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REDFIELD'S THEOREMS AND MULTILINEAR ALGEBRA

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1. Introduction. The remarkable 1927 paper by J. H. Redfield [**13**] which anticipated many recent combinatorial results in Polya counting theory and, in fact, predated Polya's theorem by ten years has been discussed at length by Harary and Palmer [**8**], Foulkes [**5**; **6**], Sheehan [**15**; **16**] and Read [**12**], not to mention de Bruijn [**3**] and others. We shall, in this paper, demonstrate how multilinear techniques may be used in this context. The Redfield superposition theorem and decomposition theorem turn out to be statements about a group acting on finite function spaces, and may thus be dealt with in multilinear terms. We shall prove Redfield's results and an extension due to Foulkes [**5**].

2. Background. We shall first sketch results which have appeared elsewhere [**19**]. Let S be a finite set, G a finite group acting on S, L a G-stable subset of S, Δ a system of distinct representatives, or transversal, on the orbits of the action of G on L. We let $G : L$ mean the action of G on the set L. Let F be a field of characteristic zero. Then F^S is an algebra under pointwise addition, multiplication, and scalar multiplication. Let $\{e_s\}_{s\in S}$ be a basis for F^S, $e_s(t) = \chi(s = t)$ where

$$\chi(\text{statement}) = \begin{cases} 1 \text{ if statement is true,} \\ 0 \text{ if statement is false.} \end{cases}$$

We define operators T_G and Q_G as follows and extend linearly:

$$T_G e_s = \frac{1}{|G|} \sum_{\sigma \in G} e_{\sigma s},$$

$$Q_G e_s = \frac{|G_s|}{|G|} e_s,$$

where G_s is the stabilizer subgroup of the point $s \in S$. Then we have

THEOREM 2.1.

$$T_G Q_G = Q_G T_G \quad \textit{and} \quad T_G{}^2 = T_G.$$

Proof. See [**19**].

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If $\hat{G}$ and H are subgroups of G, we define the operator $P_{\hat{G}}{}^H$ on the basis of F^S and extend linearly:

$$P_{\hat{G}}{}^H e_s = \frac{1}{|G_s|} \sum_{\sigma \in \hat{G}} \chi(\sigma H \sigma^{-1} \subset G_s) e_s.$$

Note that when $\hat{G} = G$, $P_{\hat{G}}{}^H e_s = M_{G_s}(H) e_s$ where $M_K(H)$ is the *mark* of K at H (see [**2**; **17**]). We then have

$$P_{\hat{G}}{}^H Q_G = Q_G P_{\hat{G}}{}^H.$$

In [**19**] we showed:

THEOREM 2.2. *$T_G I_\Delta = Q_G I_L$ where $I_A = \sum_{s \in A} e_s$ for $A \subset S$.*

Proof. See [**19**].

This theorem is a vector statement of Burnside's Lemma. The more classical versions may be obtained by applying appropriate linear functionals.

We now specialize $S = R^D$ where $R = \{1, \ldots, r\} = [1, r]$ and $D = \{1, \ldots, d\} = [1, d]$. Then we may summarize the additional structure on F^S as follows:

THEOREM 2.3. *If $S = R^D$, then F^S is the algebra of tensors of rank d and dimension r.*

We define the correspondence $\nu : M_{d,r}(F) \to F^S$ where $M_{d,r}(F) = \{d \times r$ matrices with entries in $F\}$, by

$$\nu_A(f) = \prod_{i=1}^{d} a_{if(i)},$$

where $A = (a_{ij}) \in M_{d,r}(F)$. Although ν is not one-to-one, we note that $\nu_A = \nu_B$ if and only if $A_i = \alpha_i B_i$ and $\prod_{i=1}^d \alpha_i = 1$ where A_i and B_i are the i^{th} rows of A and B respectively and $\alpha_i \in F$.

Furthermore, although ν is not onto, $\nu_{E_f} = e_f$ where $E_f \in M_{d,r}(F)$ is such that the ij^{th} coordinate of E_f is 1 if $f(i) = j$ and 0 otherwise. Thus, Im *ν spans F^S.*

Finally, we note that if $A, B, C \in M_{d,r}(F)$, $A = (a_{ij})$, $B = (b_{ij})$, $C = (c_{ij})$, then

$$\nu_C = \nu_A + \nu_B \text{ if } A_i = B_i = C_i \text{ for all } i \neq k \text{ and } A_k + B_k = C_k,$$
$$\nu_C = \alpha \nu_A \text{ if } A_i = C_i \text{ for all } i \neq k \text{ and } C_k = \alpha A_k,$$
$$\nu_C = \nu_A \cdot \nu_B \text{ if } c_{ij} = a_{ij} \cdot b_{ij} \text{ for all } i, j.$$

Proof. See [**19**].

We often let $A \in M_{d,r}(F)$ represent ν_A. The matrices $M_{d,r}(F)$ are sometimes called *pure* or *homogeneous* tensors.

For certain group actions on R^D (e.g., G acts on D and therefore on R^D) if A is a pure tensor and l is a linear functional on $F^{(R^D)}$ such that $lE_{\sigma f} = lE_f$

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for all $f \in R^D$, for all $\sigma \in G$, then lQ_GA is an easily computed quantity [**20**]. We may then restate a version of the problem of rejecting isomorphs in a G-stable subset L of a finite function space as follows: Construct pure tensors $A_1, \ldots, A_v$ such that $I_L = T_G(A_1 + \ldots + A_v)$. Then by Theorems 2.1 and 2.2, $T_GI_\Delta = T_GQ_G(A_1 + \ldots + A_v)$. Since $lT_GE_f = lE_f$, $lT_G = l$. Thus, $lI_\Delta = lQ_GA_1 + \ldots + lQ_GA_v$. For instance, if $L = R^D$, we may let $v = 1$, $A_1 = J =$ the $d \times r$ matrix of all 1's, and thus $lI_\Delta = lQ_GJ$. For some subsets L, the principle of inclusion-exclusion may be used to construct $A_1, \ldots, A_v$ [**18**]. We shall not deal with this construction problem in this paper.

We may then extend this multilinear setting as follows. If $S = R_1^{D_1} \times \ldots \times R_k^{D_k}$, then F^S is the tensor algebra of rank k of vectors from the tensor algebras $F^{(R_i^{D_i})}$ (see [**19**]). We may write a pure tensor of pure tensors as $A_1 \otimes \ldots \otimes A_k$ where $A_i \in M_{d_i,r_i}(F)$ and $|R_i| = r_i$, $|D_i| = d_i$.

We now let S_n be the symmetric group of order $n!$ acting on $[1, n]$. Let ρ be an integer partition of n. We write $\rho = 1^{j_1}2^{j_2}\ldots n^{j_n}$ where j_i denotes the number of times i appears in ρ. The following two results are well-known:

THEOREM 2.4. *There is a one-to-one correspondence between partitions of n and conjugate classes of S_n. This correspondence is as follows:*

$$\rho = 1^{j_1}\ldots n^{j_n} \leftrightarrow \textit{all elements of } S_n \textit{ with } j_i \textit{ cycles of length } i \textit{ for each } i.$$

Proof. See, for instance, [**7**].

We may discuss, then, $C_\rho =$ conjugate class of S_n corresponding to the partition ρ.

THEOREM 2.5.

$$|C_\rho| = \frac{n!}{1^{j_1}j_1!2^{j_2}j_2!\ldots n^{j_n}j_n!} = \frac{n!}{\pi_\rho} \quad \textit{where } \pi_\rho = 1^{j_1}j_1!\ldots n^{j_n}j_n!$$

Proof. See [**13**].

3. Redfield's theorems. Let $\Pi_n = \{\text{partitions of } n\}$ and let $s_1, \ldots, s_n$ be n indeterminants. We now write the cycle index polynomials (see [**4**]) $P_{G:[1,n]}(s_1, s_2, \ldots)$ as follows:

$$P_{G:[1,n]}(s_1, s_2, \ldots) = \frac{1}{|G|}\sum_{\rho \in \Pi_n} |C_\rho \cap G| s_\rho$$

where $s_\rho = s_1^{j_1}s_2^{j_2}\ldots s_n^{j_n}$. We observe that $V = \langle s_\rho \rangle_{\rho\in\Pi_n}$ forms a vector space of dimension $|\Pi_n|$. We define $*$ in V as follows:

$$s_{\rho_1} * s_{\rho_2} = \chi(\rho_1 = \rho_2)\pi_{\rho_1}s_{\rho_1},$$

and extend linearly, making V an algebra. We define a linear functional $E : V \rightarrow F$ as follows:

(3.1) $\quad E(s_\rho) = 1 \quad$ for all ρ

and extend linearly.

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In 1927 Redfield described and counted objects he called "superpositions" [**13**]. These have been further described in [**6**; **8**; **12**; **16**]. We shall describe them in terms of function spaces as follows:

Let $\{G_i\}_{i\in[1,m]}$ be subgroups of S_n, $G_i \subset S_n$, each G_i acting on $[1, n]$. Let $G = G_1 \times \ldots \times G_m \times S_n$. Let $R = S_n$, $D = [1, m]$. (We shall later have $R_1 = R$ and $D_1 = D$.) Then we define an action of G on R^D as follows:

$$(3.2) \quad ((g_1, \ldots, g_m, \sigma)f)(i) = g_i f(i)\sigma^{-1}$$

where the operation on the right hand side is function composition. Redfield's superpositions make up a system of orbit representatives Δ from $G : R^D$.

Redfield described them as m rows of n objects, each row having a group, G_i, act on the objects in the row, and two of these arrays equivalent if they could be made equal, entry by entry, by some action of the G_i's and some permutation of the columns. A moment's reflection will convince one that the action described in (3.2) yields the same objects. We shall no longer refer to Redfield's superpositions, but shall instead only use the action of G on R^D described in (3.2).

We shall use the following lemma (see Perlman [**10**]):

LEMMA 3.3. *If G acts on X and Y, two finite sets, and Δ_1 is a transversal on the orbits of $G : X$, $\Delta_2(x)$ a transversal on the orbits of $G_x : Y$, $\bar{\Delta}$ a transversal on the orbits of the induced action of G on $X \times Y(G : X \times Y)$, $W : Y \to \mathscr{A}$, a commutative algebra over F, W constant on orbits of $G : Y$, then*

$$\sum_{x\in\Delta_1}\sum_{y\in\Delta_2(x)} W(y) = \sum_{(x,y)\in\bar{\Delta}} W(y).$$

Proof. Notice that $\tilde{\Delta} = \{(x, y) \in X \times Y : x \in \Delta_1, y \in \Delta_2(x)\}$ is a transversal on the orbits of $G : X \times Y$, because

(i) If (x, y), $(x', y') \in \tilde{\Delta}$ and there exists $\sigma \in G$ such that $\sigma x = x'$ and $\sigma y = y'$, then $x = x'$ because Δ_1 is a transversal on the orbits of $G : X$ and so $\sigma \in G_x$, which means $y = y'$ since $\Delta_2(x)$ is a transversal on the orbits of $G_x : Y$. Thus, $\tilde{\Delta}$ is contained in a transversal on the orbits of $G : X \times Y$.

(ii) If $(x, y) \in \bar{\Delta}$, then there exists $x' \in \Delta_1$ and $\sigma \in G$ such that $\sigma x' = x$ since Δ_1 is a transversal on the orbits of $G : X$. Furthermore, there exists $y' \in \Delta_2(x')$ and $\tau \in G_{x'}$ such that $\tau y' = \sigma^{-1}y$, since $\Delta_2(x)$ is a transversal on the orbits of $G_x : Y$. But then $\sigma\tau(x', y') = (\sigma\tau x', \sigma\tau y') = (\sigma x', \sigma\sigma^{-1}y) = (x, y)$. Thus, $\tilde{\Delta}$ contains a transversal on the orbits of $G : X \times Y$.

But

$$\sum_{x\in\Delta_1}\sum_{y\in\Delta_2(x)} W(y) = \sum_{(x,y)\in\tilde{\Delta}} W(y) = \sum_{(x,y)\in\bar{\Delta}} W(y),$$

since W is constant on the orbits of $G : Y$ and thus is constant on the orbits of $G : X \times Y$.

We are now ready for Redfield's decomposition theorem (or, as Read [**12**] calls it, the Master Theorem).

Let $R = S_n$, $D = [1, m]$, $G = G_1 \times \ldots \times G_m \times S_n$, $G_i \subset S_n$. Then G acts on R^D as defined by (3.2). Let Δ_1 be a transversal on the orbits of $G : R^D$. Note that G acts on $[1, n]$ as follows:

$$(g_1, \ldots, g_m, \sigma)i = \sigma i \quad \text{(i.e., projection of } G \text{ to } S_n \text{ acts on } [1, n]).$$

THEOREM 3.4 (Redfield [**13**]).

$$\sum_{f_1 \in \Delta_1} P_{G_{f_1}:[1,n]}(s_1, s_2, \ldots) = P_{G_1:[1,n]}(s_1, s_2, \ldots)$$
$$* \ldots * P_{G_m:[1,n]}(s_1, s_2, \ldots).$$

Proof. There are a number of approaches to this theorem. We could use Theorem 6.3 in [**19**] which involves a homomorphism, λ, of S_n. Here, we would set $\lambda(\sigma) = s_\rho$ where $\sigma \in C_\rho$. Or we could use Theorem 2.2 directly by letting $S = R^D$ and defining a functional l where

$$lE_f = P_{G_f:[1,n]}(s_1, s_2, \ldots).$$

The proof we give exploits the multilinear aspects of F^S where $S = R_1^{D_1} \times R_2^{D_2}$, and seems, more directly, to contain the concept of summing cycle index polynomials of stabilizer subgroups over a system of orbit representatives.

For brevity, we shall denote an element $(g_1, \ldots, g_m, \sigma) \in G_1 \times \ldots \times G_m \times S_n$ by α. Let $R_1 = R$ and $D_1 = D$. The action of G on $R_1^{D_1}$ is described by (3.2). Define the linear functional l_1 on $F^{(R_1^{D_1})}$ by $l_1 E_{f_1} = 1$ for all $f_1 \in R_1^{D_1}$. In particular, $l_1 E_{\alpha f_1} = l_1 E_{f_1}$ for all $\alpha \in G$. Let $J_1 = \sum_{f_1 \in R_1^{D_1}} E_{f_1}$.

Define $R_2 = [1, r]$ and $D_2 = [1, n]$. G acts on $R_2^{D_2}$ by $(g_1, \ldots, g_m, \sigma) f_2(i) = f_2(\sigma^{-1} i)$. Define the linear functional l_2 on $F^{(R_2^{D_2})}$ by $l_2 E_{f_2} = \prod_{i=1}^n x_{f_2(i)}$ for all $f_2 \in R_2^{D_2}$ where $x_1, \ldots, x_r$ are indeterminants. In particular, $l_2 E_{\alpha f_2} = l_2 E_{f_2}$ for all $\alpha \in G$. Let $J_2 = \sum_{f_2 \in R_2^{D_2}} E_{f_2}$.

Define

$$l(E_{f_1} \otimes E_{f_2}) = (l_1 E_{f_1}) \times (l_2 E_{f_2})$$

and write $l = l_1 \otimes l_2$. Thus, $l(E_{\alpha f_1} \otimes E_{\alpha f_2}) = l(E_{f_1} \otimes E_{f_2})$ and therefore $lT_G = l$. Let Δ be a transversal on the orbits of $G : R_1^{D_1} \times R_2^{D_2}$. By Theorem 2.2,

$$lT_G I_\Delta = lQ_G(J_1 \otimes J_2). \tag{3.5}$$

We evaluate the right hand side of (3.5) first.

$$lQ_G(J_1 \otimes J_2) = \frac{1}{|G|} \sum_{\alpha=(g_1,\ldots,g_m,\sigma)\in G} \sum_{f_1 \in R_1^{D_1}} \chi(\alpha f_1 = f_1) \times \sum_{f_2 \in R_2^{D_2}} \chi(\alpha f_2 = f_2) \prod_{i=1}^n x_{f_2(i)}. \tag{3.6}$$

To evaluate the right hand side of (3.6) we first characterize all $f_1 \in R_1^{D_1}$ such that $(g_1, \ldots, g_m, \sigma) f_1 = f_1$. By (3.2), $g_i f_1(i) \sigma^{-1} = f_1(i)$ or $f_1(i)^{-1} g_i f_1(i) = \sigma$. Thus, we must have that $g_i \in C_\rho$ for all i where C_ρ is the conjugate class of

S_n containing σ. Furthermore, the number of possible values for $f_1(i)$ for each i is just the cardinality of the normalizer of σ in S_n, i.e., π_ρ. Thus, the number of possible f_1 fixed by $(g_1, \ldots, g_m, \sigma)$ is $\pi_\rho{}^m$.

Continuing with our evaluation of the right hand side of (3.6) we compute

$$\sum_{f_2 \in R_2{}^{D_2}} \chi(\alpha f_2 = f_2) \prod_{i=1}^{n} x_{f_2(i)} \quad \text{where } \alpha = (g_1, \ldots, g_m, \sigma) \in G.$$

This is easily seen to be $s_\rho = \prod_{i=1}^n s_i{}^{j_i}$ where $s_i = x_1{}^i + \ldots + x_r{}^i$ and j_i is the number of cycles of σ of length i.

Therefore, the right hand side of (3.5) is

$$\frac{1}{|G|} \sum_{\rho \in \pi_n} A_1(\rho) \cdot \ldots \cdot A_n(\rho) |C_\rho| \pi_\rho{}^m s_\rho$$

where $A_i(\rho) = |C_\rho \cap G_i|$ and this is

$$P_{G_1:[1,n]}(s_1, s_2, \ldots) * \ldots * P_{G_m:[1,n]}(s_1, s_2, \ldots).$$

We next evaluate the left hand side (3.5).

$$lT_G I_\Delta = lI_\Delta = \sum_{(f_1, f_2) \in \Delta} \prod_{i=1}^{n} x_{f_2(i)}.$$

We let $\Delta_2(f_1)$ be a transversal on the orbits of $G_{f_1} : R_2{}^{D_2}$ and apply Lemma 3.3 to get

$$lT_G I_\Delta = \sum_{f_1 \in \Delta_1} \sum_{f_2 \in \Delta_2(f_1)} \prod_{i=1}^{n} x_{f_2(i)}.$$

We now apply Polya's Theorem (see [**4**]) to the action of G_{f_1} on $R_2{}^{D_2}$ to obtain

$$lT_G I_\Delta = \sum_{f_1 \in \Delta_1} P_{G_{f_1}:[1,n]}(s_1, s_2, \ldots).$$

COROLLARY 3.7.

$$|\Delta_1| = E(P_{G_1:[1,n]}(s_1, s_2, \ldots) * \ldots * P_{G_m:[1,n]}(s_1, s_2, \ldots))$$

where E is defined in (3.1).

Proof. Apply E to both sides of Theorem 3.4 and note that $E(P_{G:S}(s_1, s_2, \ldots)) = 1$.

The idea of summing cycle index polynomials of stabilizer subgroups over a transversal was developed at length by deBruijn [**3**]. DeBruijn's results in that paper (Redfield's Superposition Theorem was one of them) may be achieved from the multilinear standpoint of a cartesian product of two function spaces, with the connecting relationship described in Lemma 3.3. Lemma 3.3, of course, may be extended as follows:

COROLLARY 3.8. *Suppose G acts on $X_1, \ldots, X_n$ and thus G acts on $X_1 \times \ldots \times X_n$. Let $\bar{\Delta}$ be a transversal on the orbits of $G : X_1 \times \ldots \times X_n$,*

$\Delta_i(x_1, \ldots, x_{i-1})$ *be a transversal on the orbits of* $G_{x_1} \cap \ldots \cap G_{x_{i-1}} : X_i$. *Let* $W : X_n \to \mathscr{A}$, *an algebra*, W *constant on orbits of* $G : X_n$. *Then*

$$\sum_{x_1 \in \Delta_1} \sum_{x_2 \in \Delta_2(x_1)} \cdots \sum_{x_n \in \Delta_n(x_1, \ldots, x_{n-1})} W(x_n) = \sum_{(x_1, \ldots, x_n) \in \bar{\Delta}} W(x_n).$$

Proof. Use repeated applications of Lemma 3.3.

4. Foulkes' extension. Redfield notes two problems which he left unsolved. First, he observed that the cycle index polynomial is not unique, i.e., two non-conjugate subgroups of S_n may have the same cycle index polynomial. Second, the decomposition of Theorem 3.4 is not unique, i.e., we may be able to find another collection of groups $H_1, \ldots, H_t$ such that

$$P_{G_1:[1,n]}(s_1, s_2, \ldots) * \ldots * P_{G_m:[1,n]}(s_1, s_2, \ldots) = \sum_{i=1}^{t} P_{H_i:[1,n]}(s_1, s_2, \ldots)$$

where $H_1, \ldots, H_t$ are not conjugate to $\{G_{f_i}\}_{f_i \in \Delta_1}$ in any order.

These problems may be overcome in the following manner (see [**2**; **5**]).

THEOREM 4.1. (Foulkes [**5**]). *For all subgroups* $H \subset S_n$, $\sum_{f \in \Delta_1} M_{G_f}(H) = \prod_{i=1}^{m} M_{G_i}(H)$ *where the marks are marks in* S_n.

Proof. Merely apply the trivial functional to $P_{S_n}{}^H T_G I_{\Delta_1} = P_{S_n}{}^H Q_G I_S$. (See [**17**].)

This overcomes Redfield's difficulties since marks are constant on conjugate subgroups and tables of marks form non-singular matrices. Thus, if we consider the free vector space over the non-conjugate subgroups of S_n, we observe that the marks, $\langle M_K \rangle$, form a basis of this vector space. Then, whereas in Theorem 3.4 we were unable to decompose uniquely $P_{G_1:[1,n]}(s_1, s_2, \ldots) * \ldots * P_{G_m:[1,n]}(s_1, s_2, \ldots)$, in Theorem 4.1 the decomposition of $\prod_{i=1}^{m} M_{G_i}$ in terms of this basis must be unique.

Furthermore, even if we were to discover the correct decomposition in Theorem 3.4, we could not in general recover the stabilizer subgroups $\{G_f\}_{f \in \Delta_1}$ from this decomposition. However, since the marks, $\langle M_K \rangle$, form a basis of the free vector space over the non-conjugate subgroups, we can recover these stabilizer subgroups from the decomposition in Theorem 4.1. In fact, we have seen in [**17**] that the decomposition into marks and subsequent reconstruction of stabilizer subgroups in Theorem 4.1 is merely an example of the general problem of enumerating orbits with a given automorphism group.

Finally, we shall note the intimate connection between marks, permutation characters, and cycle index polynomials. If α_K is the permutation representation of S_n induced by K, then we have

$$P_{K:[1,n]}(s_1, s_2, \ldots) = \frac{1}{n!} \sum_{\rho \in \pi_n} |C_\rho| \chi_{\alpha_K}(\rho) s_\rho,$$

where χ_{α_K} is the character of α_K. This formula follows from the standard fact (see [**9**; **14**]) that

$$\chi_{\alpha_K}(\rho) = \frac{|S_n|}{|G|} \cdot \frac{|C_\rho \cap G|}{|C_\rho|}.$$

Thus we see that knowledge of the permutation character χ_{α_K} of S_n is equivalent to knowledge of the cycle index polynomial $P_{K:[1,n]}$. In fact, Theorem 3.4 may be restated as

$$(4.2) \quad \sum_{f \in \Delta_1} \chi_{\alpha_{G_f}}(\sigma) = \prod_{i=1}^{m} \chi_{\alpha_{G_i}}(\sigma) \quad \text{for all } \sigma \in S_n.$$

This formula may be obtained from Theorem 4.1 by observing that

$$M_K(\langle \sigma \rangle) = \chi_{\alpha_K}(\sigma)$$

where $\sigma \in S_n$, $\langle \sigma \rangle$ is the cyclic subgroup generated by σ and the marks are marks in S_n.

5. Example. Applications of Redfield's theorems to graph theory abound [**6**; **8**; **12**; **13**; **15**]. We shall address ourselves to a simple example here and then indicate how these theorems might be further used.

Suppose we wish to know how many octagons we may construct with exactly 5 red balls and 3 blue balls at the vertices, up to the action of the dihedral group on the octagon. Suppose, further, that we wish to know, for each such pattern, the largest subgroup of the dihedral group which fixes that pattern.

Let G_1 = dihedral group on the octagon, $G_2 = S_5 \times S_3$, and $n = 8$, $m = 2$. Then the number of ways to construct such an octagon is

$$E(P_{G_1:[1,n]}(s_1, s_2, \ldots) * P_{G_2:[1,n]}(s_1, s_2, \ldots))$$

where G_2 acts on $[1, n]$ by the action of S_5 on $[1, 5]$ and the action of S_3 on $[6, 8]$. This is because

$$E(P_{G_1:[1,n]}(s_1, s_2, \ldots) * P_{G_2:[1,n]}(s_1, s_2, \ldots)) = |\Delta|$$

where Δ is a transversal on the orbits of $G_1 \times G_2 \times S_n : S_n{}^{[1,2]}$ by $(g_1, g_2, \sigma)f(i) = g_i f(i)\sigma^{-1}$ (Corollary 3.5). If we write the eight nodes in one row and the symbols $\{r, b\}$, repeating r five times and b three times, in the second row, we have G_1 acting on the first row, $S_5 \times S_3$ acting on the second row (where S_5 acts in the 5 r's and S_3 acts in the 3 b's), e.g.,

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ b & b & r & b & r & r & r & r \end{bmatrix},$$

then the patterns we wish to count are all these, up to the actions of G_1 and G_2 on the rows and up to whole permutations of columns.

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Thus,

$$\sum_{f\in\Delta} P_{G_f:[1,n]}(s_1, s_2, \ldots) = P_{G_1:[1,n]}(s_1, s_2, \ldots)$$

$$* P_{G_2:[1,n]}(s_1, s_2, \ldots) = \frac{1}{2}(7S_1^{\,8} + 3S_1^{\,2}S_2^{\,3}) \quad \text{and} \quad |\Delta| = 5.$$

Generally speaking, we cannot decompose an arbitrary $*$ -product of the cycle index polynomials into the cycle index polynomials of the stabilizer subgroups. We must, instead, use marks. However, in this case, we easily see that

$$\sum_{f\in\Delta} P_{G_f:[1,n]}(s_1, s_2, \ldots) = \frac{1}{2}(s_1^{\,8} + s_1^{\,2}s_2^{\,3})$$

$$+ \frac{1}{2}(s_1^{\,8} + s_1^{\,2}s_2^{\,3}) + \frac{1}{2}(s_1^{\,8} + s_1^{\,2}s_2^{\,3}) + s_1^{\,8} + s_1^{\,8}.$$

Again, generally speaking, we cannot say exactly to which subgroup an arbitrary cycle index polynomial is associated. But in this case, it is the cycle index polynomial of the subgroup consisting of just a reflection through a line through opposite vertices. Figure 1 lists the five figures and the dotted lines indicate the axes of reflection for the stabilizer subgroups. Note that two of the figures have trivial stabilizer subgroups and therefore cycle index polynomials equal to s_1^8.

We must remark that the result $|\Delta| = 5$ can be obtained from Polya's theorem directly by merely looking at the coefficient of $w(r)^5w(b)^3$ in the resulting polynomial, where w is the Polya weight function.

As was stated earlier, Theorem 2.2 may be applied to a G-invariant subset L of S. When S is a finite function space, this involves the construction of a list of pure tensors $A_1, \ldots, A_v$ which, up to the operator T_G, represent L. In the case at hand, we may wish to enumerate orbits from the action of $G_1 \times \ldots \times G_m \times S_n$ on some useful subset, L, of $S_n^{[1,m]}$. Certain boundary conditions might be considered (for example, no isolated red balls), or restrictions involving the unusual nature of the group action (perhaps involving the conjugacy classes of the G_i's).

We shall now indicate one approach to a graph counting problem (see [**12**; **15**]). Consider graphs with n nodes and k lines. We let $m = \binom{n}{2}$ where m corresponds to all possible lines in the graph. Then the action of S_n on $[1, n]$ induces an action on $[1, m]$. Furthermore, $G_2 = S_k \times S_{m-k}$ acts on $[1, m]$ as before. Then if Δ is a transversal on the orbits of $G_1 \times G_2 \times S_m : S_m^{[1,2]}$, Δ is also a transversal from the graphs with n nodes and k lines, up to the action of S_n on the m pairs of points.

Thus, the problem of computing $|\Delta|$ reduces to that of computing two cycle index polynomials for the special group actions above. If we also want to compute the groups under which the graphs in the transversal are stable

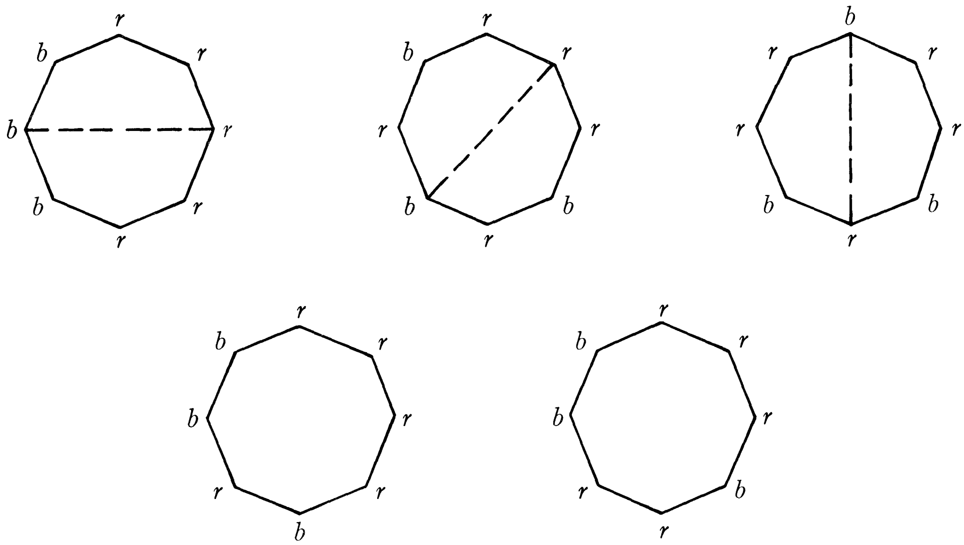

FIGURE 1. Transversal from octagons under the dihedral group of order 16 with five vertices labeled "r" and three vertices labeled "b".

(called the automorphism groups), we must, in the general case, also be able to compute the marks of the subgroups of S_m.

As before, useful restrictions on the set R^D may also be considered in this multilinear context.

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