that*

$$\{\infty, 0, 1, -r^2, re^{i\theta}, -re^{i\theta}\} \overset{T}{\mapsto} \{\infty, 0, 1, -r^2, re^{i\theta}, -re^{i\theta}\},$$

then $T = I$.

Proof. Set $\mu = re^{i\theta}$ and $\lambda = -r^2$. By direct inspection at the cross-ratios, with the restrictions $r > 1$ and $e^{i\theta} \neq \pm 1$, we may notice that the only subsets of cardinality 4 of $\{\infty, 0, 1, \lambda, \mu, -\mu\}$ contained in a generalized circle are given by

$$\{\infty, 0, 1, \lambda\}, \quad \{\infty, 0, \mu, -\mu\}, \quad \{1, \lambda, \mu, -\mu\}.$$

The respective cross-ratios are given by

$$[\infty, 0, 1, \lambda] = \lambda \notin \{-1, 1/2, 2\}$$

$$[\infty, 0, \mu, -\mu] = -1$$

$$[1, \lambda, \mu, -\mu] = -\frac{r^4 + 2(2\sin(\theta)^2 - 1)r^2 + 1)}{(r^2 + 2\cos(\theta)r + 1)^2} \notin \{-1, 1/2, 2\}.$$

Let θ and $R \in \mathbb{G}$ be fixed. The equation $[1, \lambda, \mu, -\mu] = R(\lambda)$ is equivalent to a polynomial equation $P_{\theta,R}(r) = 0$, where $P_{\theta,R}(x) \in \mathbb{R}[x]$ is a non-constant real polynomial of degree either 2 or 4. These polynomials $P_{\theta,R}(x)$ are given by the following ones:

$$x^2 + 2\cos(\theta)x - 1;\ x^2 - 2\cos(\theta)x - 1;\ 2x^4 + 3x^2 - 2\cos(\theta)x + 1;$$

$$2x^4 + 3x^2 + 2\cos(\theta)x + 1;\ x^4 + 2\cos(\theta)x^3 + 3x^2 + 2;\ x^4 - 2\cos(\theta)x^3 + 3x^2 + 2.$$

The degree four polynomials have no real zeroes greater than 1. The degree two polynomials have real zeroes greater than 1 only at r_θ. It follows that if $r \neq r_\theta$, then all the above three cross-ratios are non-equivalent under the action of $\mathbb{G}$. In particular, if

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T is a Möbius transformation so that

$$\{\infty, 0, 1, \lambda, \mu, -\mu\} \stackrel{T}{\mapsto} \{\infty, 0, 1, \lambda, \mu, -\mu\},$$

then

$$\{\infty, 0, 1, \lambda\} \stackrel{T}{\mapsto} \{\infty, 0, 1, \lambda\}, \quad \{\infty, 0, \mu, -\mu\} \stackrel{T}{\mapsto} \{\infty, 0, \mu, -\mu\}.$$

In this way,

$$\{\infty, 0\} \stackrel{T}{\mapsto} \{\infty, 0\}, \quad \{1, \lambda\} \stackrel{T}{\mapsto} \{1, \lambda\}, \quad \{-\mu, \mu\} \stackrel{T}{\mapsto} \{-\mu, \mu\}.$$

If $T \neq I$, then, from the above first two properties, we see that the only possibilities for T are given by $T(z) = \lambda z$ or $T(z) = 1/z$ or $T(z) = \lambda/z$. The possibility $T(z) = \lambda z$ asserts that $1 = T(\lambda) = \lambda^2 = r^4$, a contradiction. The possibility $T(z) = 1/z$ asserts $\lambda = T(\lambda) = 1/\lambda$, again a contradiction. The possibility $T(z) = \lambda/z$ then asserts that $\pm\mu = T(\mu) = \lambda/\mu$, from which one obtains that $r^2 = -\lambda = \mu^2 = r^2 e^{2i\theta}$, a contradiction to the assumption that $e^{i\theta} \notin \{\pm 1, \pm i\}$. □

2.3. Genus 17 **non-hyperelliptic curves.** Let $\theta \in (0, \pi)$, $\theta \neq \pi/2$, $r \in (1, +\infty)$, $r \notin \left\{\sqrt{1+\cos^2\theta} \pm \cos\theta\right\}$. In [**3, 6**] it was noticed that

$$C_{r,\theta} = \left\{ \begin{array}{rl} x_1^2 + x_2^2 + x_3^2 & = 0 \\ -r^2 x_1^2 + x_2^2 + x_4^2 & = 0 \\ re^{i\theta} x_1^2 + x_2^2 + x_5^2 & = 0 \\ -re^{i\theta} x_1^2 + x_2^2 + x_6^2 & = 0 \end{array} \right\} \subset \mathbb{P}^5$$

is an irreducible and non-singular projective algebraic curve of genus 17 so that

$$H = \langle a_1, a_2, a_3, a_4, a_5 \rangle \cong \mathbb{Z}_2^5$$

is a normal subgroup of $\mathrm{Aut}(C_{r,\theta})$, where a_j is multiplication by -1 to the x_j-coordinate. The holomorphic map

$$\pi : C_{r,\theta} \to \widehat{\mathbb{C}}; [x_1 : \cdots : x_6] \mapsto -\left(\frac{x_2}{x_1}\right)^2$$

defines a branched regular covering with H as deck group of covering maps. The branch values of π, each one of order two, are given by

$$\infty, 0, 1, -r^2, re^{i\theta}, -re^{i\theta}.$$

It can be seen from [**9**] that $C_{r,\theta}$ is a non-hyperelliptic Riemann surface.

Moreover, the curve $C_{r,\theta}$ admits the anti-conformal automorphism of order 4

$$\tau([x_1 : x_2 : x_3 : x_4 : x_5 : x_6]) = [\overline{x_2} : ir\,\overline{x_1} : \overline{x_4} : ir\,\overline{x_3} : \sqrt{r}e^{i\theta/2}\,\overline{x_6} : i\sqrt{r}e^{i\theta/2}\,\overline{x_5}].$$

So the field of moduli of $C_{r,\theta}$ is a subfield of $\mathbb{R}$.

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Let us notice that

$$\tau^2 = a_1 a_3 a_5$$

$$\tau a_1 = a_2 \tau, \ \tau a_2 = a_1 \tau,$$

$$\tau a_3 = a_4 \tau, \ \tau a_4 = a_3 \tau,$$

$$\tau a_5 = a_6 \tau, \ \tau a_6 = a_5 \tau.$$

In [7], as a direct consequence of Lemma 2.1, the following result is obtained.

THEOREM 2.2 *[7]. Let* $\theta \in (0, \pi)$*,* $\theta \neq \pi/2$*, and let* $r \in (1, +\infty)$*,* $r \notin \left\{\sqrt{1+\cos^2\theta} \pm \cos\theta\right\}$*. Then* $C_{r,\theta}$ *is a non-hyperelliptic Riemann surface of genus* 17 *which cannot be defined over* $\mathbb{R}$ *but whose field of moduli is a subfield of* $\mathbb{R}$*. In particular,* $C_{r,\theta}$ *is not definable over its field of moduli. Moreover,* $\mathrm{Aut}(C_{r,\theta}) = H$*.*

It should be said that the statement provided in [7] is slightly different than the one provided above and also in the same paper it is missing the restriction that $r \neq r_\theta$ (see the correction provided in [8]).

2.4. Connection to Earle's genus 2 example. Earle's example in [5] may be written as follows:

$$E_{r,\theta} : y^2 = x(x-1)(x+r^2)(x-re^{i\theta})(x+re^{i\theta})$$

and it can be seen as the quotient of $C_{r,\theta}$ by the subgroup of H, isomorphic to $\mathbb{Z}_2^4$ and acting freely, generated by $c_1 = a_1a_2$, $c_2 = a_2a_3$, $c_3 = a_3a_4$, $c_4 = a_4a_5$. In terms of Fuchsian groups, this covering may be seen as follows. Let $j : E_{r,\theta} \to E_{r,\theta}$ be the hyperelliptic involution. The quotient orbifold $\mathcal{O} = E_{r,\theta}/\langle j \rangle$ has signature $(0; 2, 2, 2, 2, 2, 2)$. Let Γ be a Fuchsian group acting on the hyperbolic plane $\mathbb{H}^2$ so that $\mathcal{O} = \mathbb{H}^2/\Gamma$. If Γ' denotes the derived subgroup of Γ, then it turns out that Γ' is torsion free and $C_{r,\theta} = \mathbb{H}^2/\Gamma'$. In this case, $H = \Gamma/\Gamma' \cong \mathbb{Z}_2^5$. There is an index 2 torsion-free normal subgroup F of Γ so that $E_{r,\theta} = \mathbb{H}^2/F$. Clearly, $\Gamma' \lhd F$. It is not difficult to see that Γ' is exactly the subgroup of F generated by the squares of the elements of F [1].

3. Proof of Theorem 1.1. Let $\theta \in (0, \pi)$, $\theta \neq \pi/2$, and let $r \in (1, +\infty)$, $r \notin \left\{\sqrt{1+\cos^2\theta} \pm \cos\theta\right\}$. We keep the notations of the previous section. Part (1) was already stated in [7,8] and Part (2) is a direct check. Next, we proceed to prove Part (3) of the Theorem.

Let us consider the subgroup $N \neq \{I\}$ of H with the conditions

(i) $a_j \notin N, \forall j = 1, \cdots, 6$ where $a_6 = a_1a_2a_3a_4a_5$
(ii) $\tau N \tau^{-1} = N$

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(iii) $N \cap Q_N = \emptyset$
(iv) $N \cap Q_H = \emptyset$.

By (i) there exist an unbranched regular covering $f : C_{r,\theta} \to C_{r,\theta}/N$ with N as deck group and a branched regular covering $P : C_{r,\theta}/N \to \widehat{\mathbb{C}}$ whose deck group is $H_N = H/N$.

Either H_N is a 2-Sylow subgroup of $\mathrm{Aut}(C_{r,\theta}/N)$ or there is a subgroup $K < \mathrm{Aut}(C_{r,\theta}/N)$ containing H_N as an index 2 subgroup. In the last situation, K will induce a Möbius transformation of order two keeping invariant the collection $\{\infty, 0, 1, -r^2, re^{i\theta}, -re^{i\theta}\}$ which is not possible by Lemma 2.1. So, H_N is a 2-Sylow's subgroup of $\mathrm{Aut}(C_{r,\theta}/N)$.

Next we claim $\mathrm{Aut}(C_{r,\theta}/N) = H_N$.

- $|N| = 8$: By Riemann–Hurwitz formula and condition (i) it follows that the genus of $C_{r,\theta}/N$ is 3. Furthermore $C_{r,\theta}/N$ is hyperelliptic. Looking at the table of automorphisms of hyperelliptic Riemann surfaces **[15]**, we may see that in the case that $\mathrm{Aut}(C_{r,\theta}/N)$ is different from $H_N \cong \mathbb{Z}_2^2$, there is some order two element of $\mathrm{Aut}(C_{r,\theta}/N) - H_N$ keeping H_N invariant. Such element will provide a non-trivial Möbius transformation keeping the set $\{\infty, 0, 1, -r^2, re^{i\theta}, -re^{i\theta}\}$ invariant, a contradiction. We have proved $\mathrm{Aut}(C_{r,\theta}/N) = H_N$.
- $|N| = 4$: By Riemann–Hurwitz formula and condition (i) it follows that the genus of $C_{r,\theta}/N$ is 5.
  Checking at the list of automorphism groups of compact Riemann surfaces of genus five **[14]**, one can see that H_N is contained in an abelian subgroup H'_N with index 2. Thus H'_N induces a Möbius transformation of order two keeping the set $\{\infty, 0, 1, -r^2, re^{i\theta}, -re^{i\theta}\}$ invariant, a contradiction. We have proved $\mathrm{Aut}(C_{r,\theta}/N) = H_N$.
- $|N| = 2$: By Riemann–Hurwitz formula and condition (i) it follows that the genus of $C_{r,\theta}/N$ is 9.
  In **[16]** there is a list of the automorphism groups of Riemann surfaces with genus 9. These automorphism groups have order greater than 2^5. We proved H_N is a 2-Sylow subgroup. If $\mathrm{Aut}(C_{r,\theta}/N)) \neq H_N$ it follows $[\mathrm{Aut}(C_{r,\theta}/N) : H_N] > 2$ hence $|\mathrm{Aut}(C_{r,\theta}/N)| > 2^5$. Next, by checking at the list of automorphism groups of compact Riemann surfaces of genus nine **[16]**, one can see that, they do not contain a 2-Sylow subgroup isomorphic to H_N. Therefore $\mathrm{Aut}(C_{r,\theta}/N) = H_N$.

By (ii) τ induces an anti-conformal automorphism τ_N on $C_{r,\theta}/N$. Further by (iii) τ_N has order 4. As a consequence, the field of moduli of $C_{r,\theta}/N$ is a subfield of $\mathbb{R}$.

Let us now assume that $C_{r,\theta}/N$ admits an anti-conformal involution Θ. Then $\tau_N^{-1}\Theta \in H_N$, that is, $\Theta \in \tau_N H_N$. This will ensure that some of $(\tau a n)^2$ (automorphism of $C_{r,\theta}$) must belong to N for $a \in H, n \in N$. By condition (iv) we obtain a contradiction.

4. Equations for curves.

4.1. Subgroup of order 8. First, we compute the equations for $N = T_8 = \langle a_1a_2, a_3a_5a_4a_5\rangle$.

A generating set for the N−invariant algebra $\mathbb{C}[x_1, x_2, x_3, x_4, x_5]^N$ is given by

$$y_1 = x_1^2\,, y_2 = x_2^2\,, y_3 = x_3^2\,, y_4 = x_4^2\,, y_5 = x_5^2\,, y_6 = x_1x_2\,, y_7 = x_3x_4x_5.$$

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So, if $\Phi = (y_1, y_2, y_3, y_4, y_5, y_6, y_7)$, then

$$\Phi : C^0_{r,\theta} \to \Phi\left(C^0_{r,\theta}\right) \subset \mathbb{C}^7$$

is a regular unbranched covering with N as its deck group. In particular, $\Phi\left(C^0_{r,\theta}\right)$ is an affine model for $C_{r,\theta}/N$. This curve is given by the following equations:

$$\Phi\left(C^0_{r,\theta}\right) = \left\{ \begin{array}{ll} y_1 + y_2 + y_3 & = 0 \\ -r^2 y_1 + y_2 + y_4 & = 0 \\ re^{i\theta} y_1 + y_2 + y_5 & = 0 \\ -re^{i\theta} y_1 + y_2 + 1 & = 0 \\ y_6^2 - y_1 y_2 & = 0 \\ y_7^2 - y_3 y_4 y_5 & = 0 \end{array} \right\} \subset \mathbb{C}^7.$$

The above equations imply that

$$\begin{aligned} y_2 &= -1 + re^{i\theta} y_1 \\ y_3 &= 1 - (1 + re^{i\theta}) y_1 \\ y_4 &= 1 + (r^2 - re^{i\theta}) y_1 \\ y_5 &= 1 - 2re^{i\theta} y_1. \end{aligned}$$

So, if we consider the projection

$$\Psi : \mathbb{C}^7 \to \mathbb{C}^3 : (y_1, y_2, y_3, y_4, y_5, y_6, y_7) \mapsto (y_1, y_6, y_7) = (w_1, w_2, w_3),$$

then

$$\Psi : \Phi\left(C^0_{r,\theta}\right) \to \Psi\left(\Phi\left(C^0_{r,\theta}\right)\right) = C_{r,\theta}/N$$

is an isomorphism. In particular,

$$f = \Phi \circ \Psi : C^0_{r,\theta} \to C^0_{r,\theta}/N$$

is an unbranched regular covering with N as deck group. The curve $C^0_{r,\theta}/N$ is given by the following equations:

$$\left\{ \begin{array}{l} w_2^2 = w_1\left(re^{i\theta} w_1 - 1\right) \\ w_3^2 = \left(1 - w_1\left(1 + re^{i\theta}\right)\right)\left(1 - w_1\left(re^{i\theta} - r^2\right)\right)\left(1 - 2re^{i\theta} w_1\right) \end{array} \right\} \subset \mathbb{C}^3.$$

In the following table, we resume these computations for each group in our list.

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Subgroup N	$(y_1, y_2, y_3, y_4, y_5, y_6, y_7)$	(w_1, w_2, w_3)
T_8	$(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1x_2, x_3x_4x_5)$	(y_1, y_6, y_7)
$y_1 + y_2 + y_3 = 0$ $-r^2 y_1 + y_2 + y_4 = 0$ $re^{i\theta} y_1 + y_2 + y_5 = 0$ $-re^{i\theta} y_1 + y_2 + 1 = 0$ $y_6^2 - y_1 y_2 = 0$ $y_7^2 - y_3 y_4 y_5 = 0$	$y_3 = 1 - y_1(1 + re^{i\theta})$ $y_4 = 1 - y_1(re^{i\theta} - r^2)$ $y_5 = 1 - 2re^{i\theta} y_1$ $y_2 = re^{i\theta} y_1 - 1$	$w_2^2 = w_1(re^{i\theta} w_1 - 1)$ $w_3^2 = (1 - w_1(1 + re^{i\theta}))(1 - w_1(re^{i\theta} - r^2))(1 - 2re^{i\theta} w_1)$
T_9	$(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_3x_4, x_1x_2x_5)$	(y_1, y_6, y_7)
$y_1 + y_2 + y_3 = 0$ $-r^2 y_1 + y_2 + y_4 = 0$ $re^{i\theta} y_1 + y_2 + y_5 = 0$ $-re^{i\theta} y_1 + y_2 + 1 = 0$ $y_6^2 - y_3 y_4 = 0$ $y_7^2 - y_1 y_2 y_5 = 0$	$y_3 = 1 - y_1(1 + re^{i\theta})$ $y_4 = 1 - y_1(re^{i\theta} - r^2)$ $y_5 = 1 - 2re^{i\theta} y_1$ $y_2 = re^{i\theta} y_1 - 1$	$w_2^2 = (1 - w_1(1 + re^{i\theta}))(1 - w_1(re^{i\theta} - r^2))$ $w_3^2 = w_1(re^{i\theta} w_1 - 1)(1 - 2re^{i\theta} w_1)$
T_{10}	$(x_5, x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2x_3x_4)$	(y_2, y_1, y_6)
$y_2 + y_3 + y_4 = 0$ $-r^2 y_2 + y_3 + y_5 = 0$ $re^{i\theta} y_2 + y_3 + y_1^2 = 0$ $-re^{i\theta} y_2 + y_3 + 1 = 0$ $y_6^2 - y_2 y_3 y_4 y_5 = 0$	$y_4 = 1 - y_2(1 + re^{i\theta})$ $y_5 = 1 - y_2(re^{i\theta} - r^2)$ $y_1^2 = 1 - 2re^{i\theta} y_2$ $y_3 = re^{i\theta} y_2 - 1$	$w_2^2 = (1 - 2re^{i\theta} w_1)$ $w_3^2 = w_1(re^{i\theta} w_1 - 1)(1 - w_1(1 + re^{i\theta}))(1 - w_1(re^{i\theta} - r^2))$

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4.2 Subgroup of order 4

Subgroup N	$(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8)$	$(w_1, w_2, w_3, w_4, w_5)$
S_7	$(x_5, x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3x_4, x_2x_3x_4)$	$(y_2, y_1, y_6, y_7, y_8)$
$y_2 + y_3 + y_4 = 0$ $-r^2 y_2 + y_3 + y_5 = 0$ $re^{i\theta} y_2 + y_3 + y_1^2 = 0$ $-re^{i\theta} y_2 + y_3 + 1 = 0$ $y_6^2 - y_2y_3 = 0$ $y_7^2 - y_2y_4y_5 = 0$ $y_8^2 - y_3y_4y_5 = 0$ $y_7y_8 - y_6y_4y_5 = 0$	$y_4 = 1 - y_2(1 + re^{i\theta})$ $y_5 = 1 - y_2(re^{i\theta} - r^2)$ $y_1^2 = 1 - 2re^{i\theta} y_2$ $y_3 = re^{i\theta} y_2 - 1$	$w_2^2 = 1 - 2re^{i\theta} w_1$ $w_3^2 = w_1(re^{i\theta} w_1 - 1)$ $w_4^2 = w_1(1 - w_1(1 + re^{i\theta}))(1 - w_1(re^{i\theta} - r^2))$ $w_5^2 = (re^{i\theta} w_1 - 1)(1 - w_1(1 + re^{i\theta}))(1 - w_1(re^{i\theta} - r^2))$ $w_4w_5 = w_3(1 - w_1(1 + re^{i\theta}))(1 - w_1(re^{i\theta} - r^2))$
S_8	$(x_5, x_1^2, x_2^2, x_3^2, x_4^2, x_3x_4, x_1x_2x_3, x_1x_2x_4)$	$(y_2, y_1, y_6, y_7, y_8)$
$y_2 + y_3 + y_4 = 0$ $-r^2 y_2 + y_3 + y_5 = 0$ $re^{i\theta} y_2 + y_3 + y_1^2 = 0$ $-re^{i\theta} y_2 + y_3 + 1 = 0$ $y_6^2 - y_4y_5 = 0$ $y_7^2 - y_2y_3y_4 = 0$ $y_8^2 - y_2y_3y_5 = 0$ $y_7y_8 - y_2y_3y_6 = 0$	$y_4 = 1 - y_2(1 + re^{i\theta})$ $y_5 = 1 - y_2(re^{i\theta} - r^2)$ $y_1^2 = 1 - 2re^{i\theta} y_2$ $y_3 = re^{i\theta} y_2 - 1$	$w_2^2 = 1 - 2re^{i\theta} w_1$ $w_3^2 = (1 + re^{i\theta}))(1 - w_1(re^{i\theta} - r^2))$ $w_4^2 = w_1(re^{i\theta} w_1 - 1)(1 - w_1(1 + re^{i\theta}))$ $w_5^2 = w_1(re^{i\theta} w_1 - 1)(1 - w_1(re^{i\theta} - r^2))$ $w_4w_5 = w_1(re^{i\theta} w_1 - 1)w_3$

S_9	$(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1x_2, x_3x_4, x_1x_3x_5, x_1x_4x_5, x_2x_3x_5, x_2x_4x_5)$	$(y_1, y_6, y_7, y_8, y_9, y_{10}, y_{11})$
$y_1 + y_2 + y_3 = 0$	$y_3 = 1 - y_1(1 + re^{i\theta})$	
$-r^2 y_1 + y_2 + y_4 = 0$	$y_4 = 1 - y_1(re^{i\theta} - r^2)$	
$re^{i\theta} y_1 + y_2 + y_5 = 0$	$y_5 = 1 - 2re^{i\theta} y_1$	
$-re^{i\theta} y_1 + y_2 + 1 = 0$	$y_2 = re^{i\theta} y_1 - 1$	
$y_6^2 - y_1 y_2 = 0$		$w_2^2 = w_1(re^{i\theta} w_1 - 1)$
$y_7^2 - y_3 y_4 = 0$		$w_3^2 = (1 - w_1(1 + re^{i\theta}))(1 - w_1(re^{i\theta} - r^2))$
$y_8^2 - y_1 y_3 y_5 = 0$		$w_4^2 = w_1(1 - w_1(1 + re^{i\theta}))(1 - 2re^{i\theta} w_1)$
$y_9^2 - y_1 y_4 y_5 = 0$		$w_5^2 = w_1(1 - w_1(re^{i\theta} - r^2))(1 - 2re^{i\theta} w_1)$
$y_{10}^2 - y_2 y_3 y_5 = 0$		$w_6^2 = (re^{i\theta} w_1 - 1)(1 - w_1(1 + re^{i\theta}))(1 - 2re^{i\theta} w_1)$
$y_{11}^2 - y_2 y_4 y_5 = 0$		$w_7^2 = (re^{i\theta} w_1 - 1)(1 - w_1(re^{i\theta} - r^2))(1 - 2re^{i\theta} w_1)$
S_{10}	$(x_5, x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_3x_4)$	(y_2, y_1, y_6, y_7)
$y_2 + y_3 + y_4 = 0$	$y_4 = 1 - y_2(1 + re^{i\theta})$	
$-r^2 y_2 + y_3 + y_5 = 0$	$y_5 = 1 - y_2(re^{i\theta} - r^2)$	
$re^{i\theta} y_2 + y_3 + y_1^2 = 0$	$y_1^2 = 1 - 2re^{i\theta} y_2$	$w_2^2 = 1 - 2re^{i\theta} w_1$
$-re^{i\theta} y_2 + y_3 + 1 = 0$	$y_3 = re^{i\theta} y_2 - 1$	
$y_6^2 - y_2 y_3 = 0$		$w_3^2 = w_1(re^{i\theta} w_1 - 1)$
$y_7^2 - y_4 y_5 = 0$		$w_4^2 = (1 - w_1(1 + re^{i\theta}))(1 - w_1(re^{i\theta} - r^2))$
S_{11}	$(x_4, x_1^2, x_2^2, x_3^2, x_5^2, x_1x_2, x_3x_5)$	(y_2, y_1, y_6, y_7)
$y_2 + y_3 + y_4 = 0$	$y_4 = 1 - y_2(1 + re^{i\theta})$	
$-r^2 y_2 + y_3 + y_1^2 = 0$	$y_1^2 = 1 - y_2(re^{i\theta} - r^2)$	$w_2^2 = 1 - w_1(re^{i\theta} - r^2)$
$re^{i\theta} y_2 + y_3 + y_5 = 0$	$y_5 = 1 - 2re^{i\theta} y_2$	
$-re^{i\theta} y_2 + y_3 + 1 = 0$	$y_3 = re^{i\theta} y_2 - 1$	
$y_6^2 - y_2 y_3 = 0$		$w_3^2 = w_1(re^{i\theta} w_1 - 1)$
$y_7^2 - y_4 y_5 = 0$		$w_4^2 = (1 - w_1(1 + re^{i\theta}))(1 - 2re^{i\theta} y_2)$

S_{12}	$(x_3, x_1^2, x_2^2, x_4^2, x_5^2, x_1x_2, x_4x_5)$	(y_2, y_1, y_6, y_7)
$y_2 + y_3 + y_1^2 = 0$ $-r^2y_2 + y_3 + y_4 = 0$ $re^{i\theta}y_2 + y_3 + y_5 = 0$ $-re^{i\theta}y_2 + y_3 + 1 = 0$ $y_6^2 - y_2y_3 = 0$ $y_7^2 - y_4y_5 = 0$	$y_1^2 = 1 - y_2(1 + re^{i\theta})$ $y_4 = 1 - y_2(re^{i\theta} - r^2)$ $y_5 = 1 - 2re^{i\theta}y_2$ $y_3 = re^{i\theta}y_2 - 1$	$w_2^2 = 1 - w_1(1 + re^{i\theta})$ $w_3^2 = w_1(re^{i\theta}w_1 - 1)$ $w_4^2 = (1 - w_1(re^{i\theta} - r^2))(1 - 2re^{i\theta}y_2)$
S_{13}	$(x_2, x_1^2, x_3^2, x_4^2, x_5^2, x_1x_5, x_3x_4)$	(y_2, y_1, y_6, y_7)
$y_2 + y_1^2 + y_3 = 0$ $-r^2y_2 + y_1^2 + y_4 = 0$ $re^{i\theta}y_2 + y_1^2 + y_5 = 0$ $-re^{i\theta}y_2 + y_1^2 + 1 = 0$ $y_6^2 - y_2y_5 = 0$ $y_7^2 - y_3y_4 = 0$	$y_3 = 1 - y_2(1 + re^{i\theta})$ $y_4 = 1 - y_2(re^{i\theta} - r^2)$ $y_5 = 1 - 2re^{i\theta}y_2$ $y_1^2 = re^{i\theta}y_2 - 1$	$w_2^2 = (re^{i\theta}w_1 - 1)$ $w_3^2 = w_1(1 - 2re^{i\theta}y_2)$ $w_4^2 = (1 - w_1(1 + re^{i\theta}))(1 - w_1(re^{i\theta} - r^2))$
S_{14}	$(x_1, x_2^2, x_3^2, x_4^2, x_5^2, x_2x_5, x_3x_4)$	(y_2, y_1, y_6, y_7)
$y_1^2 + y_2 + y_3 = 0$ $-r^2y_1^2 + y_2 + y_4 = 0$ $re^{i\theta}y_1^2 + y_2 + y_5 = 0$ $-re^{i\theta}y_1^2 + y_2 + 1 = 0$ $y_6^2 - y_2y_5 = 0$ $y_7^2 - y_3y_4 = 0$	$y_3 = -r^{-1}e^{-i\theta} - y_2(1 + r^{-1}e^{-i\theta})$ $y_4 = re^{-i\theta} + y_2(re^{-i\theta} - 1)$ $y_5 = 1 - 2y_2$ $y_1^2 = r^{-1}e^{-i\theta}(1 + y_2)$	$w_2^2 = r^{-1}e^{-i\theta}(1 + w_1)$ $w_3^2 = w_1(1 - 2w_1)$ $w_4^2 = (-r^{-1}e^{-i\theta} - w_1(1 + r^{-1}e^{-i\theta}))(re^{-i\theta} + w_1(re^{-i\theta} - 1))$

S_{15}	$(x_5, x_1^2, x_2^2, x_3^2, x_4^2, x_1x_3, x_2x_4)$	(y_2, y_1, y_6, y_7)
$y_2 + y_3 + y_4 = 0$	$y_4 = 1 - y_2(1 + re^{i\theta})$	
$-r^2 y_2 + y_3 + y_5 = 0$	$y_5 = 1 - y_2(re^{i\theta} - r^2)$	
$re^{i\theta} y_2 + y_3 + y_1^2 = 0$	$y_1^2 = 1 - 2re^{i\theta} y_2$	$w_2^2 = 1 - 2re^{i\theta} w_1$
$-re^{i\theta} y_2 + y_3 + 1 = 0$	$y_3 = re^{i\theta} y_2 - 1$	
$y_6^2 - y_2 y_4 = 0$		$w_3^2 = w_1(1 - w_1(1 + re^{i\theta}))$
$y_7^2 - y_3 y_5 = 0$		$w_4^2 = (re^{i\theta} w_1 - 1)(1 - w_1(re^{i\theta} - r^2))$
S_{16}	$(x_5, x_1^2, x_2^2, x_3^2, x_4^2, x_1x_4, x_2x_3)$	(y_2, y_1, y_6, y_7)
$y_2 + y_3 + y_4 = 0$	$y_4 = 1 - y_2(1 + re^{i\theta})$	
$-r^2 y_2 + y_3 + y_5 = 0$	$y_5 = 1 - y_2(re^{i\theta} - r^2)$	
$re^{i\theta} y_2 + y_3 + y_1^2 = 0$	$y_1^2 = 1 - 2re^{i\theta} y_2$	$w_2^2 = 1 - 2re^{i\theta} w_1$
$-re^{i\theta} y_2 + y_3 + 1 = 0$	$y_3 = re^{i\theta} y_2 - 1$	
$y_6^2 - y_2 y_5 = 0$		$w_3^2 = w_1(1 - w_1(re^{i\theta} - r^2))$
$y_7^2 - y_3 y_4 = 0$		$w_4^2 = (re^{i\theta} w_1 - 1)(1 - w_1(1 + re^{i\theta}))$

4.3 Subgroup of order 2

Subgroup N	$(y_1, y_2, y_3, y_4, y_5, y_6)$	$(w_1, w_2, w_3, w_4, w_5)$
R_1	$(x_3, x_4, x_5, x_1^2, x_2^2, x_1x_2)$	$(y_4, y_1, y_2, y_3, y_6)$
$y_4 + y_5 + y_1^2 = 0$ $-r^2y_4 + y_5 + y_2^2 = 0$ $re^{i\theta}y_4 + y_5 + y_3^2 = 0$ $-re^{i\theta}y_4 + y_5 + 1 = 0$ $y_6^2 - y_4y_5 = 0$	$y_1^2 = 1 - y_4(1 + re^{i\theta})$ $y_2^2 = 1 - y_4(re^{i\theta} - r^2)$ $y_3^2 = 1 - 2re^{i\theta}y_2$ $y_5 = re^{i\theta}y_4 - 1$	$w_2^2 = 1 - w_1(1 + re^{i\theta})$ $w_3^2 = 1 - w_1(re^{i\theta} - r^2)$ $w_4^2 = 1 - 2re^{i\theta}w_1$ $w_5^2 = w_1(re^{i\theta}w_1 - 1)$
R_2	$(x_1, x_2, x_5, x_3^2, x_4^2, x_3x_4)$	$(y_4, y_1, y_3, y_2, y_6)$
$y_1^2 + y_2^2 + y_4 = 0$ $-r^2y_1^2 + y_2^2 + y_5 = 0$ $re^{i\theta}y_1^2 + y_2^2 + y_3^2 = 0$ $-re^{i\theta}y_1^2 + y_2^2 + 1 = 0$ $y_6^2 - y_4y_5 = 0$	$y_4 = 1 - y_1^2(1 + re^{i\theta})$ $y_5 = 1 - y_1^2(re^{i\theta} - r^2)$ $y_3^2 = 1 - 2re^{i\theta}y_1^2$ $y_2^2 = re^{i\theta}y_1^2 - 1$	$w_2^2 = (1 + re^{i\theta})^{-1}(1 + w_1)$ $w_3^2 = 1 - 2re^{i\theta}(1 + re^{i\theta})^{-1}(w_1 - 1)$ $w_4^2 = re^{i\theta}(1 + re^{i\theta})^{-1}(w_1 - 1) - 1$ $w_5^2 = w_1(1 - (re^{i\theta} - r^2)(1 + re^{i\theta})^{-1}(1 + w_1))$
R_3	$(x_5, x_1^2, x_2^2, x_3^2, x_4^2, x_1x_2, x_1x_3,$ $x_1x_4, x_2x_3, x_2x_4, x_3x_4)$	$(y_2, y_1, y_6, y_7, y_8, y_9, y_{10}, y_{11})$
$y_2 + y_3 + y_4 = 0$ $-r^2y_2 + y_3 + y_5 = 0$ $re^{i\theta}y_2 + y_3 + y_1^2 = 0$ $-re^{i\theta}y_2 + y_3 + 1 = 0$ $y_6^2 - y_2y_3 = 0$ $y_7^2 - y_2y_4 = 0$ $y_8^2 - y_2y_5 = 0$ $y_9^2 - y_3y_4 = 0$ $y_{10}^2 - y_3y_5 = 0$ $y_{11}^2 - y_4y_5 = 0$	$y_4 = 1 - y_2(1 + re^{i\theta})$ $y_5 = 1 - y_2(re^{i\theta} - r^2)$ $y_1^2 = 1 - 2re^{i\theta}y_2$ $y_3 = re^{i\theta}y_2 - 1$	 $w_2^2 = 1 - 2re^{i\theta}w_1$ $w_3^2 = w_1(re^{i\theta}y_2 - 1)$ $w_4^2 = w_1(1 - w_1(1 + re^{i\theta}))$ $w_5^2 = w_1(1 - w_1(re^{i\theta} - r^2))$ $w_6^2 = (re^{i\theta}w_1 - 1)(1 - w_1(1 + re^{i\theta}))$ $w_7^2 = (re^{i\theta}w_1 - 1)(1 - w_1(re^{i\theta} - r^2))$ $w_8^2 = (1 - w_1(1 + re^{i\theta}))(1 - w_1(re^{i\theta} - r^2))$

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REFERENCES

1. P. Ackermann, Arithmetische Fuchssche Gruppen der Signatur (2; −), Dissertation (2005), Universität Dortmund.

2. E. Bujalance and P. Turbek, Asymmetric and pseudo-symmetric hyperelliptic surfaces, *Manuscr. Math.* **108** (2002), 1–11.

3. A. Carocca, V. Gonzalez, R. A. Hidalgo and R. Rodríguez, Generalized Humbert curves, *Israel J. Math.* **64**(1) (2008), 165–192.

4. C. J. Earle, On the moduli of closed Riemann surfaces with symmetries, in *Advances in the theory of Riemann surfaces* (L. V. Ahlfors et al., Editors) (Princeton Univ. Press, Princeton, 1971), 119–130.

5. C. J. Earle, Diffeomorphisms and automorphisms of compact hyperbolic 2-orbifolds, in *Geometry of Riemann surfaces* (Gardiner F., González-Diez G. and Kourouniotis C. Editors), London Math. Soc. Lecture Note Ser., vol. 368 (Cambridge Univ. Press, Cambridge, 2010), 139–155.

6. G. Gonzalez-Diez, R. A. Hidalgo and M. Leyton, Generalized fermat curves, *J. Algebra* **321** (2009), 1643–1660.

7. R. A. Hidalgo, Non-hyperelliptic Riemann surfaces with real field of moduli but not definable over the reals, *Arch. Math.* **93** (2009), 219–222.

8. R. A. Hidalgo, Erratum: Non-hyperelliptic Riemann surfaces with real field of moduli but not definable over the reals, *Arch. Math.* **98** (2012), 449–45.

9. R. A. Hidalgo, Homology closed Riemann surfaces, *Q. J. Math.* **63** (2012), 931–952

10. B. Huggins, Fields of moduli and fields of definition of curves, PhD Thesis (University of California, 2005).

11. B. Huggins, Fields of moduli of hyperelliptic curves, *Math. Res. Lett.* **14**(2) (2007), 249–262.

12. S. Koizumi, Fields of moduli for polarized Abelian varieties and for curves, *Nagoya Math. J.* **48** (1972), 37–55.

13. A. Kontogeorgis, Field of Moduli versus field of definition for cyclic covers of the projective line, *J. Theorie des Nombres de Bordeaux* **21** (2009), 679–693.

14. A. Kuribayashi and H. Kimura, Automorphism groups of compact Riemann surfaces of genus five, *J. Algebra* **134** (1990), 80–103.

15. I. Kuribayashi and A. Kuribayashi, Automorphism groups of compact Riemann surfaces of genera three and four, *J. Pure Appl. Algebra* **65**(3) (1990), 277–292.

16. K. Magaard, T. Shaska, S. Shpectorov and H. Völklein, The locus of curves with prescribed automorphism group, *Communications in arithmetic fundamental groups (Kyoto, 1999/2001). Surikaisekikenkyusho Kokyuroku* **No. 1267** (2002), 112–141.

17. T. Matsusaka, Polarized varieties, the field of moduli and generalized Kummer varieties of polarized abelian varieties, *Amer. J. Math.* **80** (1958), 45–82.

18. G. Shimura, On the field of rationality for an abelian variety, *Nagoya Math. J.* **45** (1971), 167–178.

19. R. Silhol, Moduli problems in real algebraic geometry, in *Real Algebraic Geometry* (M. Coste et al., Editors) (Springer-Verlag, Berlin, 1972), 110–119.

20. A. Weil, The field of definition of a variety, *Amer. J. Math.* **78** (1956), 509–524.