COPERFECT MONOIDS

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1. Introduction. Throughout this paper S will denote a given monoid, that is, a semigroup with an identity. A set A is a *right S-system* if there is a map $\phi: A \times S \rightarrow A$ satisfying

$$\phi(a, 1) = a \quad \text{and} \quad \phi(a, st) = \phi(\phi(a, s), t)$$

for any element a of A and any elements s,t of S. For $\phi(a, s)$ we write *as* and we refer to right S-systems simply as S-systems. One has the obvious definitions of an S-subsystem and an S-homomorphism.

Clearly S-systems provide the semigroup theory analogue of R-modules over a ring R. Further, many of the properties defined for S-systems are inspired by the corresponding definitions in ring theory. In particular we have projective, flat and injective S-systems, where flatness for S-systems weakens the concept of projectivity, as is the case for modules.

Many papers have been published which characterise monoids by properties of their S-systems, for example **[4]**, **[9]**, **[10]**. The properties we consider here are those of injectivity and α-injectivity, where α is any cardinal strictly greater than 1. The definition and some of the basic properties of these concepts are given in Section 2. The notion of α-injectivity was introduced for R-modules over a ring R in **[3]** and for S-systems in **[6]**. For both R-modules and S-systems the usual terminology for $\aleph_0$-injective is weakly f-injective and for 2-injective is weakly p-injective. Further, if T is a semigroup or a ring and $\gamma(T)$ is a cardinal such that every right ideal of T has a generating set of fewer than $\gamma(T)$ elements, then one writes weakly injective for $\gamma(T)$-injective. In the case of R-modules, weak injectivity coincides with injectivity, but this is not always true for S-systems **[1]**.

Monoids over which all S-systems are α-injective (for any cardinal $\alpha > 1$) are characterised in **[6]**. In Section 3 we classify monoids over which all α-injective S-systems are β-injective, for various choices of cardinals $\alpha, \beta > 1$. Our proofs are based on the construction of an α-injective S-system $A^{[\alpha]}$ containing any given S-system A. This method generalises the construction of the divisible S-system $\bar{A}$ detailed in **[7]**, where we classify monoids for whose S-systems the notions of divisibility and weak p-injectivity coincide.

The monoid S is said to be *perfect* if all flat S-systems are projective. Perfect monoids have been studied and characterised in **[5]** and **[9]**. It is clear that injectivity is a property dual to that of projectivity. By analogy with the definition of a coflat module given in **[2]**, we introduce in **[6]** the concept of coflatness for S-systems as a notion dual to that of

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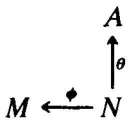
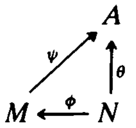
flatness. However, Corollary 3.4 of **[6]** gives that an S-system is coflat if and only if it is weakly p-injective. In Section 4 we characterise the monoids that are dual to the perfect monoids, that is, those monoids over which all coflat S-systems are injective. We call such monoids *coperfect*.

I would like to record my thanks to Dr J. B. Fountain for his advice and encouragement whilst this work was in progress.

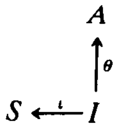
2. Preliminaries. An S-system A is *injective* if given any diagram of S-systems and S-homomorphisms,



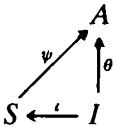
where $\phi: N \to M$ is an injection, there exists an S-homomorphism $\psi: M \to A$ such that



is commutative. By imposing conditions on M and N we weaken this definition to obtain the concept of α-injectivity, as follows. Let α be any cardinal strictly greater than 1. Then an S-system A is *α-injective* if given any diagram of the form,



where I is a right ideal of S with a generating set of fewer than α elements, $\iota: I \to S$ is the inclusion mapping and $\theta: I \to A$ is an S-homomorphism, then there exists an S-homomorphism $\psi: S \to A$ such that



is commutative.

It is clear that an injective S-system is α-injective for any α and that an α-injective S-system is β-injective for any cardinal β such that $1 < \beta \leq \alpha$. Let $\gamma = \gamma(S)$ be a cardinal such that every right ideal of S has a generating set of fewer than γ elements. As pointed out in the introduction, the usual terminology for γ-injective is *weakly injective*. Further, we write *weakly f-injective* for $\aleph_0$-injective and *weakly p-injective* for 2-injective.

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We say that an S-system A satisfies the *α-Baer criterion* for a cardinal $\alpha > 1$ if, given any right ideal I of S with a generating set of fewer than α elements, then for any S-homomorphism $\theta: I \to A$ there is an element a in A such that θ is given by $\theta(r) = ar$ for all r in I.

Given a system of equations Σ with constants from the S-system A, then Σ is *consistent* if Σ has a solution in some S-system containing A. If all equations in Σ are of the form $xs = a$, where $s \in S$ and $a \in A$, and if the same variable appears in each, then Σ is an *α-system over A*, where α is any cardinal larger than that of Σ. Thus Σ is an α-system over A if and only if Σ has the form

$$\Sigma = \{xs_j = a_j : j \in J, |J| < \alpha, s_j \in S, a_j \in A\}.$$

We will rely on the following two results from [**6**].

LEMMA 2.1. *Let A be an S-system and let*

$$\Sigma = \{xs_j = a_j : j \in J, |J| < \alpha, s_j \in S, a_j \in A\}$$

be an α-system over A. Then the following conditions are equivalent:

(i) *Σ is consistent,*

(ii) *for all elements h, k of S and for all elements i, j of J,*

$$s_ih = s_jk \Rightarrow a_ih = a_jk.$$

PROPOSITION 2.2. *Let $\alpha > 1$ be a cardinal. Then the following conditions are equivalent for an S-system A*:

(i) *every consistent α-system over A has a solution in A,*

(ii) *A satisfies the α-Baer criterion,*

(iii) *A is α-injective.*

For an S-system A and a subset H of $A \times A$, then by $\rho(H)$ we denote the congruence generated by H, that is, the smallest congruence relation ν over A such that $H \subseteq \nu$.

LEMMA 2.3 [**10**]. *The ordered pair (a, b) is in $\rho(H)$ if and only if $a = b$ or there exists a natural number n and a sequence*

$$a = c_1t_1,\ d_1t_1 = c_2t_2, \ldots, d_{n-1}t_{n-1} = c_nt_n,\ d_nt_n = b,$$

where $t_1, \ldots, t_n$ are elements of S and for each $i \in \{1, \ldots, n\}$ either (c_i, d_i) or (d_i, c_i) is in H.

A sequence as in Lemma 3.3 will be referred to as a *$\rho(H)$-sequence of length n*. For any congruence ρ on A the set of congruence classes of ρ can be made into an S-system, with the obvious action of S. We write A/ρ to denote this S-system and $[a]_\rho$, or simply $[a]$ where ρ is understood, for the ρ-class of an element a of A.

3. Characterising monoids by their α-injective *S*-systems. Let α be any cardinal with $1 < \alpha \leq \aleph_0$. We begin this section by detailing a construction of an α-injective S-system $A^{[\alpha]}$ containing an arbitrary given S-system A.

Firstly, we define Σ_0, F_0, H_0 and A_1 as follows: for any natural number n, where $1 \leq n < \alpha$, let

$$\Sigma_0^n = \{((s_1, a_1), \ldots, (s_n, a_n)) \in (S \times A)^n: \\ s, t \in S,\ i, j \in \{1, \ldots, n\},\ s_i s = s_j t \text{ implies that } a_i s = a_j t\}.$$

Then we put

$$\Sigma_0 = \bigcup_{n<\alpha} \Sigma_0^n,$$

$$F_0 = \bigcup \{x_\sigma S : \sigma \in \Sigma_0\}$$

that is, F_0 is the free S-system on $\{x_\sigma : \sigma \in \Sigma_0\}$,

$$H_0 = \{(x_\sigma s_i, a_i) : \sigma \in \Sigma_0^n,\ n < \alpha,\ (s_i, a_i) = \sigma_i,\ i \in \{1, \ldots, n\}\},$$

where σ_i is the ith component of the row vector σ. Now let

$$A_1 = (A \cup F_0)/\rho(H_0).$$

Suppose now that $a_1, a_2 \in A$ and $[a_1] = [a_2]$ in A_1. Thus $a_1 = a_2$ or a_1 and a_2 are connected via a $\rho(H_0)$-sequence, which it is easy to see must be of even length. If

$$a_1 = c_1 t_1,\ d_1 t_1 = c_2 t_2,\ d_2 t_2 = a_2$$

is a $\rho(H_0)$-sequence, then $c_1 \in A$ and so $(c_1, d_1) = (a_i, x_\sigma s_i)$ for some $(x_\sigma s_i, a_i) \in H_0$, where $\sigma \in \Sigma_0^n$ say, $n < \alpha$. From $d_1 t_1 = c_2 t_2$ it follows that there exists a $j \in \{1, \ldots, n\}$ with $c_2 = x_\sigma s_j$, $d_2 = a_j$ and $(s_j, a_j) = \sigma_j$. Then $x_\sigma s_i t_1 = x_\sigma s_j t_2$ gives $s_i t_1 = s_j t_2$ and so from the definition of Σ_0, $a_i t_1 = a_j t_2$. Hence

$$a_1 = c_1 t_1 = a_i t_1 = a_j t_2 = d_2 t_2 = a_2.$$

We now let $m \in \mathbb{N}$, $m > 1$ and make the inductive assumption that if b_1, b_2 are elements of A connected by a $\rho(H_0)$-sequence of (necessarily even) length less than $2m$, then $b_1 = b_2$.

Suppose that

$$a_1 = c_1 t_1,\ d_1 t_2 = c_2 t_2, \ldots, d_{2m} t_{2m} = a_2$$

is a $\rho(H_0)$-sequence connecting a_1 and a_2. As above, $a_1 = d_2 t_2$ and so

$$a_1 = c_3 t_3, \ldots, d_{2m} t_{2m} = a_2$$

is a $\rho(H_0)$-sequence of length $2(m-1)$ connecting a_1 and a_2, thus $a_1 = a_2$ by the inductive assumption. Hence A is embedded in A_1 and we may identify the element $a \in A$ with the element $[a]$ of A_1.

In a similar manner one constructs a sequence $A_1 \subseteq A_2 \subseteq \ldots$ using $\Sigma_1, \Sigma_2, \ldots, F_1, F_2, \ldots$ and $H_1, H_2, \ldots$, where Σ_i, F_i and H_i are defined using A_i in the same way that Σ_0, F_0 and H_0 are defined in terms of A. Although $\Sigma_0 \subseteq \Sigma_1 \subseteq \ldots$, at each

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stage we choose a basis for F_i which is disjoint from the bases used for $F_0, F_1, \ldots, F_{i-1}$. For ease of notation we make the convention that for $n \in \mathbb{N}$ the $\rho(H_n)$-class of an element a of $A_n \cup F_n$ will be denoted by $[a]_n$.

Now put $A^{[\alpha]} = \bigcup_{i \in \mathbb{N}} A_i$, where A_0 is identified with A. We claim that $A^{[\alpha]}$ is α-injective.

Let $I = \bigcup_{k \in K} s_k S$ be a right ideal of S where $|K| < \alpha$. Suppose that $\theta : I \to A^{[\alpha]}$ is an S-homomorphism. Then for any $i, j \in K$ and $s, t \in S$, $s_j s = s_k t$ implies that $\theta(s_j)s = \theta(s_k)t$, since θ is well-defined. Since $\alpha \leq \aleph_0$, K is a finite set and so we may assume that $K = \{1, \ldots, m\}$ for some $m \in \mathbb{N}$ with $m < \alpha$. Clearly there is an $n \in \mathbb{N}$ with $\theta(s_k) \in A_n$ for all $k \in K$. Thus

$$\sigma = ((s_1, \theta(s_1)), \ldots, (s_m, \theta(s_m))$$

is an element of Σ_n and $[y_\sigma]_n$ is an element of A_{n+1}, where $\{y_\sigma : \sigma \in \Sigma_n\}$ is the basis of F_n. Since $A_{n+1} \subseteq A^{[\alpha]}$, $[y_\sigma]_n$ is an element of $A^{[\alpha]}$. Further, for any $k \in K$,

$$\theta(s_k) = [\theta(s_k)]_n = [y_\sigma s_k]_n = [y_\sigma]_n s_k$$

and it follows that $\theta(s) = [y_\sigma]_n s$ for all $s \in I$. Thus $A^{[\alpha]}$ has the α-Baer criterion and so by Proposition 2.2, $A^{[\alpha]}$ is α-injective.

The results of this paper are all dependent upon the structure of $A^{[\alpha]}$.

Proposition 3.1. *Let $\alpha > 1$ be a cardinal. Then the following conditions are equivalent for the monoid S:*

(i) *all right ideals of S with a generating set of fewer than α elements are principal,*
(ii) *all weakly p-injective S-systems are α-injective.*

Proof. (i)$\Rightarrow$(ii). Given (i) it is clear that the notions of weak p-injectivity and α-injectivity coincide for S-systems; thus (ii) holds.

(ii)$\Rightarrow$(i). To show that this implication holds we need a technical lemma.

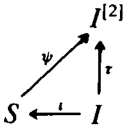
Lemma 3.2. *Let A be an S-system and let $A^{[2]}$ be constructed as above. Suppose that there exists an element b of A_n, $n > 0$, such that $A \subseteq bS$. Then there exists an element c in A_{n-1} with $A \subseteq cS$.*

Proof. We may assume that $b \in A_n \backslash A_{n-1}$, otherwise there is nothing to prove. If $b \in A_n \backslash A_{n-1}$ then b has the form $b = [y_\sigma u]_{n-1}$ where $u \in S$, $\sigma \in \Sigma_{n-1}$ and $\{y_\sigma : \sigma \in \Sigma_{n-1}\}$ is the basis of F_{n-1}. Given any $a \in A$ there exists $v \in S$ with $a = bv$, that is, $[a]_{n-1} = [y_\sigma uv]_{n-1}$. Since $a \neq y_\sigma uv$ in $A_{n-1} \cup F_{n-1}$, we have that a and $y_\sigma uv$ are connected by a $\rho(H_{n-1})$-sequence

$$y_\sigma uv = c_1 t_1,\ d_1 t_1 = c_2 t_2,\ \ldots,\ d_m t_m = a.$$

Thus $c_1 = y_\sigma s$, $d_1 = c$, where $c \in A_{n-1}$ and $\sigma = (s, c)$. Then a, ct_1 are $\rho(H_{n-1})$-related elements of A_{n-1} and from the construction of A_{n-1}, $a = ct_1$. Hence $A \subseteq cS$ and the lemma holds.

Returning to the proof of Proposition 3.1, let $I = \bigcup_{k \in K} u_k S$ be a right ideal of S where $|K| < \alpha$. We form the weakly p-injective S-system $I^{[2]}$, which by assumption is α-injective. Thus there exists an S-homomorphism $\psi : S \to I^{[2]}$ such that



is commutative, where ι, τ are the appropriate inclusion mappings. Then for any $k \in K$,

$$u_k = \tau(u_k) = \psi\iota(u_k) = \psi(u_k) = \psi(1)u_k.$$

Hence

$$I = \bigcup_{k \in K} u_k S = \bigcup_{k \in K} \psi(1) u_k S \subseteq \psi(1) S.$$

Now $\psi(1) \in I_n$ for some $n \in \mathbb{N}$. If $n \neq 0$ then we may apply Lemma 3.2 successively n times and obtain an element c in I such that $I \subseteq cS$. Hence in either case I is contained in a principal right ideal sS of S, where $s \in I$. It follows that $I = sS$ and so I is principal.

COROLLARY 3.3. *Let α be any cardinal such that $2 < \alpha \leqq \aleph_0$. Then the following conditions are equivalent for the monoid S:*

(i) *all weakly p-injective S-systems are weakly f-injective,*
(ii) *all weakly p-injective S-systems are α-injective,*
(iii) *all weakly p-injective S-systems are 3-injective,*
(iv) *all right ideals of S with a generating set of 2 elements are principal,*
(v) *finitely generated right ideals of S are principal.*

Proof. (i)$\Rightarrow$(ii), (ii)$\Rightarrow$(iii). These implications are immediate.

(iii)$\Rightarrow$(iv), (v)$\Rightarrow$(i). These follow from Proposition 3.1.

(iv)$\Rightarrow$(v). Let $a, b \in S$. Then $aS \cup bS$ is a principal right ideal by (iv) and it follows that $aS \subseteq bS$ or $bS \subseteq aS$. Hence the principal right ideals of S are linearly ordered, giving that finitely generated right ideals of S are principal.

COROLLARY 3.4. *The following conditions are equivalent for the monoid S:*

(i) *S is a principal right ideal monoid,*
(ii) *all weakly p-injective S-systems are weakly injective.*

Proof. This is immediate from Proposition 3.1, with $\alpha = \gamma(S)$.

In order to establish our next result we need the following technical lemma.

LEMMA 3.5. *Let A be an S-system and let $A^{[\aleph_0]}$ be constructed as above. Suppose that A is contained in a finitely generated S-subsystem of A_n for some $n > 0$. Then A is contained in a finitely generated S-subsystem of A_{n-1}.*

Proof. Let $b_1, \ldots, b_m \in A_n$, $n > 0$, be such that $A \subseteq \bigcup_{i=1}^{m} b_i S$. If each b_i is in A_{n-1} then there is nothing to prove. Thus we may assume that there is an $r \in \{1, \ldots, m\}$ such that $b_1, \ldots, b_r \in A_n \backslash A_{n-1}$ and $b_{r+1}, \ldots, b_m \in A_{n-1}$. From the form of A_n we have

$$b_i = [y_{\sigma_i} u_i]_{n-1} \qquad (1 \leqq i \leqq r),$$

where $\{y_\sigma : \sigma \in \Sigma_{n-1}\}$ is the basis of F_{n-1}, $\sigma_1, \ldots, \sigma_r \in \Sigma_{n-1}$ and $u_1, \ldots, u_r \in S$. Suppose further that for $i \in \{1, \ldots, r\}$, $\sigma_i \in \Sigma_{n-1}^{p(i)}$ and

$$\sigma_i = ((s_{i1}, c_{i1}), \ldots, (s_{i,p(i)}, c_{i,p(i)})).$$

Let $a \in A$. If $a \in b_i S$ for $i \in \{1, \ldots, r\}$ then there exists an element v of S with

$$a = [a]_{n-1} = [y_{\sigma_i} u_i v]_{n-1}.$$

As $a \neq y_{\sigma_i} u_i v$ in $A_{n-1} \cup F_{n-1}$, there is a $\rho(H_{n-1})$-sequence

$$y_{\sigma_i} u_i v = c_1 t_1,\ d_1 t_1 = c_2 t_2, \ldots, d_l t_l = a$$

connecting $y_{\sigma_i} u_i v$ and a. Then there exists an element $j \in \{1, \ldots, p(i)\}$ such that $c_1 = y_{\sigma_i} s_{ij}$, $d_1 = c_{ij}$. Thus a, $c_{ij} t_1$ are $\rho(H_{n-1})$-related elements of A_{n-1}, giving that $a = c_{ij} t_1$. It follows that

$$A \subseteq \Big(\bigcup_{\substack{1 \leqq i \leqq r \\ 1 \leqq j \leqq p(i)}} c_{ij} S \Big) \cup \Big(\bigcup_{r < k \leqq m} b_k S \Big),$$

so proving the lemma.

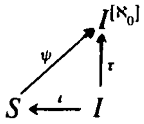
PROPOSITION 3.6. *Let α be a cardinal no less than $\aleph_0$. Then the following conditions are equivalent for the monoid S:*

(i) *all right ideals of S with a generating set of fewer than α elements are finitely generated,*

(ii) *all weakly f-injective S-systems are α-injective.*

Proof. (i)$\Rightarrow$(ii). Given (i) we see that the concepts of α injectivity and weak f-injectivity coincide for S-systems; hence (ii) holds.

(ii)$\Rightarrow$(i). Let I be a right ideal of S with a generating set of fewer than α elements. We may form $I^{[\aleph_0]}$ which is an α-injective S-system by assumption. Thus there is an S-homomorphism $\psi : S \to I^{[\aleph_0]}$ such that



is commutative, where ι, τ are the appropriate inclusion mappings. Let r be any element of I. Then

$$r = \tau(r) = \psi\iota(r) = \psi(r) = \psi(1)r$$

and so $I \subseteq \psi(1)S$. If $\psi(1) \in I$ then

$$I \subseteq \psi(1)S \subseteq IS \subseteq I$$

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and so I is finitely generated (indeed principal). Otherwise, $\psi(1) \in I_n \setminus I_{n-1}$ for some $n > 0$. Then $\psi(1)S \subseteq I_n$ and so $\psi(1)S$ is a finitely generated S-subsystem of I_n. Applying Lemma 3.5 n times, one sees that I is contained in a finitely generated S-subsystem of I. Clearly then I is finitely generated.

The monoid S is *noetherian* if S satisfies the ascending chain condition on right ideals. It is well known that this is equivalent to all right ideals of S being finitely generated.

Corollary 3.7. *Let β be a cardinal with $\gamma(S) \geqq \beta \geqq \aleph_1$. Then the following conditions are equivalent for the monoid S:*

(i) *S is noetherian,*
(ii) *all weakly f-injective S-systems are weakly injective,*
(iii) *all weakly f-injective S-systems are β-injective,*
(iv) *all weakly f-injective S-systems are $\aleph_1$-injective,*
(v) *all countably generated right ideals of S are finitely generated.*

Proof. (i)$\Rightarrow$(ii). This is immediate from Proposition 3.6, with $\alpha = \gamma(S)$.

(ii)$\Rightarrow$(iii), (iii)$\Rightarrow$(iv). These are clear.

(iv)$\Rightarrow$(v). This follows from Proposition 3.6, with $\alpha = \aleph_1$.

(v)$\Rightarrow$(i). Let I be a right ideal of S. If I is not finitely generated then we may form a strictly increasing sequence of right ideals of S

$$a_1S \subset a_1S \cup a_2S \subset a_1S \cup a_2S \cup a_3S \subset \dots,$$

where $a_i \in I$, $i \in \mathbb{N}$. Let $J = \bigcup_{i \in \mathbb{N}} a_iS$. Then J is a countably generated right ideal of S and so by assumption J is finitely generated. Thus there exist $m, n \in \mathbb{N}$ and elements $b_1, \dots, b_m$ of $a_1S \cup \dots \cup a_nS$ such that $J = \bigcup_{i=1}^{m} b_iS$. Then

$$\bigcup_{j=1}^{n} a_jS \subset J = \bigcup_{i=1}^{m} b_iS \subseteq \bigcup_{j=1}^{n} a_jS,$$

a contradiction. Hence I is finitely generated and as I was chosen arbitrarily, S is noetherian.

4. Coperfect monoids. The concept of a coflat module over a ring is introduced by Damiano in [**2**]. He develops in Proposition 1.3 of that paper an elementary criterion for a module to be coflat; we take the non-additive analogue of this criterion to define a coflat S-system. Thus an S-system is *coflat* if, given any elements a of A and s of S with $a \notin As$, then there exist elements h, k in S such that $sh = sk$ but $ah \neq ak$. However, using Lemma 2.1 and Proposition 2.2, it is easy to see that an S-system A is coflat if and only if it is weakly p-injective. This fact enables us to use the structure of the coflat S-system $A^{[2]}$ to prove Proposition 4.1.

Before stating the result we give some definitions. For any element a of an S-system A,

$$\mathrm{ann}_r(a) = \{(u, v) \in S \times S : au = av\}.$$

Clearly $\mathrm{ann}_r(a)$ is a right congruence on S, the *right annihilator congruence of a*.

Conversely, given any right congruence ρ on S one defines

$$\mathrm{Ann}_l(\rho) = \{s \in S : (u, v) \in \rho \text{ implies } su = sv\}.$$

Then $\mathrm{Ann}_l(\rho)$ is empty or is a left ideal of S, the *left annihilator ideal of* ρ. However, this concept is too strong for our purposes and weaken it to fit our requirements, as follows.

Let ρ, ν be right congruences on S and let t be an element of S. Then $\mathrm{Ann}(\rho, t, \nu)$ is defined by

$$\mathrm{Ann}(\rho, t, \nu) = \mathrm{Ann}_l(\rho) \cup \{s \in S : \text{if } (u, v) \in \rho \text{ and } su \neq sv, \text{ then there exist } h, k \in S \text{ with } su = th,\ h\nu k,\ tk = sv\}.$$

Let s, t be elements of S. Then an *n-link from s to t in S* consists of n-tuples $\vec{p} = (p_1, \ldots, p_n)$, $\vec{q} = (q_1, \ldots, q_n)$, $\vec{r} = (r_1, \ldots, r_n)$ with $r_n = t$ and

$$p_1 s = q_1 r_1,\ p_{i+1} r_i = q_{i+1} r_{i+1} \qquad (1 \leqq i \leqq n-1).$$

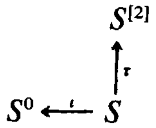
We remind the reader that S is *coperfect* if all its coflat S-systems are injective.

PROPOSITION 4.1. *The monoid S is coperfect if and only if S is a principal right ideal monoid with a left zero and S satisfies condition (CI)*:

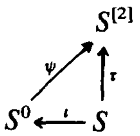
(CI) *For any element s of S and any right congruence ρ on S, there are elements t, u in S and right congruences $\nu_0 = \rho$, $\nu_1, \ldots, \nu_n$ on S such that there is an n-link from s to t satisfying* $\mathrm{ann}_r(q_i) \subseteq \nu_i$, $p_i \in \mathrm{Ann}(\nu_{i-1}, q_i, \nu_i)$ $(1 \leqq i \leqq n)$, *$sut\rho s$ and* $\nu_n = \{(h, k) : suh\rho suk\}$.

Proof. Assume first that S is coperfect. Since all coflat S-systems are weakly injective Corollary 3.4 gives that S is a principal right ideal monoid.

To show that S has a left zero, regard S as an S-system and consider the diagram,



where S^0 is S with a zero adjoined and τ, ι are inclusion mappings. By assumption, $S^{[2]}$ is injective and so there is an S-homomorphism $\psi : S^0 \to S^{[2]}$ which makes the diagram



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commute. For any $s \in S$,

$$\psi(0) = \psi(0s) = \psi(0)s$$

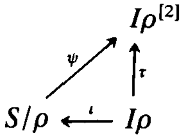
and so if $\psi(0) \in S$ it is immediate that S has a left zero. Otherwise, $\psi(0) \in S_n \backslash S_{n-1}$ for some $n \in \mathbb{N}$ and so $\psi(0)$ has the form $\psi(0) = [y_\sigma t]_{n-1}$, where $\{y_\delta : \delta \in \Sigma_{n-1}\}$ is the basis of F_{n-1}, $\sigma \in \Sigma_{n-1}$ and $t \in S$. Now $\sigma = (u, a)$ for some $u \in S$ and $a \in S_{n-1}$. If $t \in uS$, say $t = uv$, then

$$\psi(0) = [y_\sigma uv]_{n-1} = [av]_{n-1}$$

and so $\psi(0) \in S_{n-1}$, a contradiction. Thus $t \notin uS$.

For any $s \in S$, $\psi(0) = \psi(0)s$ gives $[y_\sigma ts]_{n-1} = [y_\sigma t]_{n-1}$ and as $t \notin uS$ one sees that $y_\sigma ts$, $y_\sigma t$ cannot be related by a $\rho(H_{n-1})$-sequence. Hence $y_\sigma ts = y_\sigma t$ so that $t = ts$ and t is a left zero of S.

Let $I = sS$ be a principal right ideal of S and let ρ be a right congruence on S. The S-system $I\rho = \{a\rho : a \in I\}$ is an S-subsystem of S/ρ and as $I\rho^{[2]}$ is injective there is an S-homomorphism $\psi : S/\rho \to I\rho^{[2]}$ which makes the diagram



commute.

For any $(h, k) \in \rho$ we have

$$\psi(1\rho)h = \psi((1\rho)h) = \psi(h\rho) = \psi(k\rho) = \psi((1\rho)k) = \psi(1\rho)k.$$

Further,

$$s\rho = \tau(s\rho) = \psi\iota(s\rho) = \psi(s\rho) = \psi((1\rho)s) = \psi(1\rho)s.$$

If $\psi(1\rho) \in I\rho$, it follows that there exists an element u of S such that $sus\rho s$ and for any $(h, k) \in \rho$, $suh\rho suk$. It is then easy to see that (CI) is satisfied, with $n = 1$, $p_1 = q_1 = 1$ and $r_1 = s$.

We now suppose that $\psi(1\rho) \in (I\rho)_n$, where $n > 0$. From the construction of $(I\rho)_n$ we have $\psi(1\rho) = [y_\sigma p_1]_{n-1}$ or $\psi(1\rho) = [m]_n$, where $\{y_\delta : \delta \in \Sigma_{n-1}\}$ is the basis of F_{n-1}, $\sigma \in \Sigma_{n-1}$, $p_1 \in S$ and $m \in (I\rho)_{n-1}$. In the latter case, $(1, m) \in \Sigma_{n-1}$ and so

$$\psi(1\rho) = [m]_n = [y_{(1,m)}]_n.$$

Thus we may assume that $\psi(1\rho)$ has the former expression.

If $h, k \in S$ and $h\rho k$ then $\psi(1\rho)h = \psi(1\rho)k$ and so $[y_\sigma p_1 h]_{n-1} = [y_\sigma p_1 k]_{n-1}$. Thus $p_1 h = p_1 k$ or $y_\sigma p_1 h$, $y_\sigma p_1 k$ are connected by a $\rho(H_{n-1})$-sequence

$$y_\sigma p_1 h = c_1 t_1,\ d_1 t_1 = c_2 t_2, \ldots, d_l t_l = y_\sigma p_1 k.$$

Now $\sigma \in \Sigma_{n-1}$ and so $\sigma = (q_1, m_1)$ for some $q_1 \in S$ and $m_1 \in (I\rho)_{n-1}$. It follows that $p_1 h = p_1 k$ or

$$p_1 h = q_1 t_1,\ m_1 t_1 \rho(H_{n-1}) m_1 t_l,\ q_1 t_l = p_1 k.$$

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But since m_1t_1, m_1t_l are $\rho(H_{n-1})$-related elements of $(I\rho)_{n-1}$, $m_1t_1 = m_1t_l$. Define the right congruence ν_1 on S by

$$\nu_1 = \mathrm{ann_r}(m_1).$$

Hence $p_1h = p_1k$ or

$$p_1h = q_1t_1,\ t_1\nu_1t_l,\ q_1t_l = p_1k$$

and so $p_1 \in \mathrm{Ann}(\nu_0, q_1, \nu_1)$, where $\nu_0 = \rho$. Further, if $(h, k) \in \mathrm{ann_r}(q_1)$, then $q_1h = q_1k$ so that $m_1h = m_1k$ (for $\sigma \in \Sigma_{n-1}$) and $(h, k) \in \nu_1$, thus $\mathrm{ann_r}(q_1) \subseteq \nu_1$.

Now $s\rho = [s\rho]_{n-1} = [y_\sigma p_1 s]_{n-1}$ and as $s\rho \neq y_\sigma p_1 s$ in $F_{n-1} \cup (I\rho)_{n-1}$ we have that $s\rho$, $y_\sigma p_1 s$ are connected by a $\rho(H_{n-1})$-sequence. This gives that $p_1s = q_1r_1$, $m_1r_1 = s\rho$ for some $r_1 \in S$.

One may express m_1 as $m_1 = [z_\mu p_2]_{n-2}$, where $\{z_\delta : \delta \in \Sigma_{n-2}\}$ is the basis of F_{n-2}, $p_2 \in S$ and $\mu = (q_2, m_2) \in \Sigma_{n-2}$. Again we define a right congruence ν_2 on S by

$$\nu_2 = \mathrm{ann_r}(m_2).$$

Suppose that $h\nu_1k$, that is, $m_1h = m_1k$. Hence $p_2h = p_2k$ or $z_\mu p_2h$, $z_\mu p_2k$ are related by a $\rho(H_{n-2})$-sequence. It follows that $p_2h = p_2k$ or there exist $t, t' \in S$ with $p_2h = q_2t$, $t\nu_2t'$, $q_2t' = p_2k$, that is, $p_2 \in \mathrm{Ann}(\nu_1, q_2, \nu_2)$. Since $(q_2, m_2) \in \Sigma_{n-2}$, it is clear that $\mathrm{ann_r}(q_2) \subseteq \nu_2$. Further, $[s\rho]_{n-2} = [z_\mu p_2 r_1]_{n-2}$ gives that $p_2r_1 = q_2r_2$, $m_2r_2 = s\rho$ for some $r_2 \in S$.

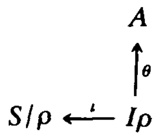
Clearly we may continue in this manner to obtain elements p_i, q_i, r_i of S and elements m_i of $(I\rho)_{i-1}$ $(1 \leqq i \leqq n)$, such that

$$p_1s = q_1r_1,\ p_{i+1}r_1 = q_{i+1}r_{i+1} \qquad (1 \leqq i \leqq n-1).$$

Further, defining $\nu_0 = \rho$ and $\nu_i = \mathrm{ann_r}(m_i)$, we have $\mathrm{ann_r}(q_i) \subseteq \nu_i$ and $p_i \in \mathrm{Ann}(\nu_{i-1}, q_i, \nu_i)$ $(1 \leqq i \leqq n)$. Also, $m_nr_n = s\rho$, where $m_n \in I\rho$. Thus there exists an element u of S with $m_n = su\rho$. This gives $s\rho = su\rho r_n = sur_n\rho$, that is, $s\rho sur_n$. Finally, for $h, k \in S$, $(h, k) \in \nu_n$ if and only if $m_nh = m_nk$, that is, $su\rho h = su\rho k$. Hence $(h, k) \in \nu_n$ if and only if $suh\rho suk$.

Thus S satisfies condition (CI).

Conversely, assume that S is a principal right ideal monoid with a left zero satisfying condition (CI). Let A be a coflat S-system. We show first that given any diagram of the form,



where $I = sS$ is a principal right ideal of S and $\theta : I\rho \to A$ is an S-homomorphism, there exists an S-homomorphism $\psi : S/\rho \to A$ such that $\psi i = \theta$.

Suppose that I, ρ and θ are given as above. By assumption there exist $n \in \mathbb{N}$, elements p_i, q_i, r_i of S and right congruences ν_i on S $(1 \leqq i \leqq n)$, satisfying the conditions of (CI).

Let $\phi_n : q_nS \to A$ be defined by

$$\phi_n(q_nt) = \theta(sut\rho).$$

Then ϕ_n is well-defined, for if $q_nt = q_nt'$, then $(t, t') \in \text{ann}_r(q_n)$ so that $(t, t') \in \nu_n$. Then the definition of ν_n gives $sut\rho sut'$. Clearly ϕ_n is an S-homomorphism and since A is coflat we may extend ϕ_n to an S-homomorphism $\bar{\phi}_n : S \to A$. Now define $\xi_n : S/\nu_{n-1} \to A$ by

$$\xi_n(t\nu_{n-1}) = \bar{\phi}_n(p_nt).$$

If $t\nu_{n-1}t'$, then as $p_n \in \text{Ann}(\nu_{n-1}, q_n, \nu_n)$, either (a) $p_nt = p_nt'$, or (b) $p_nt = q_nv$, $v\nu_nv'$, $q_nv' = p_nt'$ for some $v, v' \in S$.

If (a) holds, then clearly $\xi_n(t\nu_{n-1}) = \xi_n(t'\nu_{n-1})$. If (b) holds, by the definition of ν_n, $suv\rho suv'$ and so

$$\begin{aligned}\xi_n(t\nu_{n-1}) &= \bar{\phi}_n(p_nt) = \bar{\phi}_n(q_nv) = \phi_n(q_nv) = \theta(suv\rho) \\ &= \theta(suv'\rho) = \phi_n(q_nv') = \bar{\phi}_nq_nv') = \bar{\phi}_n(p_nt') = \xi_n(t'\nu_{n-1}).\end{aligned}$$

Thus ξ_n is well-defined and obviously is an S-homomorphism.

We now define $\phi_{n-1} : q_{n-1}S \to A$ by

$$\phi_{n-1}(q_{n-1}t) = \xi_n(t\nu_{n-1});$$

then, as $\text{ann}_r(q_{n-1}) \subseteq \nu_{n-1}$, ϕ_{n-1} is a well-defined S-homomorphism. Again using the coflatness of A, we may extend ϕ_{n-1} to an S-homomorphism $\bar{\phi}_{n-1} : S \to A$. We now use $\bar{\phi}_{n-1}$ to define an S-homomorphism $\xi_{n-1} : S/\nu_{n-2} \to A$ by putting

$$\xi_{n-1}(t\nu_{n-2}) = \bar{\phi}_{n-1}(p_{n-1}t).$$

To see that ξ_{n-1} is well-defined, suppose that $t\nu_{n-2}t'$. As above we have that either (a) $p_{n-1}t = p_{n-1}t'$ or (b) $p_{n-1}t' = q_{n-1}v$, $v\nu_{n-1}v'$, $q_{n-1}v' = p_{n-1}t'$ for some $v, v' \in S$. If (a) holds, it is immediate that $\xi_{n-1}(t\nu_{n-2}) = \xi_{n-1}(t'\nu_{n-2})$. If (b) holds, then

$$\begin{aligned}\xi_{n-1}(t\nu_{n-2}) &= \bar{\phi}_{n-1}(p_{n-1}t) = \bar{\phi}_{n-1}(q_{n-1}v) = \phi_{n-1}(q_{n-1}v) \\ &= \xi_n(v\nu_{n-1}) = \xi_n(v'\nu_{n-1}) = \phi_{n-1}(q_{n-1}v') = \bar{\phi}_{n-1}(q_{n-1}v') \\ &= \bar{\phi}_{n-1}(p_{n-1}t') = \xi_{n-1}(t'\nu_{n-2}).\end{aligned}$$

Clearly we may continue in this way to obtain S-homomorphisms $\phi_i : q_iS \to A$, $\bar{\phi}_i : S \to A$, $\xi_i : S/\nu_{i-1} \to A$ $(1 \leqq i \leqq n)$, such that

$$\phi_n(q_nt) = \theta(sut\rho),$$

$$\phi_i(q_it) = \xi_{i+1}(t\nu_i) \qquad (1 \leqq i \leqq n-1)$$

and for $i \in \{1, \ldots, n\}$, $\bar{\phi}_i$ is an S-homomorphism extending ϕ_i and

$$\xi_i(t\nu_{i-1}) = \bar{\phi}_i(p_it).$$

Thus we obtain an S-homomorphism $\psi = \xi_1 : S/\nu_0 = S/\rho \to A$. It remains to show that ψ extends θ.

We have $\psi\iota(s\rho) = \psi(s\rho) = \xi_1(s\rho) = \xi_1(sv_0) = \bar{\phi}_1(p_1 s) = \bar{\phi}_1(q_1 r_1) = \phi_1(q_1 r_1) = \xi_2(r_1 v_1) = \bar{\phi}_2(p_2 r_1) = \bar{\phi}_2(q_2 r_2) = \phi_2(q_2 r_2) = \xi_3(r_2 v_2) = \ldots = \xi_n(r_{n-1} v_{n-1}) = \bar{\phi}_n(p_n r_{n-1}) = \bar{\phi}_n(q_n r_n) = \phi_n(q_n r_n) = \theta(sur_n\rho) = \theta(s\rho)$. Hence for any $st \in I$, $\psi\iota(st\rho) = \theta(st\rho)$, that is, $\psi\iota = \theta$.

Now suppose that N is an S-subsystem of an S-system M and $\phi: N \to A$ is an S-homomorphism. Consider the partially ordered set whose members are pairs (N', ϕ'), where N' is an S-subsystem of M containing N and $\phi': N' \to A$ is an S-homomorphism extending ϕ and $\leqq$ is defined by

$$(N', \phi') \leqq (N'', \phi'') \quad \text{if and only if } N' \subseteq N'' \text{ and } \phi'' \text{ extends } \phi'.$$

By Zorn's lemma, this set has a maximal member, say (P, θ). If $P \neq M$, choose $m \in M \setminus P$ and put $I = \{s \in S : ms \in P\}$.

If $I = \varnothing$, then $mS \cap P = \varnothing$ and we may define a function $\xi: mS \cup P \to A$ by

$$\xi(ms) = as_0,$$
$$\xi(p) = \theta(p) \qquad (p \in P),$$

where s_0 is a left zero of S and a is a fixed element of A. We have

$$\xi(mst) = as_0 = as_0 t = \xi(ms)t$$

and it follows that ξ is an S-homomorphism strictly extending θ, that is, $(P, \theta) < (mS \cup P, \xi)$, contradicting the maximality of (P, θ). Thus $I \neq \varnothing$ and it follows that I is a principal right ideal of S, say $I = sS$.

Define a right congruence ρ on S by

$$h\rho k \quad \text{if and only if} \quad mh = mk,$$

that is, $\rho = \operatorname{ann}_r(m)$. Let $\lambda: I\rho \to A$ be defined by $\lambda(st\rho) = \theta(mst)$. Since $\rho = \operatorname{ann}_r(m)$, it is clear that λ is a well-defined S-homomorphism. Hence there is an S-homomorphism $\mu: S/\rho \to A$ which extends λ. Now define $\psi: mS \cup P \to A$ by

$$\psi(mt) = \mu(t\rho),$$
$$\psi(p) = \theta(p) \qquad (p \in P).$$

If $mt = mt'$, then $t\rho t'$ so that $\psi(mt) = \psi(mt')$. If $mt = p$ for some $p \in P$, then $t \in I$ and so $t = st'$ for some $t' \in S$. Thus

$$\psi(mt) = \mu(t\rho) = \mu(st'\rho) = \lambda(st'\rho) = \theta(mst') = \theta(mt) = \theta(p) = \psi(p)$$

and so ψ is a well-defined S-homomorphism. But $(P, \theta) < (mS \cup P, \psi)$, a contradiction. Hence $P = M$ and A is injective. Since A is an arbitrary coflat S-system, the monoid S is coperfect.

To establish our next corollary we need a technical lemma.

LEMMA 4.2. *Let $I = sS$ be a principal right ideal of the monoid S and ρ a right congruence on S. Suppose that $n \in \mathbb{N}$ and there exists elements p_i, q_i, r_i of S and right*

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congruences ν_i *on* S $(1 \leqq i \leqq n)$ *satisfying the conditions of* (*CI*). *Suppose further that* q_i *is regular for* $i \in \{1, \ldots, n\}$. *Then there exists an element* x *of* S *such that if* $h,k \in S$ *and* $h\rho k$, *then* $suxh\rho suxk$ *and further*, $st\rho suxst$, *for any* $st \in I$.

Proof. Let $i \in \{1, \ldots, n\}$. We show that for any $h,k \in S$,

$$h\nu_{i-1}k \Rightarrow q_i'p_ih\nu_iq_i'p_ik,$$

where $q_iq_i'q_i = q_i$.

Given $q_iq_i'q_i = q_i$, $(q_iq_i', 1) \in \mathrm{ann}_r(q_i)$ and so $q_i'q_i\nu_i1$. Now since $p_i \in \mathrm{Ann}(\nu_{i-1}, q_i, \nu_i)$, either (a) $p_ih = p_ik$ or (b) $p_ih = q_ih'$, $h'\nu_ik'$, $q_ik' = p_ik$ for some $h', k' \in S$.

If (a) holds, then $q_i'p_ih = q_i'p_ik$ and so certainly $q_i'p_ih\nu_iq_i'p_ik$. If (b) holds, then

$$q_i'p_ih = q_i'q_ih'\nu_ih'\nu_ik'\nu_iq_i'q_ik' = q_i'p_ik$$

and so our claim is correct. It follows that if $h\rho k$ then $xh\nu_nxk$, where $x = q_n'p_nq_{n-1}'p_{n-1} \cdots q_1'p_1$. Hence if $h\rho k$, then $suxh\rho suxk$.

Now $s\rho s$, that is, $s\nu_0s$, so $q_1'p_1s\nu_1q_1'p_1s$, which gives $q_1'q_1r_1\nu_1q_1'p_1s$. But $1\nu_1q_1'q_1$, so that $r_1\nu_1q_1'q_1r_1$, hence $r_1\nu_1q_1'p_1s$. Thus $q_2'p_2r_1\nu_2q_2'p_2q_1'p_1s$ and so $q_2'q_2r_2\nu_2q_2'p_2q_1'p_1s$, giving $r_2\nu_2q_2'p_2q_1'p_1s$. Clearly we may continue in this manner to obtain $r_n\nu_nxs$. Thus $sur_n\rho suxs$, hence $s\rho suxs$ and so for any $st \in I$, $st\rho suxst$.

If all S-systems are injective, then S is a *completely right injective* monoid. We may now deduce the following result which appears in **[4]**, **[8]** and **[11]**.

COROLLARY 4.3. *The monoid* S *is completely right injective if and only if*

(a) *S has a left zero, and*

(b) *for any right ideal* I *of* S *and right congruence* ρ *on* S, *there is an element* y *of* I *such that for any* $t \in I$, $yt\rho t$ *and for any* h, $k \in S$ *with* $h\rho k$, $yh\rho yk$.

Proof. If S is completely right injective, then clearly all coflat S-systems are injective. Thus S has a left zero, all right ideals of S are principal and S satisfies condition (CI). Further, all S-systems are coflat and so by Proposition 4.1 of **[6]**, S is regular.

Let I be a right ideal of S and ρ a right congruence on S. Then $I = sS$ for some $s \in S$ and since S is regular and satisfies (CI), it follows from Lemma 4.2 that there is an element x of S such that $h\rho k$ implies $suxh\rho suxk$ and $t\rho suxt$ for any $t \in I$. Putting $y = sux$, we see that (b) holds.

Conversely, suppose that S satisfies (a) and (b). Let I be a right ideal of S and ρ the equality relation on S. Then there is an element y of I with $ys = s$ for any $s \in I$. Hence

$$I = yI \subseteq yS \subseteq I,$$

so that $I = yS$ and I is principal.

As in the proof of Proposition 4.1, S satisfies condition (CI). Thus all coflat S-systems are injective.

Let $s \in S$. Then as above there is an element y of sS with $ys = s$; hence s is a regular element and so S is a regular monoid. Thus all S-systems are coflat and hence injective, that is, S is a completely right injective monoid.

We end this section by using Proposition 4.1 to give an example of a coperfect monoid that is not completely right injective.

COROLLARY 4.4. *Let S be the infinite cyclic monoid generated by the element a. Then S^0 is a coperfect monoid which is not completely right injective.*

Proof. Since the only regular elements of S^0 are 0 and 1 ($=a^0$), S^0 is not a regular monoid and so, by Proposition 4.1 of [**6**], not all S-systems are coflat. Hence S^0 is certainly not completely right injective.

The monoid S^0 is commutative and is a principal ideal monoid. Further, S^0 is 0-cancellative and has no zero-divisors.

Let $s \in S^0$ and let ρ be a congruence on S^0. If $s=0$, take $n=1$ and put $p_1=q_1=u=1$ and $r_1=0$. Then $p_1s=q_1r_1$ and $sur_1=0$ so that $sur_1\rho s$. Further, $(h, k)\in \mathrm{ann}_r(q_1)$ if and only if $h=k$ and so $\mathrm{ann}_r(q_1)$ is contained in every congruence on S. Let $\nu_1=\{(h, k): suh\rho suk\}$; as $s=0$ we have that ν_1 is the trivial congruence $S^0\times S^0$. If $h, k\in S^0$ and $h\rho k$, then $1h=1h$, $h\nu_1 k$, $1k=1k$ and so $1\in \mathrm{Ann}(\rho, 1, \nu_1)$.

We now suppose that $s\neq 0$. If $\rho=I_{S^0}$, the identity relation on S^0, then we again take $n=1$ and put $p_1=r_1=u=1$ and $q_1=s$. Letting $\nu_1=\{(h, k): sh\rho sk\}$, we have $\nu_1=\{(h, k): sh=sk\}=I_{S^0}$. Now $p_1s=q_1r_1$ and $sur_1\rho s$. Also, $\mathrm{ann}_r(q_1)=\mathrm{ann}_r(s)=I_{S^0}$ and so $\mathrm{ann}_r(q_1)\subseteq \nu_1$. Since $\rho=I_{S^0}$, it is clear that $1\in \mathrm{Ann}(\rho, s, \nu_1)$.

If $\rho\neq I_{S^0}$, we may choose an element t of S^0 such that there is an element z of S^0 with $t\rho z$, $t\neq z$ and tS^0 is the maximal ideal with this property. Clearly $t\neq 0$. If $z=0$, then $t\rho 0$ so that $t^2\rho 0\rho t$ and $t\rho t^2$. Now $t=t^2$ if and only if $t=1$. If $t=1$, then $1\rho 0$ and so $b\rho 0$ for all elements b of S^0. This gives that ρ is trivial. However, if ρ is trivial, then putting $n=1$, $p_1=q_1=r_1=u=0$ and $\nu_1=S^0\times S^0$, it is easy for us to see that the conditions of (CI) are satisfied.

Thus we may assume that $\rho\neq I_{S^0}$, $\rho\neq S^0\times S^0$ and there exist non-zero elements t, z of S^0 such that $t\rho z$, $t\neq z$ and tS^0 is maximal with respect to this property.

Since S^0 is a principal ideal monoid, either $tS^0\subseteq sS^0$, or $sS^0\subseteq tS^0$. Suppose firstly that $tS^0\subseteq sS^0$. Take $n=1$ and put $p_1=r_1=u=1$ and $q_1=s$. Then $p_1s=q_1r_1$ and $sur_1\rho s$. Let $\nu_1=\{(h, k): sh\rho sk\}$. Then $\mathrm{ann}_r(q_1)=\mathrm{ann}_r(s)=I_{S^0}$ and so $\mathrm{ann}_r(q_1)\subseteq \nu_1$. It remains to prove that $1\in \mathrm{Ann}(\rho, s, \nu_1)$. Let $v, v'\in S^0$ and suppose that $v\rho v'$. If $v=v'$, then clearly $1v=1v'$. If $v\neq v'$, then $v, v'\in tS^0$ and so $v=sh$, $v'=sk$ for some $h, k\in S^0$. Then from $sh\rho sk$ we have that $h\nu_1 k$ and so $1\in \mathrm{Ann}(\rho, s, \nu_1)$, as required.

Assume now that $sS^0\subseteq tS^0$. We know that there are natural numbers c, d, e with $t=a^c$, $z=a^d$, $d=c+e$ and $e>0$. Then $t\rho a^{c+me}$ for all $m\in \mathbb{N}$ and so we may choose an element w of S such that $wS^0\subseteq sS^0\subseteq tS^0$ and $t\rho w$.

Let y, k be the elements of S with $s=ty$, $w=sk$. Then $s\rho wy$ and $wy=sky$. Take $n=2$ and put $u=1$,

$$p_1=1,\ q_1=t,\ r_1=y,$$

$$p_2=w,\ q_2=s,\ r_2=ky,$$

$$\nu_1=\{(h, h'): th\rho th'\},$$

$$\nu_2=\{(h, h'): sh\rho sh'\}.$$

Then $p_1s=s=ty=q_1r_1$, $p_2r_1=wy=sky=q_2r_2$ and $sur_2=sky=wy\rho s$. Since q_1, q_2 are non-zero, $\mathrm{ann}_r(q_1)\subseteq \nu_1$ and $\mathrm{ann}_r(q_2)\subseteq \nu_2$.

If $v, v' \in S^0$, $v\rho v'$ and $v \neq v'$, then $v = th$, $v' = th'$ for some $h, h' \in S^0$. Thus $th\rho th'$ and so hv_1h', which gives that $1 \in \mathrm{Ann}(\rho, t, v_1)$, that is, $p_1 \in \mathrm{Ann}(v_0, q_0, v_1)$.

Finally, if v, v' are elements of S^0 such that vv_1v', then $tv\rho tv'$ and so $wv\rho wv'$ as $wS^0 \subseteq tS^0$. Now $wv = skv$ and $wv' = skv'$, giving $skv\rho skv'$ and kvv_2kv'. Thus $w \in \mathrm{Ann}(v_1, s, v_2)$, that is, $p_2 \in \mathrm{Ann}(v_1, q_2, v_2)$. This completes the proof that S^0 satisfies condition (CI).

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