J. Aust. Math. Soc. **105** (2018), 257–288
doi:10.1017/S1446788717000349

ON PSEUDO-EHRESMANN SEMIGROUPS

SHOUFENG WANG

(Received 28 March 2016; accepted 9 November 2017; first published online 18 June 2018)

Communicated by M. Jackson

Abstract

As generalizations of inverse semigroups, Ehresmann semigroups are introduced by Lawson and investigated by many authors extensively in the literature. In particular, Lawson has proved that the category of Ehresmann semigroups and admissible morphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors, which generalizes the well-known Ehresmann–Schein–Nambooripad (ESN) theorem for inverse semigroups. From a varietal perspective, Ehresmann semigroups are derived from reducts of inverse semigroups. In this paper, inspired by the approach of Jones ['A common framework for restriction semigroups and regular $*$-semigroups', *J. Pure Appl. Algebra* **216** (2012), 618–632], Ehresmann semigroups are extended from a varietal perspective to *pseudo-Ehresmann semigroups* derived instead from reducts of regular semigroups with a multiplicative inverse transversal. Furthermore, motivated by the method used by Gould and Wang ['Beyond orthodox semigroups', *J. Algebra* **368** (2012), 209–230], we introduce the notion of *inductive pseudocategories over admissible quadruples* by which pseudo-Ehresmann semigroups are described. More precisely, we show that the category of pseudo-Ehresmann semigroups and (2,1,1,1)-morphisms is isomorphic to the category of inductive pseudocategories over admissible quadruples and pseudofunctors. Our work not only generalizes the result of Lawson for Ehresmann semigroups but also produces a new approach to characterize regular semigroups with a multiplicative inverse transversal.

2010 *Mathematics subject classification*: primary 20M10.

Keywords and phrases: pseudo-Ehresmann semigroup, inductive pseudocategory, admissible quadruple, Ehresmann semigroup, multiplicative inverse transversal.

1. Introduction

Let S be a semigroup. We denote the set of all idempotents of S by $E(S)$ and the set of all inverses of $x \in S$ by $V(x)$. Recall that

$$V(x) = \{a \in S \mid xax = x, axa = a\}$$

for all $x \in S$. A semigroup S is called *regular* if $V(x) \neq \emptyset$ for any $x \in S$, and a regular semigroup S is called *inverse* if $E(S)$ is a commutative subsemigroup (that is,

The author is supported by the National Natural Science Foundations of China (11661082,11301470).

a subsemilattice) of S, or, equivalently, the cardinal of $V(x)$ is equal to 1 for all $x \in S$. On inverse semigroups, we have the following Ehresmann–Schein–Nambooripad or ESN theorem, due to its varied authorship.

THEOREM 1.1 (ESN theorem; see Lawson [21]). *The category of inverse semigroups and morphisms is isomorphic to the category of inductive groupoids and inductive functors.*

The above ESN theorem have been extended in various directions by many authors. Nambooripad [23] introduced regular biordered sets and obtained a generalization of Theorem 1.1 from a semilattice to a regular biordered set which established the structure of regular semigroups. Meanwhile, Meakin [22] characterized regular semigroups by using structure mappings. In 1988, Nambooripad's result was extended by Armstrong [1] from regular to concordant semigroups. On the other hand, Theorem 1.1 was generalized in a different direction to Ehresmann semigroups by Lawson [20] (see section 2 for a definition). Here is Lawson's result on Ehresmann semigroups.

THEOREM 1.2 (Lawson [20]). *The category of Ehresmann semigroups and admissible morphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors.*

It is worth remarking that the class of Ehresmann semigroups and its subclasses are investigated extensively in the literature by many authors (see [6, 8, 9, 18, 19], for example). In particular, Jones [18] provided a common framework for Ehresmann semigroups and regular *-semigroups from a varietal perspective. More recent developments in this area can be found in good survey articles by Gould [10, 11] and Hollings [14–16].

The results of Lawson and Armstrong have been further generalized in recent years by several authors. In particular, by introducing the notion of generalized categories over bands, Gould–Wang [12] went a step further to extend Lawson's result on Ehresmann semigroups to the class of weakly B-orthodox semigroups which extend the class of Ehresmann semigroups by replacing semilattices by bands. In 2016, using the notion of so-called regular biordered sets categories over regular biordered sets, Wang [25] generalized Armstrong's work to weakly U-regular semigroups, which is a wide class containing all regular semigroups and all abundant semigroups with a regular biordered set of idempotents.

On the other hand, Blyth–McFadden [5] introduced the concept of inverse transversals for regular semigroups. A subsemigroup S° of a regular semigroup S is called an *inverse transversal* of S if $V(x) \cap S^\circ$ contains exactly one element for all $x \in S$. Clearly, in this case, S° is an inverse subsemigroup of S. Since an inverse semigroup can be regarded as an inverse transversal of itself, the class of regular semigroups with inverse transversals contains the class of inverse semigroups as a proper subclass. Regular semigroups with inverse transversals are investigated extensively by many authors (see [2–4, 24] and their references) and some generalizations of inverse transversals are proposed (see [7, 13, 26], for example).

Inspired by the approach used in Jones [18], in this paper, a common framework, termed *pseudo-Ehresmann semigroups*, for Ehresmann semigroups and regular semigroups with a multiplicative inverse transversal is introduced from a varietal perspective and some properties of this class of semigroups are explored. Moreover, motivated by the method in Gould–Wang [12], we introduce *pseudocategories over admissible quadruples* and extend Lawson's result on Ehresmann semigroups to pseudo-Ehresmann semigroups by using pseudocategories; see Theorem 5.7 below. Our work also produces a new approach to characterize regular semigroups with a multiplicative inverse transversal. It is worth pointing out that the class of pseudo-Ehresmann semigroups considered in this paper is different from the class of weakly B-orthodox semigroups studied in Gould–Wang [12] (see Example 2.8 in this paper).

2. Pseudo-Ehresmann semigroups

In this section, after giving some preliminaries on Ehresmann semigroups and inverse transversals, we introduce pseudo-Ehresmann semigroups and consider some basic properties of this class of semigroups. Firstly, we consider Ehresmann semigroups. Let S be a semigroup and let $E \subseteq E(S)$. The relation $\widetilde{\mathcal{R}}_E$ is defined on S by the rule that for any $x, y \in S$, we have $x\widetilde{\mathcal{R}}_E y$ if

$$ex = x \quad \text{if and only if} \quad ey = y \text{ for all } e \in E.$$

Dually, we have the relation $\widetilde{\mathcal{L}}_E$ on S. Observe that both $\widetilde{\mathcal{R}}_E$ and $\widetilde{\mathcal{L}}_E$ are equivalences on S but $\widetilde{\mathcal{R}}_E$ (respectively, $\widetilde{\mathcal{L}}_E$) may not be a left congruence (respectively, a right congruence). Recall that a *band* is a semigroup in which every element is idempotent and a *semilattice* is a commutative band. From Gould–Wang [12], (S, E) is called a *weakly E-orthodox semigroup* if:

(i) E is a subband of S;
(ii) every $\widetilde{\mathcal{R}}_E$-class contains an element of E and $\widetilde{\mathcal{R}}_E$ is a left congruence; and
(iii) every $\widetilde{\mathcal{L}}_E$-class contains an element of E and $\widetilde{\mathcal{L}}_E$ is a right congruence.

If this is the case, E is called *the distinguished band* of S. In view of Lawson [20], a weakly E-orthodox semigroup (S, E) is called *E-Ehresmann* if E is a subsemilattice of S and, in this case, E is called the *distinguished semilattice* of S. We say that a semigroup S is *Ehresmann* in the subsequent work if (S, E) is an E-Ehresmann semigroup for some $E \subseteq E(S)$. From Lemma 2.2 and its dual in Gould [11], we have the following characterization of Ehresmann semigroups from a varietal perspective.

LEMMA 2.1. *A semigroup $(S, \cdot)$ is Ehresmann if and only if there are two unary operations '$+$' and '$*$' on S such that the following identities hold.*

$$x^+x = x, x^+y^+ = y^+x^+, (x^+y^+)^+ = x^+y^+, (xy)^+ = (xy^+)^+;$$
$$xx^* = x, x^*y^* = y^*x^*, (x^*y^*)^* = x^*y^*, (xy)^* = (x^*y)^*; (x^+)^* = x^+, (x^*)^+ = x^*.$$

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

In this case, S is an Ehresmann semigroup with distinguished semilattice $\{x^+ \mid x \in S\}$ (or, equivalently, $\{x^* \mid x \in S\}$), and we also call $\mathbf{S} = (S, \cdot, +, *)$ an Ehresmann semigroup.

Now we consider regular semigroups with an inverse transversal. Recall that an inverse subsemigroup S° of a regular semigroup S is called an *inverse transversal* of S if every element $x \in S$ has exactly one inverse in S°. In this case, for any $x \in S$, we use x° to denote the unique inverse of x in S° and let $x^{\circ\circ} = (x^\circ)^\circ$. Observe that

$$x^{\circ\circ\circ} = x^\circ, \quad S^\circ = \{x^\circ \mid x \in S\} \quad \text{and} \quad x^{\circ\circ}x^\circ, x^\circ x^{\circ\circ} \in E(S^\circ).$$

On inverse transversals of regular semigroups, we have the following known results.

Lemma 2.2 (Blyth–Almeida Santos [4]). *Let S be a regular semigroup with an inverse transversal S°. Then:*

(i) $(xy)^\circ = y^\circ(x^\circ xyy^\circ)^\circ x^\circ$ *for all $x, y \in S$; and*
(ii) $(xy)^\circ = y^\circ x^\circ$ *for all $x, y \in S$ with $\{x, y\} \cap S^\circ \neq \emptyset$.*

Here we are only interested in the case when S° is *multiplicative* in the sense that

$$x^\circ xyy^\circ \in E(S^\circ) \quad \text{for all } x, y \in S.$$

For this kind of inverse transversal, we have the following lemma.

Lemma 2.3. *If S° is a multiplicative inverse transversal of a regular semigroup S, then $xyz = xy^{\circ\circ}z$ for all $x, z \in S^\circ$ and $y \in S$.*

Proof. Since S° is multiplicative, it follows that

$$xyz = x(x^\circ xyy^\circ)y^{\circ\circ}(y^\circ yzz^\circ)z \in S^\circ,$$

which implies that $xyz = (xyz)^{\circ\circ} = xy^{\circ\circ}z$ by using Lemma 2.2(ii). □

Let S be a regular semigroup with an inverse transversal S°. Then we can consider the induced triunary semigroup $(S, \cdot, +, *, -)$, where

$$x^+ = xx^\circ, \quad x^* = x^\circ x \quad \text{and} \quad \overline{x} = x^{\circ\circ}.$$

Proposition 2.4. *Let S be a regular semigroup with a multiplicative inverse transversal S°. The triunary semigroup $\mathbf{S} = (S, \cdot, +, *, -)$ satisfies the identities given in Table 1.*

Proof. By symmetry, we only need to show (1)–(10). Identity (1) is equivalent to the equality $(xx^\circ)x = x$. Using Lemma 2.3,

$$\begin{aligned} x^+y^+x^+ &= xx^\circ yy^\circ xx^\circ = xx^\circ y^{\circ\circ}y^\circ x^{\circ\circ}x^\circ \\ &= xx^\circ x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ = xx^\circ y^{\circ\circ}y^\circ = xx^\circ yy^\circ = x^+y^+, \end{aligned}$$

which gives (2). Using Lemma 2.3 and Lemma 2.2(ii),

$$\begin{aligned} (x^+y^+)^+ &= x^+y^+(x^+y^+)^\circ = (xx^\circ yy^\circ)(xx^\circ yy^\circ)^\circ \\ &= (xx^\circ y^{\circ\circ}y^\circ)(xx^\circ y^{\circ\circ}y^\circ)^\circ = xx^\circ y^{\circ\circ}y^\circ y^{\circ\circ}y^\circ x^{\circ\circ}x^\circ \\ &= xx^\circ y^{\circ\circ}y^\circ x^{\circ\circ}x^\circ = xx^\circ x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ = xx^\circ y^{\circ\circ}y^\circ = xx^\circ yy^\circ = x^+y^+. \end{aligned}$$

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

Table 1. Pseudo-Ehresmann conditions.

(1)	$x^+x = x$	(1)′	$xx^* = x$
(2)	$x^+y^+x^+ = x^+y^+$	(2)′	$x^*y^*x^* = y^*x^*$
(3)	$(x^+y^+)^+ = x^+y^+$	(3)′	$(x^*y^*)^* = x^*y^*$
(4)	$(xy)^+ = (xy^+)^+$	(4)′	$(xy)^* = (x^*y)^*$
(5)	$(x^+)^* = \overline{x}^+$	(5)′	$(x^*)^+ = \overline{x}^*$
(6)	$\overline{x}^+ = \overline{x^+}$	(6)′	$\overline{x}^* = \overline{x^*}$
(7)	$x = x^+\overline{x}x^*$	(8)	$\overline{x} = \overline{x}^+x\overline{x}^*$
(9)	$(x^*y^+)^+ = x^*y^+$	(10)	$(x^*y^+)^* = x^*y^+$

This shows (3). Moreover, using Lemma 2.2(ii) and Lemma 2.3 again,

$$(xy^+)^+ = (xyy^\circ)(xyy^\circ)^\circ = xyy^\circ y^{\circ\circ}(xy)^\circ = xyy^\circ y(xy)^\circ = xy(xy)^\circ = (xy)^+,$$

which shows that (4) is true. On the other hand, by Lemma 2.2(ii),

$$(x^+)^* = (xx^\circ)^\circ(xx^\circ) = x^{\circ\circ}x^\circ xx^\circ = x^{\circ\circ}x^\circ = x^{\circ\circ}x^{\circ\circ\circ} = \overline{x}^+$$

and

$$\overline{x^+} = (xx^\circ)^{\circ\circ} = x^{\circ\circ}x^\circ = x^{\circ\circ}(x^{\circ\circ})^\circ = \overline{x}^+.$$

This deduces the identities (5) and (6). The identity (7) is equivalent to the statement $x = (xx^\circ)x^{\circ\circ}(x^\circ x)$ and (8) is equivalent to the statement $x^{\circ\circ} = (x^{\circ\circ}x^\circ)x(x^\circ x^{\circ\circ})$. The identities (9) and (10) follow from the fact that S° is multiplicative. □

We shall term any triunary semigroup $\mathbf{S} = (S, \cdot, +, *, -)$ that satisfies the identities in Table 1 a *pseudo-Ehresmann semigroup*. By Proposition 2.4, any regular semigroup S with a multiplicative inverse transversal S° induces the pseudo-Ehresmann semigroup $\mathbf{S} = (S, \cdot, +, *, -)$ by setting $x^+ = xx^\circ$, $x^* = x^\circ x$ and $\overline{x} = x^{\circ\circ}$. The following example gives a very special case of this kind of pseudo-Ehresmann semigroups.

Example 2.5. Let S be a rectangular band. Fix an element u in S. Consider the triunary semigroup $\mathbf{S} = (S, \cdot, +, *, -)$ where

$$x^+ = xu, \quad x^* = ux, \quad \overline{x} = u \quad \text{for all } x \in S.$$

Then it is routine to check that the identities in Table 1 are satisfied and so $\mathbf{S} = (S, \cdot, +, *, -)$ is a pseudo-Ehresmann semigroup. In fact, $\{u\}$ is a multiplicative inverse transversal of S.

Example 2.6. Any Ehresmann semigroup $(S, \cdot, +, *)$ also induces a pseudo-Ehresmann semigroup, which justifies our term '*pseudo-Ehresmann semigroups*'. In fact, for an Ehresmann semigroup $(S, \cdot, +, *)$, we define the third unary operation '$-$' on S by $\overline{x} = x$ for all $x \in S$. Then we have triunary semigroup $\mathbf{S} = (S, \cdot, +, *, -)$ and it is easy to see that the identities in Table 1 are all satisfied by Lemma 2.1.

Since a rectangular band having more than one element must not be an Ehresmann semigroup, the class of pseudo-Ehresmann semigroups contains the class

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

of Ehresmann semigroups and the class of rectangular bands as proper subclasses by the above two examples. We observe that a pseudo-Ehresmann semigroup which is also regular may not contain any inverse transversal. In fact, any monoid S with the identity 1 is always a (pseudo-)Ehresmann semigroup by setting

$$x^+ = x^* = 1 \quad \text{and} \quad \overline{x} = x \text{ for all } x \in S.$$

Obviously, a regular monoid may not contain any inverse transversal. Here is an example.

EXAMPLE 2.7. Let $M = \{1, b, c, x\}$ (taken from [17, Exercise 10, Ch. VI]) with the multiplication

M	1	b	c	x
1	1	b	c	x
b	b	b	b	b
c	c	c	c	c
x	x	c	b	1.

Then M is monoid and

$$E(M) = \{1, b, c\}, \quad V(1) = \{1\}, \quad V(b) = \{b, c\} = V(c), \quad V(x) = \{x\}.$$

It is easy to check that M contains no inverse transversal.

The following example shows that there is a pseudo-Ehresmann semigroup S which is not a weakly E-orthodox semigroup for any $E \subseteq E(S)$.

EXAMPLE 2.8. Consider the semigroup S with the Cayley table

S	0	e	f	g	a
0	0	0	0	0	0
e	0	e	f	e	f
f	0	e	f	0	0
g	0	g	a	g	a
a	0	g	a	0	0.

In view of the fact that $E(S) = \{0, e, f, g\}$, it is routine to check that S is not a weakly E-orthodox semigroup for any $E \subseteq E(S)$. However, we can see that $\{0, e\}$ is a multiplicative inverse transversal of S and so S is a pseudo-Ehresmann semigroup by Proposition 2.4.

Now, we consider some properties associated with pseudo-Ehresmann semigroups which will be used in the next sections. Let $\mathbf{S} = (S, \cdot, +, *, -)$ be a pseudo-Ehresmann semigroup. Denote

$$I_{\mathbf{S}} = \{x^+ \mid x \in S\}, \quad \Lambda_{\mathbf{S}} = \{x^* \mid x \in S\}, \quad E^{\circ}_{\mathbf{S}} = \{\overline{x}^+ \mid x \in S\}.$$

Recall that a band B is *left normal* (respectively, *right normal*) if $efg = egf$ (respectively, $efg = feg$) for all $e, f, g \in B$.

LEMMA 2.9. *Let* $\mathbf{S} = (S, \cdot, +, *, -)$ *be a pseudo-Ehresmann semigroup.*

(a) $e^+ = e, e^* = \overline{e}$ *and* $f^* = f, f^+ = \overline{f}$ *for all* $e \in I_{\mathbf{S}}$ *and* $f \in \Lambda_{\mathbf{S}}$*, respectively.*

(b) $E^\circ_{\mathbf{S}} = \{\overline{x}^* \mid x \in S\} = I_{\mathbf{S}} \cap \Lambda_{\mathbf{S}}$*, and so* $E^\circ_{\mathbf{S}}$ *is a subsemilattice of* S *and* $x^*y^+ \in E^\circ_{\mathbf{S}}$ *for all* $x, y \in S$ *(or, equivalently,* $fg \in E^\circ_{\mathbf{S}}$ *for all* $f \in \Lambda_{\mathbf{S}}$ *and* $g \in I_{\mathbf{S}}$*).*

(c) $(xy)^+ = x^+(\overline{x}x^*y^+)^+$ *and* $(xy)^* = (x^*y^+\overline{y})^*y^*$.

(d) $x^+\mathcal{L}\overline{x}^+$ *and* $x^*\mathcal{R}\overline{x}^*$.

(e) $I_{\mathbf{S}}$ *is a left normal band and* $\Lambda_{\mathbf{S}}$ *is a right normal band, respectively.*

PROOF. (a) Using identities (1), (4) and (3) in Table 1,

$$x^+ = (x^+x)^+ = (x^+x^+)^+ = x^+x^+$$

and so $x^+ \in E(S)$. Take $e \in I_{\mathbf{S}}$. Then $e = x^+$ for some $x \in S$. This implies that

$$e^+ = (x^+)^+ = (x^+x^+)^+ = x^+x^+ = x^+$$

by $x^+ \in E(S)$ and the identity (3). Moreover,

$$e^* = (x^+)^* = \overline{x}^+ = \overline{x^+} = \overline{e}$$

by the identities (5) and (6). Dually, $f^* = f$ and $f^+ = \overline{f}$ for all $f \in \Lambda_{\mathbf{S}}$.

(b) By identity (5), $\overline{x}^+ = (x^+)^* \in I_{\mathbf{S}} \cap \Lambda_{\mathbf{S}}$ for all $x \in S$. Now let $u = x^+ \in I_{\mathbf{S}} \cap \Lambda_{\mathbf{S}}$ for some $x \in S$. Using item (a) and the identity (5), we have $u = u^* = (x^+)^* = \overline{x}^+ \in E^\circ_{\mathbf{S}}$. Thus $E^\circ_{\mathbf{S}} = I_{\mathbf{S}} \cap \Lambda_{\mathbf{S}}$. Dually, $\{\overline{x}^* \mid x \in S\} = I_{\mathbf{S}} \cap \Lambda_{\mathbf{S}}$. By the proof of item (a) in the current lemma, we can see that every element in $I_{\mathbf{S}}$ is idempotent. Moreover, by the identity (3), $x^+y^+ = (x^+y^+)^+ \in I_{\mathbf{S}}$ for all $x^+, y^+ \in I_{\mathbf{S}}$. So $I_{\mathbf{S}}$ is a subband of S. Dually, $\Lambda_{\mathbf{S}}$ is also a subband of S. In view of the identities (2) and (2)′, $E^\circ_{\mathbf{S}}$ is a semilattice. Finally, the identities (9) and (10) give that $x^*y^+ \in I_{\mathbf{S}} \cap \Lambda_{\mathbf{S}} = E^\circ_{\mathbf{S}}$ for all $x, y \in S$.

(c) Using (4), (7), (4) and (3) in Table 1 one by one,

$$(xy)^+ = (xy^+)^+ = (x^+\overline{x}x^*y^+)^+ = (x^+(\overline{x}\,x^*y^+)^+)^+ = x^+(\overline{x}\,x^*y^+)^+.$$

Dually, $(xy)^* = (x^*y^+\overline{y})^*y^*$.

(d) Using identities (7), (4), (3), (4), (5)′ and (1)′ in Table I one by one, we have

$$\begin{aligned} x^+ &= (x^+\overline{x}x^*)^+ = (x^+(\overline{x}x^*)^+)^+ \\ &= x^+(\overline{x}x^*)^+ = x^+(\overline{x}(x^*)^+)^+ = x^+(\overline{x}\,\overline{x}^*)^+ = x^+\overline{x}^+. \end{aligned}$$

On the other hand, in view of identities (8), (4), (3) and the item (c) of this lemma,

$$\overline{x}^+ = (\overline{x}^+x\overline{x}^*)^+ = (\overline{x}^+(x\overline{x}^*)^+)^+ = \overline{x}^+(x\overline{x}^*)^+ = \overline{x}^+(x^+(x\overline{x}^*)^+),$$

which implies that

$$\overline{x}^+x^+ = (\overline{x}^+x^+(x\overline{x}^*)^+)x^+ = \overline{x}^+x^+(x\overline{x}^*)^+ = \overline{x}^+$$

by identity (2). This shows that $x^+\mathcal{L}\overline{x}^+$. Dually, $x^*\mathcal{R}\overline{x}^*$.

(e) By the proof of item (b) in the current lemma, $I_{\mathbf{S}}$ is a subband of S. Let $x, y, z \in S$. By the identity (5), $\overline{x}^+ = (x^+)^*$ and hence $(x^+)^* y^+, (x^+)^* z^+ \in E^\circ_{\mathbf{S}}$ by item (b), and so $\overline{x}^+ y^+, \overline{x}^+ z^+ \in E^\circ_{\mathbf{S}}$. By item (d), the identity (2) and the fact that $E^\circ_{\mathbf{S}}$ is a semilattice,

$$\begin{aligned} x^+ y^+ z^+ &= x^+ (\overline{x}^+ y^+) z^+ = x^+ (\overline{x}^+ y^+ \overline{x}^+) z^+ \\ &= x^+ (\overline{x}^+ y^+)(\overline{x}^+ z^+) = x^+ (\overline{x}^+ z^+)(\overline{x}^+ y^+) = x^+ z^+ y^+, \end{aligned}$$

which yields that $I_{\mathbf{S}}$ is a left normal band. Dually, $\Lambda_{\mathbf{S}}$ is a right normal band. □

The following result gives a condition under which a pseudo-Ehresmann semigroup becomes an Ehresmann semigroup.

Proposition 2.10. *Let* $\mathbf{S} = (S, \cdot, +, *, -)$ *be a pseudo-Ehresmann semigroup. Then* $(S, \cdot, +, *)$ *is an Ehresmann semigroup if and only if* $\overline{x} = x$ *for all* $x \in S$.

Proof. If $(S, \cdot, +, *)$ is an Ehresmann semigroup and $x \in S$, then $x^+ = (x^+)^* = \overline{x}^+$ by Lemma 2.1 and the identity (5) in Table 1. Dually, $x^* = \overline{x}^*$. This implies that

$$\overline{x} = \overline{x}^+ x \overline{x}^* = x^+ x x^* = x$$

by the identities (8), (1) and (1)′. Conversely, assume that $\overline{x} = x$ for all $x \in S$. In view of Lemma 2.1 and the identities in Table 1, we can see that $(S, \cdot, +, *)$ is an Ehresmann semigroup. □

The following proposition shows that every pseudo-Ehresmann semigroup contains an Ehresmann subsemigroup.

Proposition 2.11. *Let* $\mathbf{S} = (S, \cdot, +, *, -)$ *be a pseudo-Ehresmann semigroup. Then*

$$\overline{S} = \{\overline{x} \mid x \in S\}$$

is a $(2, 1, 1, 1)$*-subalgebra of* S. *In fact,* $(\overline{S}, \cdot, +, *)$ *is an Ehresmann semigroup with distinguished semilattice* $E^\circ_{\mathbf{S}}$.

Proof. We assert that

$$\overline{\overline{x}} = \overline{x}, \quad \overline{xy} = \overline{x} x^* y^+ \overline{y} \quad \text{for all } x, y \in S. \tag{2.1}$$

In fact, we have $\overline{x}^+ \mathcal{L} \overline{\overline{x}}^+$ and $\overline{x}^* \mathcal{R} \overline{\overline{x}}^*$ by Lemma 2.9(d). In view of Lemma 2.9(b), $\overline{x}^+, \overline{\overline{x}}^+, \overline{x}^*, \overline{\overline{x}}^*$ are all in the semilattice $E^\circ_{\mathbf{S}}$, and this implies that $\overline{x}^+ = \overline{\overline{x}}^+$ and $\overline{x}^* = \overline{\overline{x}}^*$. It follows that

$$\overline{\overline{x}} = \overline{\overline{x}}^+ \overline{x}\, \overline{\overline{x}}^* = \overline{x}^+ \overline{x}\, \overline{x}^* = \overline{x}$$

by the identities (8), (1) and (1)′. On the other hand, by Lemma 2.9(c), (d) and the identity (4) in Table 1,

$$(xy)^+ = x^+ (\overline{x} x^* y^+)^+ = x^+ (\overline{x} x^* y^+ \overline{y}^+)^+ = x^+ (\overline{x} x^* y^+ \overline{y})^+.$$

Using the identities (5), (4)′, (10), (5) and (2) one by one, we obtain that

$$\begin{aligned} \overline{xy}^+ &= ((xy)^+)^* = (x^+ (\overline{x} x^* y^+ \overline{y})^+)^* = ((x^+)^* (\overline{x} x^* y^+ \overline{y})^+)^* \\ &= (x^+)^* (\overline{x} x^* y^+ \overline{y})^+ = \overline{x}^+ (\overline{x} x^* y^+ \overline{y})^+ = \overline{x}^+ (\overline{x} x^* y^+ \overline{y})^+ \overline{x}^+. \end{aligned}$$

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

Dually, $\overline{xy}^{*} = \bar{y}^{*}(\bar{x}x^{*}y^{+}\bar{y})^{*}\bar{y}^{*}$. Thus

$$\begin{aligned}
\overline{xy} &= \overline{xy}^{+}\,xy\,\overline{xy}^{*} \quad \text{(by the identity (8) in Table 1)}\\
&= \bar{x}^{+}(\bar{x}x^{*}y^{+}\bar{y})^{+}\bar{x}^{+}x^{+}\bar{x}x^{*}y^{+}\bar{y}y^{*}\bar{y}^{*}(\bar{x}x^{*}y^{+}\bar{y})^{*}\bar{y}^{*} \quad \text{(by the identity (7) in Table 1)}\\
&= \bar{x}^{+}(\bar{x}x^{*}y^{+}\bar{y})^{+}\bar{x}^{+}\bar{x}x^{*}y^{+}\bar{y}\,\bar{y}^{*}(\bar{x}x^{*}y^{+}\bar{y})^{*}\bar{y}^{*} \quad \text{(by Lemma 2.9(d))}\\
&= \bar{x}x^{*}y^{+}\bar{y}. \quad \text{(by the identities (1) and (1)}').
\end{aligned}$$

Now let $x, y \in S$. By (2.1) and the identities (1) and (1)′,

$$\overline{\bar{x}\,\bar{y}} = \bar{\bar{x}}\,\overline{\bar{x}^{*}}\,\overline{\bar{y}^{+}}\,\bar{\bar{y}} = \bar{x}\,\bar{x}^{*}\bar{y}^{+}\bar{y} = \bar{x}\,\bar{y}.$$

This yields that $\bar{S}$ is closed with respect to the binary operation. Moreover, by the identities (6) and (6)′, $\bar{x}^{+} = \overline{x^{+}} \in \bar{S}$ and $\bar{x}^{*} = \overline{x^{*}} \in \bar{S}$ for all $x \in S$. This shows that $\bar{S}$ is closed under '+' and '∗'. From (2.1), $\bar{S}$ is also closed under '−'. Thus $\bar{S} = \{\bar{x} \mid x \in S\}$ is a (2, 1, 1, 1)-subalgebra of S.

To see the remainder part of this proposition, let $\bar{x} \in \bar{S}$ for some $x \in S$. By identity (5) in Table 1 and (2.1), $(\bar{x}^{+})^{*} = \bar{\bar{x}}^{+} = \bar{x}^{+}$. Dually, $(\bar{x}^{*})^{+} = \bar{\bar{x}}^{*} = \bar{x}^{*}$. In view of the identities (1)–(4) in Table 1 and their duals, the result follows from Lemma 2.1 and the first part of this proposition. □

As pointed out in Example 2.7, a pseudo-Ehresmann semigroup being regular may contain no inverse transversal. However, we have the following proposition.

Proposition 2.12. *Let* $\mathbf{S} = (S, \cdot, +, *, -)$ *be a pseudo-Ehresmann semigroup. If there exists* $x' \in S$ *such that* $xx' = x^{+}$ *and* $x'x = x^{*}$ *for all* $x \in S$, *then* $\bar{S} = \{\bar{x} \mid x \in S\}$ *is a multiplicative inverse transversal of* $(S, \cdot)$ *and induces the pseudo-Ehresmann semigroup* $\mathbf{S}$.

Proof. By Proposition 2.11, $\bar{S}$ is a subsemigroup of S with $E^{\circ}_{\mathbf{S}} \subseteq \bar{S}$. In view of the hypothesis and the identity (1) in Table 1, $xx'x = x^{+}x = x$ for all $x \in S$. This implies that S is regular and

$$x'xx' \in V(x), x(x'xx') = xx' = x^{+}, (x'xx')x = x'x = x^{*}$$

for all $x \in S$. Thus there exists $x^{\circ} \in V(x)$ such that

$$xx^{\circ} = x^{+} \quad \text{and} \quad x^{\circ}x = x^{*} \quad \text{for all } x \in S. \tag{2.2}$$

We observe that the above x° is unique. In fact, if $y \in V(x)$ and $xy = x^{+}, yx = x^{*}$, then

$$y = yxy = x^{*}y = x^{\circ}xy = x^{\circ}x^{+} = x^{\circ}xx^{\circ} = x^{\circ}.$$

Now let $x \in S$ and x° be the element satisfying (2.2) and let $x^{\circ\circ} = (x^{\circ})^{\circ}$. By Lemma 2.9(d), $x^{+}\mathcal{L}\bar{x}^{+}$ and $x^{*}\mathcal{R}\bar{x}^{*}$. Therefore, we have the following diagram.

x	x^{+}		
x^{*}	x°	$(x^{\circ})^{+}$	$\bar{x}^{*}$
	$(x^{\circ})^{*}$	$x^{\circ\circ}$	
	$\bar{x}^{+}$		

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

This implies that $\overline{x}^*\mathcal{R}x^\circ\mathcal{L}\overline{x}^+$ and hence

$$x^\circ = \overline{x}^*x^\circ\overline{x}^+ = (\overline{x}^*(x^\circ)^+)\overline{x^\circ}((x^\circ)^*\overline{x}^+) \in E^\circ_{\mathbf{S}}\overline{S}E^\circ_{\mathbf{S}} \subseteq \overline{S}$$

by the identity (7), Lemma 2.9(b) and the fact that $\overline{S}$ is a subsemigroup with $E^\circ_{\mathbf{S}} \subseteq \overline{S}$. Thus, every element x in S has an inverse x° in $\overline{S}$. Furthermore, since $(x^\circ)^*, \overline{x}^+ = (x^+)^* \in \Lambda_{\mathbf{S}}$ by the identity (5) and $\Lambda_{\mathbf{S}}$ is a right normal band (by Lemma 2.9(e)), we have $(x^\circ)^* = \overline{x}^+$ by the fact that $(x^\circ)^*\mathcal{L}\overline{x}^+$. Dually, $(x^\circ)^+ = \overline{x}^*$. This implies that $\overline{x}^+ = (x^\circ)^*\mathcal{R}x^{\circ\circ}\mathcal{L}(x^\circ)^+ = \overline{x}^*$ and hence

$$x^{\circ\circ} = \overline{x}^+x^{\circ\circ}\overline{x}^* = \overline{x}^+x^+x^{\circ\circ}x^*\overline{x}^* = \overline{x}^+xx^\circ x^{\circ\circ}x^\circ x\overline{x}^* = \overline{x}^+x\overline{x}^* = \overline{x} \tag{2.3}$$

by the identity (8) in Table 1.

Let $y \in S$ with $\overline{y} \in V(x)$. Since $\overline{y}^*x^+, x^*\overline{y}^+ \in E^\circ_{\mathbf{S}}$ and $x^+\overline{y}^* \in I_{\mathbf{S}}, \overline{y}^+x^* \in \Lambda_{\mathbf{S}}$ by Lemma 2.9(b) and (e),

x	x^+	$x\overline{y} = x^+\overline{y}^*$	
x^*	x°		$x^*\overline{y}^+$
$\overline{y}^+x^* = \overline{y}x$		$\overline{y}$	$\overline{y}^+$
	$\overline{y}^*x^+$	$\overline{y}^*$	$\overline{y}^\circ$

Since $I_{\mathbf{S}}$ is a left normal band and $x^+ \in I_{\mathbf{S}}$, $x\overline{y} = x^+\overline{y}^* \in I_{\mathbf{S}}$, it follows that $x^+ = x^+\overline{y}^* = x\overline{y}$. Dually, $\overline{y}x = \overline{y}^+x^* = x^*$. This implies that $x^\circ\mathcal{H}\overline{y}$ and so $x^\circ = \overline{y}$ by the fact that $x^\circ, \overline{y} \in V(x)$. Thus x° is the unique inverse of x in $\overline{S}$. Moreover,

$$x^+ = xx^\circ, \quad x^* = x^\circ x, \quad \overline{x} = x^{\circ\circ}, \quad x^\circ xyy^\circ = x^*y^+ \in E^\circ_{\mathbf{S}} \subseteq E(\overline{S})$$

by (2.2), (2.3) and Lemma 2.9(b). Therefore, $\mathbf{S} = (S, \cdot, +, *, -)$ is exactly the pseudo-Ehresmann semigroup induced by the multiplicative inverse transversal $\overline{S}$ of $(S, \cdot)$. □

To give more properties of pseudo-Ehresmann semigroups, we need the notion of *admissible quadruples* which is defined in the text [26]. Let I (respectively, Λ) be a left normal band (respectively, a right normal band), let $E^\circ = I \cap \Lambda$ be a subsemilattice of I and Λ, and let '$\diamond$' be a mapping

$$\diamond : \Lambda \times I \to E^\circ, (f, g) \mapsto f \diamond g.$$

The quadruple $(I, \Lambda, E^\circ, \diamond)$ is called *admissible* if, for all $g \in I$ and $f \in \Lambda$, there exist $g^\circ, f^\circ \in E^\circ$ such that $g\mathcal{L}g^\circ$, $f\mathcal{R}f^\circ$ and, for all $i \in E^\circ$,

$$i(f \diamond g) = (if) \diamond g, \quad (f \diamond g)i = f \diamond (gi), \quad f \diamond i = fi, \quad i \diamond g = ig. \tag{2.4}$$

Since E° is a subsemilattice, the elements g° and f° above are uniquely determined by g and f, respectively. In particular, $i \in E^\circ$ if and only if $i^\circ = i$. Thus, $(f \diamond g)^\circ = f \diamond g$ for all $f \in \Lambda$ and $g \in I$. By Lemma 2.9(b), (d) and (e), we can obtain the following result easily.

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

Lemma 2.13. *Let* $\mathbf{S} = (S, \cdot, +, *, -)$ *be a pseudo-Ehresmann semigroup. Then*

$$Q_{\mathbf{S}} = (I_{\mathbf{S}}, \Lambda_{\mathbf{S}}, E^{\circ}_{\mathbf{S}}, \diamond_{\mathbf{S}})$$

forms an admissible quadruple, called the admissible quadruple of $\mathbf{S}$, *where*

$$(x^+)^{\circ} = (x^+)^* = \overline{x}^+, \quad (y^*)^{\circ} = (y^*)^+ = \overline{y}^*, \quad y^* \diamond_{\mathbf{S}} x^+ = y^* x^+$$

for all $x^+ \in I_{\mathbf{S}}$ *and* $y^* \in \Lambda_{\mathbf{S}}$.

We say that $\varphi = (\varphi_I, \varphi_\Lambda)$ is a *morphism* from an admissible quadruple $Q_1 = (I_1, \Lambda_1, E^{\circ}_1, \diamond_1)$ to an admissible quadruple $Q_2 = (I_2, \Lambda_2, E^{\circ}_2, \diamond_2)$ if φ_I is a semigroup morphism from I_1 to I_2 and φ_Λ is a semigroup morphism from Λ_1 to Λ_2 such that

$$\varphi_I|_{E^{\circ}_1} = \varphi_\Lambda|_{E^{\circ}_1}, \quad E^{\circ}_1\varphi_I \subseteq E^{\circ}_2, \quad (f \diamond_1 g)\varphi_I = (f\varphi_\Lambda) \diamond_2 (g\varphi_I)$$

for all $f \in \Lambda_1$ and $g \in I_1$. Observe that

$$g^{\circ}\varphi_I = (g\varphi_I)^{\circ} \quad \text{and} \quad f^{\circ}\varphi_\Lambda = (f\varphi_\Lambda)^{\circ} \tag{2.5}$$

for all $f \in \Lambda_1$ and $g \in I_1$ in this case. Moreover, for all $f \in \Lambda_1$ and $g \in I_1$, we always denote $g\varphi_I = g\varphi$, $f\varphi_\Lambda = f\varphi$ in the subsequent work. For example, the equation $(f \diamond_1 g)\varphi_I = (f\varphi_\Lambda) \diamond_2 (g\varphi_I)$ will be written as $(f \diamond_1 g)\varphi = (f\varphi) \diamond_2 (g\varphi)$.

Let $\mathbf{S} = (S, \cdot, +, *, -)$ and $\mathbf{T} = (T, \cdot, +, *, -)$ be two pseudo-Ehresmann semigroups. A mapping θ from S to T is called a *(2,1,1,1)-morphism* from $\mathbf{S}$ to $\mathbf{T}$ if θ preserves '$\cdot$', '$+$', '$*$' and '$-$', respectively. By Lemma 2.9, it is not difficult to check that the following lemma is true.

Lemma 2.14. *Let* $\mathbf{S} = (S, \cdot, +, *, -)$ *and* $\mathbf{T} = (T, \cdot, +, *, -)$ *be two pseudo-Ehresmann semigroups and let* θ *be a (2,1,1,1)-morphism from* $\mathbf{S}$ *to* $\mathbf{T}$. *Then* θ *induces a morphism from* $(I_{\mathbf{S}}, \Lambda_{\mathbf{S}}, E^{\circ}_{\mathbf{S}}, \diamond_{\mathbf{S}})$ *to* $(I_{\mathbf{T}}, \Lambda_{\mathbf{T}}, E^{\circ}_{\mathbf{T}}, \diamond_{\mathbf{T}})$.

To state our next observation, we need the notion of a category. The following concept of categories is standard.

Definition 2.15. A *category* $\mathbf{C}$ consists of a class $\mathrm{Ob}(\mathbf{C})$ of objects, a class $\mathrm{Mor}(\mathbf{C})$ of morphisms between objects and two assignments, $\mathbf{d}$ and $\mathbf{r}$, from $\mathrm{Mor}(\mathbf{C})$ to $\mathrm{Ob}(\mathbf{C})$, such that the following conditions hold.

(i) If $A, B, C, D \in \mathrm{Ob}(\mathbf{C})$, then there is a binary operation

$$\mathrm{Mor}_{\mathbf{C}}(A, B) \times \mathrm{Mor}_{\mathbf{C}}(B, C) \to \mathrm{Mor}_{\mathbf{C}}(A, C), (f, g) \mapsto f \star g$$

called *composition* of morphisms such that if $f \in \mathrm{Mor}_{\mathbf{C}}(A, B)$, $g \in \mathrm{Mor}_{\mathbf{C}}(B, C)$ and $h \in \mathrm{Mor}_{\mathbf{C}}(C, D)$, then $(f \star g) \star h = f \star (g \star h)$, where

$$\mathrm{Mor}_{\mathbf{C}}(A, B) = \{f \in \mathrm{Mor}(\mathbf{C}) | \mathbf{d}(f) = A, \mathbf{r}(f) = B\}$$

for all $A, B \in \mathrm{Ob}(\mathbf{C})$.

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

(ii) For each $A \in \mathrm{Ob}(\mathbf{C})$, there exists a *local identity* 1_A such that $1_A \star f = f$ and $g \star 1_A = g$ for all $f, g \in \mathrm{Mor}(\mathbf{C})$ with $\mathbf{d}(f) = A = \mathbf{r}(g)$. The class of local identities of $\mathbf{C}$ is denoted by $\mathbf{C}_o$, that is to say, $\mathbf{C}_o = \{1_A \mid A \in \mathrm{Ob}(\mathbf{C})\}$.

It is routine to check that the following lemma is valid.

LEMMA 2.16. *The class of pseudo-Ehresmann semigroups together with (2,1,1,1)-morphisms forms a category in the sense of Definition 2.15, denoted by* **PES**.

In the remainder of this section, we shall recall some concepts and results on the theory of category which are necessary in the discussion later. We first give the definition of a functor. This is a structure preserving mapping between two categories, which allows us to compare categories.

DEFINITION 2.17. Let $\mathbf{C}$ and $\mathbf{D}$ be two categories. A *functor* $\mathcal{F} = (\Phi, \Psi)$ from $\mathbf{C}$ to $\mathbf{D}$ is a pair of maps

$$\Phi : \mathrm{Ob}(\mathbf{C}) \to \mathrm{Ob}(\mathbf{D}), A \mapsto A\Phi; \quad \Psi : \mathrm{Mor}(\mathbf{C}) \to \mathrm{Mor}(\mathbf{D}), f \mapsto f\Psi$$

satisfying the following conditions.

(i) Ψ maps an element of $\mathrm{Mor}_{\mathbf{C}}(A, B)$ to $\mathrm{Mor}_{\mathbf{D}}(A\Phi, B\Phi)$ for all $A, B \in \mathrm{Mor}(\mathbf{C})$.

(ii) If $f, g \in \mathrm{Mor}(\mathbf{C})$ and $f \star g$ is defined in $\mathbf{C}$, then $f\Psi \star g\Psi$ is defined in $\mathbf{D}$ and $(f \star g)\Psi = f\Psi \star g\Psi$.

(iii) For all $A \in \mathrm{Ob}(\mathbf{C})$, $1_A\Psi = 1_{A\Phi}$.

As usual, if $\mathbf{C}$ and $\mathbf{D}$ are categories and $\mathcal{F} = (\Phi, \Psi)$ is a functor from $\mathbf{C}$ to $\mathbf{D}$, then we denote $A\Phi$ and $f\Psi$ by $A\mathcal{F}$ and $f\mathcal{F}$ for all $A \in \mathrm{Ob}(\mathbf{C})$ and $f \in \mathrm{Mor}(\mathbf{C})$, respectively.

DEFINITION 2.18. Let $\mathbf{C}$ and $\mathbf{D}$ be categories. Then $\mathbf{C}$ and $\mathbf{D}$ are isomorphic if there exist functors $\mathcal{F} : \mathbf{C} \to \mathbf{D}$ and $\mathcal{G} : \mathbf{D} \to \mathbf{C}$ such that $\mathcal{F}\mathcal{G} = 1_{\mathbf{C}}$ and $\mathcal{G}\mathcal{F} = 1_{\mathbf{D}}$, where $1_{\mathbf{C}}$ and $1_{\mathbf{D}}$ are identity functors on $\mathbf{C}$ and $\mathbf{D}$, respectively.

Recall that a category $\mathbf{C}$ is *small* if both $\mathrm{Ob}(\mathbf{C})$ and $\mathrm{Mor}(\mathbf{C})$ are sets. We now present the algebraic definition of *a small category* which will be used in the next sections.

DEFINITION 2.19 (Lawson [20, 21]). Let C be a set, let '$\cdot$' be a partial binary operation on C, and let $\mathbf{d}$ and $\mathbf{r}$ be two mappings from C to C. Then $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r})$ is called a *small category* if the following statements hold.

(i) $x \cdot y$ is defined if and only if $\mathbf{r}(x) = \mathbf{d}(y)$ and then $\mathbf{d}(x \cdot y) = \mathbf{d}(x)$, $\mathbf{r}(x \cdot y) = \mathbf{r}(y)$.

(ii) If $x, y, z \in C$ such that both $x \cdot y$ and $y \cdot z$ are defined, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(iii) For all $x \in C$, $\mathbf{d}(x) \cdot x$ and $x \cdot \mathbf{r}(x)$ are defined and $\mathbf{d}(x) \cdot x = x = x \cdot \mathbf{r}(x)$.

In this case, it is easy to see that $\{\mathbf{d}(x) \mid x \in C\} = \{\mathbf{r}(x) \mid x \in C\}$, denoted by $E_{\mathbf{C}}$. One can check routinely that $\mathbf{C}$ is indeed a category in the sense of Definition 2.15, where

$$\mathrm{Ob}(\mathbf{C}) = E_{\mathbf{C}}, \quad \mathrm{Mor}(\mathbf{C}) = C, \quad 1_u = u = \mathbf{d}(u) = \mathbf{r}(u)$$

for all $u \in \mathrm{Ob}(\mathbf{C})$. In this case, the set of local identities $\mathbf{C}_o$ is also $E_{\mathbf{C}}$ (see Lawson [20] for details).

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

The following results can be proved easily.

PROPOSITION 2.20. *Let* $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r})$ *and* $\mathbf{D} = (D, \cdot, \mathbf{d}, \mathbf{r})$ *be two small categories in the sense of Definition 2.19 and let* $\mathcal{F} = (\Phi, \Psi)$ *be a pair of maps*

$$\Phi : E_{\mathbf{C}} \to E_{\mathbf{D}}, u \mapsto u\Phi; \quad \Psi : C \to D, x \mapsto x\Psi.$$

Then $\mathcal{F}$ *is a functor from* $\mathbf{C}$ *to* $\mathbf{D}$ *in the sense of Definition 2.17 if and only if:*

(i) *for all* $x \in C$, $\mathbf{d}(x\Psi) = (\mathbf{d}(x))\Phi$ *and* $\mathbf{r}(x\Psi) = (\mathbf{r}(x))\Phi$*;*
(ii) *for all* $x, y \in C$ *satisfying that* $x \cdot y$ *is defined in* $\mathbf{C}$, $(x\Psi) \cdot (y\Psi) = (x \cdot y)\Psi$*; and*
(iii) *for all* $u \in E_{\mathbf{C}}$, $u\Psi = u\Phi$.

Observe that Φ *can be regarded as the restriction of* Ψ *to* $E_{\mathbf{C}}$ *by the above items (i) and (iii) in this case.*

COROLLARY 2.21. *Let* $\mathbf{C} = (C, \cdot, \mathbf{d}, \mathbf{r})$ *and* $\mathbf{D} = (D, \cdot, \mathbf{d}, \mathbf{r})$ *be two small categories in the sense of Definition 2.19. Then a functor from* $\mathbf{C}$ *to* $\mathbf{D}$ *in the sense of Definition 2.17 is essentially a mapping* ψ *from* C *to* D *such that:*

(i) *for all* $x \in C$, $\mathbf{d}(x\psi) = (\mathbf{d}(x))\psi$ *and* $\mathbf{r}(x\psi) = (\mathbf{r}(x))\psi$*; and*
(ii) *for all* $x, y \in C$ *satisfying that* $x \cdot y$ *is defined in* $\mathbf{C}$, $(x\psi) \cdot (y\psi) = (x \cdot y)\psi$.

3. Inductive pseudocategories over admissible quadruples

This section is devoted to the introduction of the category of inductive pseudocategories over admissible quadruples and obtains some elementary properties of this category. To this aim, we need a result on admissible quadruples, which will be used frequently in the subsequent work. The following lemma appears in the text [26], but we also give its proof here for the sake of completeness.

LEMMA 3.1 [26, Lemma 3.3]. *Let* $(I, \Lambda, E^\circ, \diamond)$ *be an admissible quadruple and* $e, g \in I$, $f, h \in \Lambda$. *Then*

$$eg = eg^\circ, \quad (eg)^\circ = e^\circ g^\circ, \quad fh = f^\circ h, \quad (fh)^\circ = f^\circ h^\circ. \tag{3.1}$$

PROOF. Since $g\mathcal{L}g^\circ$ and I is a left normal band, we have $eg = egg^\circ = eg^\circ g = eg^\circ$. This implies that $e^\circ g = e^\circ g^\circ$, and so $eg\mathcal{L}e^\circ g = e^\circ g^\circ \in E^\circ$ by the fact that $e\mathcal{L}e^\circ$. This yields that $(eg)^\circ = e^\circ g^\circ$. The remaining facts of this lemma can be proved by symmetry. □

Now we can give the notion of pseudocategories over admissible quadruples.

DEFINITION 3.2. Let P be a set which contains the underlying set $I \cup \Lambda$ of an admissible quadruple $Q = (I, \Lambda, E^\circ, \diamond)$ and let '$\cdot$' be a partial binary operation on P. Assume that

$$\mathbf{d} : P \to I, x \mapsto \mathbf{d}(x), \quad \mathbf{r} : P \to \Lambda, x \mapsto \mathbf{r}(x)$$

are maps such that

$$\mathbf{d}(e) = e, \quad \mathbf{r}(e) = e^\circ, \quad \mathbf{d}(f) = f^\circ, \quad \mathbf{r}(f) = f \tag{3.2}$$

for all $e \in I$ and $f \in \Lambda$. Then $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, Q)$ is called a *pseudocategory* over an admissible quadruple Q if the following conditions hold.

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

(PC1) For all $x, y \in P$, $x \cdot y$ is defined if and only if $\mathbf{r}^\circ(x) = \mathbf{r}(x) \diamond \mathbf{d}(y) = \mathbf{d}^\circ(y)$ and then $\mathbf{d}(x \cdot y) = \mathbf{d}(x)$ and $\mathbf{r}(x \cdot y) = \mathbf{r}(y)$, where $\mathbf{r}^\circ(x) = (\mathbf{r}(x))^\circ$ and $\mathbf{d}^\circ(y) = (\mathbf{d}(y))^\circ$.

(PC2) If $x, y, z \in P$ such that both $x \cdot y$ and $y \cdot z$ are defined, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.

(PC3) For all $x \in P$, $\mathbf{d}(x) \cdot x$ and $x \cdot \mathbf{r}(x)$ are defined and $\mathbf{d}(x) \cdot x = x = x \cdot \mathbf{r}(x)$.

The following proposition can be deduced easily from (3.2) and (PC3) in the above Definition 3.2.

Proposition 3.3. *Let $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, Q)$ be a pseudocategory over an admissible quadruple $Q = (I, \Lambda, E^\circ, \diamond)$. Then the following results hold.*

(i) *$I = \{\mathbf{d}(x) \mid x \in P\}$ and $\Lambda = \{\mathbf{r}(x) \mid x \in P\}$.*

(ii) *For all $e \in I$ and $f \in \Lambda$, we have $\mathbf{d}(\mathbf{d}(e)) = \mathbf{d}(e), \mathbf{r}(\mathbf{r}(f)) = \mathbf{r}(f)$ and*

$$\mathbf{r}(\mathbf{d}(e)) = \mathbf{d}^\circ(e) = e^\circ = \mathbf{r}(e), \quad \mathbf{d}(\mathbf{r}(f)) = \mathbf{r}^\circ(f) = f^\circ = \mathbf{d}(f).$$

(iii) *For all $e \in I$ and $f \in \Lambda$, we have $e \cdot e = e = e \cdot e^\circ$ and $f \cdot f = f = f^\circ \cdot f$.*

Remark 3.4. In Definition 3.2, if $I = \Lambda = E^\circ$, then $I \cup \Lambda = E^\circ$ is a semilattice. So $u^\circ = u$ for all $u \in I \cup \Lambda = E^\circ$ and $\mathbf{d}(u) = \mathbf{r}(u) = u$ for all $u \in E^\circ$. In this case, by (2.4) the condition (PC1) reduces to the following (PC1)$^\#$: for all $x, y \in P$, $x \cdot y$ is defined if and only if $\mathbf{r}(x) = \mathbf{d}(y)$ and then $\mathbf{d}(x \cdot y) = \mathbf{d}(x)$ and $\mathbf{r}(x \cdot y) = \mathbf{r}(y)$. Thus, in the case that $I = \Lambda = E^\circ$, $\mathbf{P}$ becomes a small category in the sense of Definition 2.19. In this case, the set of local identities of $\mathbf{P}$ is E°.

Now, we focus on a class of special pseudocategories, namely inductive pseudocategories. Recall that *the natural partial order* '$\leq_B$' on a band B is defined as follows: for all $e, f \in B$,

$$e \leq_B f \quad \text{if and only if} \quad ef = fe = e.$$

Definition 3.5. A pseudocategory $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, Q)$ over an admissible quadruple $Q = (I, \Lambda, E^\circ, \diamond)$ is called *inductive* if the following conditions and the duals (IPC1)′, (IPC2)′ and (IPC3)′ of (IPC1), (IPC2) and (IPC3) hold.

(IPC1) For $x \in P$ and $e \in I$ with $e \leq_I \mathbf{d}(x)$, there exists ${}_e|x \in P$, which is called *the restriction* of x to e, such that $\mathbf{d}({}_e|x) = e$, $\mathbf{r}({}_e|x) \leq_\Lambda \mathbf{r}(x)$ and ${}_{\mathbf{d}(x)}|x = x$.

(IPC2) If $e, g \in I$ and $x \in P$ with $e \leq_I g \leq_I \mathbf{d}(x)$, then ${}_e|({}_g|x) = {}_e|x$.

(IPC3) If $e \in I, x, y \in P, e \leq_I \mathbf{d}(x)$ and $x \cdot y$ is defined, then

$${}_e|(x \cdot y) = {}_e|x \cdot {}_{\mathbf{d}(y)(\mathbf{r}({}_e|x) \diamond \mathbf{d}(y))}|y.$$

(IPC4) If $e, g \in I, f, h \in \Lambda, u \in E^\circ$ and $e^\circ u = u, uf^\circ = u$, then

$$e|_u = eu, \quad {}_{eg}|e = eg, \quad eg \cdot ge = eg; \quad {}_u|f = uf, \quad f|_{hf} = hf, \quad hf \cdot fh = fh.$$

Moreover, if $f \cdot g$ is defined in $\mathbf{P}$ (i.e. $f^\circ = f \diamond g = g^\circ$), then $f \cdot g = f \diamond g$.

(IPC5) If $g \in I, f \in \Lambda, x \in P$ and $p = x|_{(\mathbf{r}(x) \diamond g)\mathbf{r}(x)}$, $q = {}_{\mathbf{d}(x)(f \diamond \mathbf{d}(x))}|x$, then ${}_{\mathbf{d}(p)(f \diamond \mathbf{d}(p))}|p = q|_{(\mathbf{r}(q) \diamond g)\mathbf{r}(q)}$.

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

At this stage, it is worth remarking that (IPC3) makes sense. Suppose that $e \in I, x, y \in P, e \leq_I \mathbf{d}(x)$ and $x \cdot y$ is defined. Then $\mathbf{d}(x \cdot y) = \mathbf{d}(x)$ and so $e \leq_I \mathbf{d}(x \cdot y)$. This shows that ${}_e|(x \cdot y)$ is meaningful. Moreover, we have $\mathbf{r}^\circ(x) = \mathbf{r}(x) \diamond \mathbf{d}(y)$. By (IPC1), $\mathbf{r}({}_e|x) \leq_\Lambda \mathbf{r}(x)$. This implies that $\mathbf{r}({}_e|x)\mathbf{r}(x) = \mathbf{r}({}_e|x)$. We deduce by (3.1) and (2.4) that

$$\begin{aligned}\mathbf{r}^\circ({}_e|x) &= \mathbf{r}^\circ({}_e|x)\mathbf{r}^\circ(x) = \mathbf{r}^\circ({}_e|x)(\mathbf{r}(x) \diamond \mathbf{d}(y)) \\ &= (\mathbf{r}^\circ({}_e|x)\mathbf{r}(x)) \diamond \mathbf{d}(y) = (\mathbf{r}({}_e|x)\mathbf{r}(x)) \diamond \mathbf{d}(y) = \mathbf{r}({}_e|x) \diamond \mathbf{d}(y).\end{aligned}$$

(IPC1) also gives that $\mathbf{d}({}_{\mathbf{d}(y)(\mathbf{r}({}_e|x)\diamond\mathbf{d}(y))}|y) = \mathbf{d}(y)(\mathbf{r}({}_e|x) \diamond \mathbf{d}(y))$. By (2.4),

$$\mathbf{r}({}_e|x) \diamond [\mathbf{d}(y)(\mathbf{r}({}_e|x) \diamond \mathbf{d}(y))] = (\mathbf{r}({}_e|x) \diamond \mathbf{d}(y))(\mathbf{r}({}_e|x) \diamond \mathbf{d}(y)) = \mathbf{r}({}_e|x) \diamond \mathbf{d}(y).$$

Finally, in view of (IPC1), (3.1) and (2.4),

$$\begin{aligned}\mathbf{d}^\circ({}_{\mathbf{d}(y)(\mathbf{r}({}_e|x)\diamond\mathbf{d}(y))}|y) &= (\mathbf{d}(y)(\mathbf{r}({}_e|x) \diamond \mathbf{d}(y)))^\circ = (\mathbf{r}({}_e|x) \diamond \mathbf{d}(y))^\circ\mathbf{d}^\circ(y) \\ &= (\mathbf{r}({}_e|x) \diamond \mathbf{d}(y))\mathbf{d}^\circ(y) = \mathbf{r}({}_e|x) \diamond (\mathbf{d}(y)\mathbf{d}^\circ(y)) = \mathbf{r}({}_e|x) \diamond \mathbf{d}(y).\end{aligned}$$

Hence the right-hand side of the equality in (IPC3) is meaningful. We also remark that the equations in (IPC4) and (IPC5) make sense according to (2.4), (3.1) and (PC1).

Remark 3.6. From Definition 3.5, we can deduce the notion of an inductive pseudocategory (in fact a category by Remark 3.4) over a semilattice. A pseudocategory $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, E)$ over a semilattice E is called *inductive* if the following conditions and the duals (IC1)′, (IC2)′ and (IC3)′ of (IC1), (IC2) and (IC3) hold.

(IC1) For $x \in P$ and $e \in E$ with $e \leq_E \mathbf{d}(x)$, there exists ${}_e|x \in P$, which is called *the restriction* of x to e, such that $\mathbf{d}({}_e|x) = e$, $\mathbf{r}({}_e|x) \leq_E \mathbf{r}(x)$ and ${}_{\mathbf{d}(x)}|x = x$.

(IC2) If $e, g \in E$ and $x \in P$ with $e \leq_E g \leq_E \mathbf{d}(x)$, then ${}_e|({}_g|x) = {}_e|x$.

(IC3) If $e \in E, x, y \in P, e \leq_E \mathbf{d}(x)$ and $x \cdot y$ is defined, then

$${}_e|(x \cdot y) = {}_e|x \cdot {}_{\mathbf{r}({}_e|x)\mathbf{d}(y)}|y.$$

(IC4) If $e, u \in E$ with $u \leq_E e$, then $e|_u = u = {}_u|e$.

(IC5) If $g, f \in E, x \in P$ and $p = x|_{\mathbf{r}(x)g}, q = {}_{f\mathbf{d}(x)}|x$, then ${}_{f\mathbf{d}(p)}|p = q|_{\mathbf{r}(q)g}$.

We pause to introduce a pair of partial orders '$\leq_l$' and '$\leq_r$' on an inductive pseudocategory $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, Q)$ over an admissible $Q = (I, \Lambda, E^\circ, \diamond)$. For any $x, y \in P$, define

$$x \leq_l y \quad \text{if and only if} \quad x = {}_e|y \text{ for some } e \in I,$$
$$x \leq_r y \quad \text{if and only if} \quad x = y|_f \text{ for some } f \in \Lambda.$$

The following lemma shows that $\leq_l$ and $\leq_r$ defined as above are indeed partial orders.

Lemma 3.7. *$\leq_l$ and $\leq_r$ defined as above are partial orders.*

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

Proof. Firstly, for all $x \in P$, we have $x =_{\mathbf{d}(x)} | x$, where $\mathbf{d}(x) \in I$ by (IPC1). Next, if $x, y \in P$ and $x =_e | y$ and $y =_g | x$ for some $e, g \in I$, then

$$\mathbf{d}(x) = e \leq_I \mathbf{d}(y) = g \leq_I \mathbf{d}(x)$$

by (IPC1). This implies that $e = \mathbf{d}(y)$ and so $x =_{\mathbf{d}(y)} | y = y$ by (IPC1) again. Finally, if $x, y, z \in P$ and $x =_e | y$ and $y =_g | z$ for some $e, g \in I$, then

$$\mathbf{d}(x) = e \leq_I \mathbf{d}(y) = g \leq_I \mathbf{d}(z)$$

and so $x =_e |({}_g|z) =_e | z$ by (IPC2). Thus $x \leq_l z$. This proves that $\leq_l$ is a partial order. Dually, we can see that $\leq_r$ is also a partial order. □

We end this section by proving that the class of inductive pseudocategories over admissible quadruples forms a category, together with certain maps referred to as pseudofunctors, which appear in the next definition.

Definition 3.8. Let $\mathbf{P}_1 = (P_1, \cdot, \mathbf{d}, \mathbf{r}, Q_1)$ and $\mathbf{P}_2 = (P_2, \cdot, \mathbf{d}, \mathbf{r}, Q_2)$ be inductive pseudocategories over admissible quadruples $Q_1 = (I_1, \Lambda_1, E_1^\circ, \diamond)$ and $Q_2 = (I_2, \Lambda_2, E_2^\circ, \diamond)$, respectively. A pseudofunctor φ from $\mathbf{P_1}$ to $\mathbf{P_2}$ is a mapping from P_1 to P_2 satisfying the following conditions.

(PF1) φ induces a morphism from Q_1 to Q_2.
(PF2) For all $x \in P_1$, $\mathbf{d}(x\varphi) = (\mathbf{d}(x))\varphi$ and $\mathbf{r}(x\varphi) = (\mathbf{r}(x))\varphi$.
(PF3) If $x, y \in P_1$ and $x \cdot y$ is defined in $\mathbf{P}_1$, then $(x\varphi) \cdot (y\varphi) = (x \cdot y)\varphi$.
(PF4) If $x \in P_1, e \in I_1, f \in \Lambda_1$ and $e \leq_{I_1} \mathbf{d}(x), f \leq_{\Lambda_1} \mathbf{r}(x)$, then

$$({}_e|x)\varphi =_{e\varphi} | x\varphi \quad \text{and} \quad (x|_f)\varphi = x\varphi|_{f\varphi}.$$

To see that (PF3) makes sense, let $x, y \in P_1$ and $x \cdot y$ be defined in $\mathbf{P}_1$. Then

$$\mathbf{r}^\circ(x) = \mathbf{r}(x) \diamond_1 \mathbf{d}(y) = \mathbf{d}^\circ(y). \tag{3.3}$$

By (PF1), (PF2) and (2.5), $\mathbf{r}^\circ(x\varphi) = (\mathbf{r}(x)\varphi)^\circ = (\mathbf{r}^\circ(x))\varphi$. Dually, $\mathbf{d}^\circ(y\varphi) = (\mathbf{d}^\circ(y))\varphi$. Moreover,

$$\mathbf{r}(x\varphi) \diamond_2 \mathbf{d}(y\varphi) = (\mathbf{r}(x))\varphi \diamond_2 (\mathbf{d}(y))\varphi = (\mathbf{r}(x) \diamond_1 \mathbf{d}(y))\varphi$$

by (PF1) and (PF2). In view of the equation (3.3),

$$\mathbf{r}^\circ(x\varphi) = \mathbf{r}(x\varphi) \diamond_2 \mathbf{d}(y\varphi) = \mathbf{d}^\circ(y\varphi)$$

and so $x\varphi \cdot y\varphi$ is defined in $\mathbf{P}_2$. Condition (PF4) also makes sense. In fact, let $e \in I_1$ and $e \leq_{I_1} \mathbf{d}(x)$. Since φ is a morphism from I_1 to I_2 by (IPC1), we have $e\varphi \leq_{I_2} (\mathbf{d}(x))\varphi = \mathbf{d}(x\varphi)$ by (PF2), which implies that ${}_{e\varphi}|x\varphi$ is meaningful. Dually, we can see that $x\varphi|_{f\varphi}$ is meaningful.

The following two lemmas now are straightforward.

Lemma 3.9. *Let $\mathbf{P}_1, \mathbf{P}_2$ and $\mathbf{P}_3$ be inductive pseudocategories and let $\varphi_1 : \mathbf{P}_1 \to \mathbf{P}_2$ and $\varphi_2 : \mathbf{P}_2 \to \mathbf{P}_3$ be pseudofunctors. Then $\varphi_1\varphi_2 : \mathbf{P}_1 \to \mathbf{P}_3$ is also a pseudofunctor.*

Lemma 3.10. *The class of inductive pseudocategories over admissible quadruples together with pseudofunctors forms a category in the sense of Definition 2.15, denoted by* **IPC**.

4. A construction

Our primary aim in this section will be a construction of pseudo-Ehresmann semigroups, built from an inductive pseudocategory over some admissible quadruple. Let $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, Q)$ be a pseudocategory over an admissible quadruple $Q = (I, \Lambda, E^\circ, \diamond)$. We define the pseudoproduct '$\otimes$' on P by

$$x \otimes y = x|_{(\mathbf{r}(x)\diamond\mathbf{d}(y))\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond\mathbf{d}(y))}|y$$

for all $x, y \in P$. Observe that the above pseudoproduct is well defined. In fact, let $x, y \in P$. Firstly, since I is a left normal band,

$$[\mathbf{d}(y)(\mathbf{r}(x) \diamond \mathbf{d}(y))]\mathbf{d}(y) = \mathbf{d}(y)(\mathbf{r}(x) \diamond \mathbf{d}(y)) = \mathbf{d}(y)[\mathbf{d}(y)(\mathbf{r}(x) \diamond \mathbf{d}(y))].$$

Dually,

$$\mathbf{r}(x)[(\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x)] = (\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x) = [(\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x)]\mathbf{r}(x).$$

This implies that $\mathbf{d}(y)(\mathbf{r}(x) \diamond \mathbf{d}(y)) \leq_I \mathbf{d}(y)$ and $(\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x) \leq_\Lambda \mathbf{r}(x)$. Secondly, by (IPC1), (2.4) and (3.1),

$$\begin{aligned}\mathbf{d}^\circ({}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond\mathbf{d}(y))}|y) &= (\mathbf{d}(y)(\mathbf{r}(x) \diamond \mathbf{d}(y)))^\circ \\ &= (\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{d}^\circ(y) = \mathbf{r}(x) \diamond (\mathbf{d}(y)\mathbf{d}^\circ(y)) = \mathbf{r}(x) \diamond \mathbf{d}(y).\end{aligned}$$

Dually, $\mathbf{r}^\circ(x|_{(\mathbf{r}(x)\diamond\mathbf{d}(y))\mathbf{r}(x)}) = \mathbf{r}(x) \diamond \mathbf{d}(y)$. Finally,

$$\begin{aligned}&((\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x)) \diamond (\mathbf{d}(y)(\mathbf{r}(x) \diamond \mathbf{d}(y))) \\ &\quad = (\mathbf{r}(x) \diamond \mathbf{d}(y))(\mathbf{r}(x) \diamond \mathbf{d}(y))(\mathbf{r}(x) \diamond \mathbf{d}(y)) = \mathbf{r}(x) \diamond \mathbf{d}(y)\end{aligned}$$

by (2.4). This implies that

$$x|_{(\mathbf{r}(x)\diamond\mathbf{d}(y))\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond\mathbf{d}(y))}|y$$

is defined in $\mathbf{P}$ and so '$\otimes$' is well defined. Moreover, we have the following result.

Lemma 4.1. *With the above notation, $(P, \otimes)$ is a semigroup.*

Proof. Let $x, y, z \in P$. For convenience, we denote

$$s = y|_{(\mathbf{r}(y)\diamond\mathbf{d}(z))\mathbf{r}(y)}, \quad t = {}_{\mathbf{d}(z)(\mathbf{r}(y)\diamond\mathbf{d}(z))}|z$$

and

$$u = x|_{(\mathbf{r}(x)\diamond\mathbf{d}(s))\mathbf{r}(x)}, \quad v = {}_{\mathbf{d}(s)(\mathbf{r}(x)\diamond\mathbf{d}(s))}|s, \quad w = {}_{\mathbf{d}(t)(\mathbf{r}(v)\diamond\mathbf{d}(t))}|t.$$

This implies that

$$\mathbf{d}(v) = \mathbf{d}(s)(\mathbf{r}(x) \diamond \mathbf{d}(s)) \tag{4.1}$$

by (IPC1). On the one hand, by (IPC3),

$$x \otimes (y \otimes z) = u \cdot (v \cdot w). \tag{4.2}$$

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

Denote $a = {}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond\mathbf{d}(y))}|y$. In view of (IPC5),

$$v = a|_{(\mathbf{r}(a)\diamond\mathbf{d}(z))\mathbf{r}(a)}, \tag{4.3}$$

which gives

$$\mathbf{r}(v) = (\mathbf{r}(a) \diamond \mathbf{d}(z))\mathbf{r}(a)$$

by (IPC1)′. Since

$$\mathbf{d}(t) = \mathbf{d}({}_{\mathbf{d}(z)(\mathbf{r}(y)\diamond\mathbf{d}(z))}|z) = \mathbf{d}(z)(\mathbf{r}(y) \diamond \mathbf{d}(z)) \tag{4.4}$$

by (IPC1),

$$\begin{aligned}\mathbf{r}(v) \diamond \mathbf{d}(t) &= [(\mathbf{r}(a) \diamond \mathbf{d}(z))\mathbf{r}(a)] \diamond [\mathbf{d}(z)(\mathbf{r}(y) \diamond \mathbf{d}(z))]\\ &= (\mathbf{r}(a) \diamond \mathbf{d}(z))(\mathbf{r}(a) \diamond \mathbf{d}(z))(\mathbf{r}(y) \diamond \mathbf{d}(z)) \quad \text{(by (2.4))}\\ &= (\mathbf{r}(a) \diamond \mathbf{d}(z))(\mathbf{r}(y) \diamond \mathbf{d}(z)) \quad \text{(since } E^\circ \text{ is a semilattice).}\end{aligned}$$

In view of the equation (4.4),

$$\begin{aligned}w &= {}_{\mathbf{d}(t)(\mathbf{r}(v)\diamond\mathbf{d}(t))}|t = {}_{[\mathbf{d}(z)(\mathbf{r}(y)\diamond\mathbf{d}(z))][(\mathbf{r}(a)\diamond\mathbf{d}(z))(\mathbf{r}(y)\diamond\mathbf{d}(z))]}|t\\ &= {}_{[\mathbf{d}(z)(\mathbf{r}(y)\diamond\mathbf{d}(z))]((\mathbf{r}(a)\diamond\mathbf{d}(z))}|t \quad \text{(since } E^\circ \text{ is a semilattice)}\\ &= {}_{[\mathbf{d}(z)(\mathbf{r}(y)\diamond\mathbf{d}(z))]((\mathbf{r}(a)\diamond\mathbf{d}(z))}|z \quad \text{(by } t = {}_{\mathbf{d}(z)(\mathbf{r}(y)\diamond\mathbf{d}(z))}|z \text{ and (IPC2)).}\end{aligned}$$

Since $\mathbf{r}(a) = \mathbf{r}({}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond\mathbf{d}(y))}|y) \leq_\wedge \mathbf{r}(y)$ by (IPC1), we have $\mathbf{r}^\circ(a)\mathbf{r}(y) = \mathbf{r}(a)\mathbf{r}(y) = \mathbf{r}(a)$ by (3.1). By (2.4), this implies that

$$\mathbf{r}(a) \diamond \mathbf{d}(z) = (\mathbf{r}^\circ(a)\mathbf{r}(y)) \diamond \mathbf{d}(z) = \mathbf{r}^\circ(a)(\mathbf{r}(y) \diamond \mathbf{d}(z)).$$

Thus

$$\begin{aligned}w &= {}_{[\mathbf{d}(z)(\mathbf{r}(y)\diamond\mathbf{d}(z))](\mathbf{r}(a)\diamond\mathbf{d}(z))}|z = {}_{[\mathbf{d}(z)(\mathbf{r}(y)\diamond\mathbf{d}(z))]\mathbf{r}^\circ(a)(\mathbf{r}(y)\diamond\mathbf{d}(z))}|z\\ &= {}_{\mathbf{d}(z)\mathbf{r}^\circ(a)(\mathbf{r}(y)\diamond\mathbf{d}(z))}|z \quad \text{(since } E^\circ \text{ is a semilattice)}\\ &= {}_{\mathbf{d}(z)(\mathbf{r}(a)\diamond\mathbf{d}(z))}|z.\end{aligned} \tag{4.5}$$

On the other hand, write $c = x|_{(\mathbf{r}(x)\diamond\mathbf{d}(y))\mathbf{r}(x)}$. By (IPC3)′, (4.3) and (4.5),

$$(x \otimes y) \otimes z = (c|_{(\mathbf{r}(c)\diamond\mathbf{d}(v))\mathbf{r}(c)} \cdot v) \cdot w. \tag{4.6}$$

Since

$$\mathbf{r}(c) = \mathbf{r}(x|_{(\mathbf{r}(x)\diamond\mathbf{d}(y))\mathbf{r}(x)}) = (\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x) \tag{4.7}$$

by (IPC1)′, it follows that

$$\begin{aligned}\mathbf{r}(c) \diamond \mathbf{d}(v) &= [(\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x)] \diamond [\mathbf{d}(s)(\mathbf{r}(x) \diamond \mathbf{d}(s))] \quad \text{(by (4.1))}\\ &= (\mathbf{r}(x) \diamond \mathbf{d}(y))(\mathbf{r}(x) \diamond \mathbf{d}(s))(\mathbf{r}(x) \diamond \mathbf{d}(s)) \quad \text{(by (2.4))}\\ &= (\mathbf{r}(x) \diamond \mathbf{d}(y))(\mathbf{r}(x) \diamond \mathbf{d}(s)) \quad \text{(since } E^\circ \text{ is a semilattice)}\end{aligned}$$

which implies that

$$\begin{aligned}(\mathbf{r}(c) \diamond \mathbf{d}(v))\mathbf{r}(c) &= (\mathbf{r}(x) \diamond \mathbf{d}(y))(\mathbf{r}(x) \diamond \mathbf{d}(s))[(\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x)] \quad \text{(by (4.7))}\\ &= (\mathbf{r}(x) \diamond \mathbf{d}(s))[(\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x)] \quad \text{(since } E^\circ \text{ is a semilattice)}\end{aligned}$$

and so

$$\begin{aligned} c|_{(\mathbf{r}(c)\diamond \mathbf{d}(v))\mathbf{r}(c)} &= c|_{(\mathbf{r}(x)\diamond \mathbf{d}(s))[(\mathbf{r}(x)\diamond \mathbf{d}(y))\mathbf{r}(x)]} \\ &= (x|_{(\mathbf{r}(x)\diamond \mathbf{d}(y))\mathbf{r}(x)})|_{(\mathbf{r}(x)\diamond \mathbf{d}(s))[\mathbf{r}(x)\diamond \mathbf{d}(y))\mathbf{r}(x)]} \\ &= x|_{(\mathbf{r}(x)\diamond \mathbf{d}(s))[(\mathbf{r}(x)\diamond \mathbf{d}(y))\mathbf{r}(x)]} \quad \text{(by (IPC2)}'\text{)}. \end{aligned} \tag{4.8}$$

Since $\mathbf{d}(s) = \mathbf{d}(y|_{(\mathbf{r}(y)\diamond \mathbf{d}(z))\mathbf{r}(y)}) \leq \mathbf{d}(y)$ by (IPC1)′, it follows from (3.1) that $\mathbf{d}(y)\mathbf{d}^\circ(s) = \mathbf{d}(y)\mathbf{d}(s) = \mathbf{d}(s)$. By (2.4), this implies that

$$\mathbf{r}(x) \diamond \mathbf{d}(s) = [\mathbf{r}(x) \diamond (\mathbf{d}(y)\mathbf{d}^\circ(s))] = [\mathbf{r}(x) \diamond \mathbf{d}(y)]\mathbf{d}^\circ(s),$$

which gives

$$\begin{aligned} c|_{(\mathbf{r}(c)\diamond \mathbf{d}(v))\mathbf{r}(c)} &= x|_{(\mathbf{r}(x)\diamond \mathbf{d}(s))(\mathbf{r}(x)\diamond \mathbf{d}(y))\mathbf{r}(x)} \quad \text{(by (4.8))} \\ &= x|_{[(\mathbf{r}(x)\diamond \mathbf{d}(y))\mathbf{d}^\circ(s)][(\mathbf{r}(x)\diamond \mathbf{d}(y))\mathbf{r}(x)]} \\ &= x|_{(\mathbf{r}(x)\diamond \mathbf{d}(y))\mathbf{d}^\circ(s)\ \mathbf{r}(x)} \quad \text{(since } E^\circ \text{ is a semilattice)} \\ &= x|_{(\mathbf{r}(x)\diamond \mathbf{d}(s))\ \mathbf{r}(x)} \\ &= u. \end{aligned} \tag{4.9}$$

This implies that

$$x \otimes (y \otimes z) = u \cdot (v \cdot w) = (u \cdot v) \cdot w = (x \otimes y) \otimes z$$

by (4.2), (PC2), (4.6), (4.8) and (4.9). Thus $(S, \otimes)$ is a semigroup. □

To obtain our desired pseudo-Ehresmann semigroups, we need the following lemmas.

Lemma 4.2. *In the semigroup $(P, \otimes)$, if $x, y \in P$ and $x \cdot y$ is defined in $\mathbf{P}$, then $x \otimes y = x \cdot y$.*

Proof. By hypothesis, we have $\mathbf{r}^\circ(x) = \mathbf{r}(x) \diamond \mathbf{d}(y) = \mathbf{d}^\circ(y)$, and hence

$$\begin{aligned} x \otimes y &= x|_{(\mathbf{r}(x)\diamond \mathbf{d}(y))\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond \mathbf{d}(y))}|y \\ &= x|_{\mathbf{r}^\circ(x)\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)\mathbf{d}^\circ(y)}|y = x|_{\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)}|y = x \cdot y \end{aligned}$$

by (2.4) and (IPC1) and its dual. □

Lemma 4.3. *In the semigroup $(P, \otimes)$, $e \otimes g = eg$ and $f \otimes h = fh$ for all $e, g \in I$ and $f, h \in \Lambda$.*

Proof. By Definition 3.2, (2.4), (3.1) and (IPC4),

$$\begin{aligned} e \otimes g &= e|_{(\mathbf{r}(e)\diamond \mathbf{d}(g))\mathbf{r}(e)} \cdot {}_{\mathbf{d}(g)(\mathbf{r}(e)\diamond \mathbf{d}(g))}|g = e|_{(e^\circ \diamond g)e^\circ} \cdot {}_{g(e^\circ \diamond g)}|g \\ &= e|_{e^\circ g e^\circ} \cdot {}_{g e^\circ g}|g = e|_{e^\circ g^\circ e^\circ} \cdot {}_{geg}|g = e|_{e^\circ g^\circ} \cdot {}_{ge}|g = ee^\circ g^\circ \cdot ge = eg \cdot ge = eg \end{aligned}$$

for all $e, g \in I$. Dually, $f \otimes h = fh$. □

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

THEOREM 4.4. *In the semigroup* $(P, \otimes)$, *define*

$$x^{\clubsuit} = \mathbf{d}(x), \quad x^{\spadesuit} = \mathbf{r}(x), \quad \widehat{x} = \mathbf{d}^{\circ}(x) \otimes x \otimes \mathbf{r}^{\circ}(x). \tag{4.10}$$

Then $\mathbf{P}\mathcal{S} = (P, \otimes, \clubsuit, \spadesuit, \widehat{\ })$ *forms a pseudo-Ehresmann semigroup.*

PROOF. By symmetry, it suffices to show that the identities (1)–(10) in Table 1 are valid. Firstly, since $\mathbf{d}(x) \cdot x$ is defined and $\mathbf{d}(x) \cdot x = x$ by (PC3),

$$x^{\clubsuit} \otimes x = \mathbf{d}(x) \otimes x = \mathbf{d}(x) \cdot x = x \tag{4.11}$$

by Lemma 4.2. This gives the identity (1) in Table 1. Since I is a left normal band, by Lemma 4.3,

$$x^{\clubsuit} \otimes y^{\clubsuit} \otimes x^{\clubsuit} = \mathbf{d}(x) \otimes \mathbf{d}(y) \otimes \mathbf{d}(x) = \mathbf{d}(x)\mathbf{d}(y)\mathbf{d}(x) = \mathbf{d}(x)\mathbf{d}(y) = x^{\clubsuit} \otimes y^{\clubsuit}.$$

Thus the identity (2) holds. By (3.2) in Definition 3.2,

$$(x^{\clubsuit} \otimes y^{\clubsuit})^{\clubsuit} = (\mathbf{d}(x)\mathbf{d}(y))^{\clubsuit} = \mathbf{d}(\mathbf{d}(x)\mathbf{d}(y)) = \mathbf{d}(x)\mathbf{d}(y) = x^{\clubsuit} \otimes y^{\clubsuit},$$

which proves the identity (3). Since

$$x \otimes y = x|_{(\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)(\mathbf{r}(x) \diamond \mathbf{d}(y))}|y,$$

we have $\mathbf{d}(x \otimes y) = \mathbf{d}(x|_{(\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x)})$. Similarly, we have $\mathbf{d}(x \otimes y^{\clubsuit}) = \mathbf{d}(x|_{(\mathbf{r}(x) \diamond \mathbf{d}(y^{\clubsuit}))\mathbf{r}(x)})$. By Proposition 3.3, $\mathbf{d}(y^{\clubsuit}) = \mathbf{d}(\mathbf{d}(y)) = \mathbf{d}(y)$. This implies that

$$(x \otimes y)^{\clubsuit} = \mathbf{d}(x \otimes y) = \mathbf{d}(x \otimes y^{\clubsuit}) = (x \otimes y^{\clubsuit})^{\clubsuit}.$$

That is, the identity (4) is satisfied. To verify the identity (5), we first observe that

$$(x^{\clubsuit})^{\spadesuit} = (\mathbf{d}(x))^{\spadesuit} = \mathbf{r}(\mathbf{d}(x)) = \mathbf{d}^{\circ}(x)$$

by Proposition 3.3. Furthermore, since

$$\mathbf{r}(x) \diamond \mathbf{d}(\mathbf{r}^{\circ}(x)) = \mathbf{r}(x) \diamond \mathbf{r}^{\circ}(x) = \mathbf{r}(x)\mathbf{r}^{\circ}(x) = \mathbf{r}^{\circ}(x) = \mathbf{d}^{\circ}(\mathbf{r}^{\circ}(x))$$

by Proposition 3.3 and (2.4), it follows that $x \cdot \mathbf{r}^{\circ}(x)$ is defined and so $x \otimes \mathbf{r}^{\circ}(x) = x \cdot \mathbf{r}^{\circ}(x)$ by Lemma 4.2, and hence $\mathbf{d}(x \otimes \mathbf{r}^{\circ}(x)) = \mathbf{d}(x \cdot \mathbf{r}^{\circ}(x)) = \mathbf{d}(x)$. By Proposition 3.3, (2.4) and (IPC4),

$$\begin{aligned} \widehat{x} &= \mathbf{d}^{\circ}(x) \otimes x \otimes \mathbf{r}^{\circ}(x) = \mathbf{d}^{\circ}(x) \otimes (x \otimes \mathbf{r}^{\circ}(x)) \\ &= \mathbf{d}^{\circ}(x)|_{(\mathbf{r}(\mathbf{d}^{\circ}(x)) \diamond \mathbf{d}(x))\mathbf{r}(\mathbf{d}^{\circ}(x))} \cdot {}_{\mathbf{d}(x)(\mathbf{r}(\mathbf{d}^{\circ}(x)) \diamond \mathbf{d}(x))}|(x \otimes \mathbf{r}^{\circ}(x)) \\ &= \mathbf{d}^{\circ}(x)|_{\mathbf{d}^{\circ}(x)} \cdot {}_{\mathbf{d}(x)}|(x \otimes \mathbf{r}^{\circ}(x)) = \mathbf{d}^{\circ}(x) \cdot {}_{\mathbf{d}(x)}|(x \otimes \mathbf{r}^{\circ}(x)) \end{aligned}$$

and hence $\mathbf{d}(\widehat{x}) = \mathbf{d}(\mathbf{d}^{\circ}(x)) = \mathbf{d}^{\circ}(x)$ by Proposition 3.3 again, and so

$$\widehat{x}^{\clubsuit} = \mathbf{d}(\widehat{x}) = \mathbf{d}^{\circ}(x). \tag{4.12}$$

Hence $(x^{\clubsuit})^{\spadesuit} = \widehat{x}^{\clubsuit}$. This is the identity (5). Moreover, the identity (6) follows from

$$\begin{aligned} \widehat{x^{\clubsuit}} &= \widehat{\mathbf{d}(x)} = \mathbf{d}^{\circ}(\mathbf{d}(x)) \otimes \mathbf{d}(x) \otimes \mathbf{r}^{\circ}(\mathbf{d}(x)) \\ &= \mathbf{d}^{\circ}(x) \otimes \mathbf{d}(x) \otimes \mathbf{d}^{\circ}(x) = \mathbf{d}^{\circ}(x)\mathbf{d}(x)\mathbf{d}^{\circ}(x) = \mathbf{d}^{\circ}(x) = \widehat{x}^{\clubsuit} \end{aligned}$$

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

by Proposition 3.3, Lemma 4.3 and (4.12). Furthermore, by Lemma 4.3 and (4.11) and its dual,

$$\begin{aligned} x^{\clubsuit} \otimes \widehat{x} \otimes x^{\spadesuit} &= (\mathbf{d}(x) \otimes \mathbf{d}^{\circ}(x)) \otimes x \otimes (\mathbf{r}^{\circ}(x) \otimes \mathbf{r}(x)) \\ &= (\mathbf{d}(x)\mathbf{d}^{\circ}(x)) \otimes x \otimes (\mathbf{r}^{\circ}(x)\mathbf{r}(x)) = \mathbf{d}(x) \otimes x \otimes \mathbf{r}(x) = x. \end{aligned}$$

This is exactly the identity (7). In view of (4.12) and its dual,

$$\widehat{x}^{\clubsuit} \otimes x \otimes \widehat{x}^{\spadesuit} = \mathbf{d}^{\circ}(x) \otimes x \otimes \mathbf{r}^{\circ}(x) = \widehat{x}.$$

This proves the identity (8). Finally,

$$\begin{aligned} x^{\spadesuit} \otimes y^{\clubsuit} = \mathbf{r}(x) \otimes \mathbf{d}(y) &= \mathbf{r}(x)|_{(\mathbf{r}(x)\diamond\mathbf{d}(y))\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond\mathbf{d}(y))}|\mathbf{d}(y) \\ &= (\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x) \cdot \mathbf{d}(y)(\mathbf{r}(x) \diamond \mathbf{d}(y)) \quad \text{(by (IPC4))} \\ &= [(\mathbf{r}(x) \diamond \mathbf{d}(y))\mathbf{r}(x)] \diamond [\mathbf{d}(y)(\mathbf{r}(x) \diamond \mathbf{d}(y))] \quad \text{(by (IPC4))} \\ &= (\mathbf{r}(x) \diamond \mathbf{d}(y))(\mathbf{r}(x) \diamond \mathbf{d}(y))(\mathbf{r}(x) \diamond \mathbf{d}(y)) \quad \text{(by (2.4))} \\ &= \mathbf{r}(x) \diamond \mathbf{d}(y) \in E^{\circ} \quad (\text{since } \mathbf{r}(x) \diamond \mathbf{d}(y) \in E^{\circ}). \end{aligned} \tag{4.13}$$

This implies that

$$(x^{\spadesuit} \otimes y^{\clubsuit})^{\clubsuit} = \mathbf{d}(\mathbf{r}(x) \diamond \mathbf{d}(y)) = \mathbf{r}(x) \diamond \mathbf{d}(y) = \mathbf{r}(\mathbf{r}(x) \diamond \mathbf{d}(y)) = (x^{\spadesuit} \otimes y^{\clubsuit})^{\spadesuit}$$

by Proposition 3.3, which states that the identities (9) and (10) hold. □

The next lemma shows that a pseudofunctor between inductive pseudocategories provides a (2,1,1,1)-morphism between the corresponding pseudo-Ehresmann semigroups.

Lemma 4.5. *Let* $\mathbf{P}_1 = (P_1, \cdot, \mathbf{d}, \mathbf{r}, Q_1), \mathbf{P}_2 = (P_2, \cdot, \mathbf{d}, \mathbf{r}, Q_2)$ *and* $\mathbf{P}_3 = (P_3, \cdot, \mathbf{d}, \mathbf{r}, Q_3)$ *be three inductive pseudocategories. If* φ *is a pseudofunctor from* $\mathbf{P_1}$ *to* $\mathbf{P_2}$*, then* $\varphi\mathcal{S} : P_1 \to P_2, \quad x \mapsto x\varphi$ *provides a (2,1,1,1)-morphism from* $\mathbf{P}_1\mathcal{S}$ *to* $\mathbf{P}_2\mathcal{S}$*. Moreover, if* φ_1 *and* φ_2 *are pseudofunctors from* $\mathbf{P_1}$ *to* $\mathbf{P_2}$ *and from* $\mathbf{P_2}$ *to* $\mathbf{P_3}$*, respectively, then* $(\varphi_1\varphi_2)\mathcal{S} = (\varphi_1\mathcal{S})(\varphi_2\mathcal{S})$*.*

Proof. Let $x, y \in P_1$. Firstly,

$$\begin{aligned} (x \otimes y)\varphi &= (x|_{(\mathbf{r}(x)\diamond\mathbf{d}(y))\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond\mathbf{d}(y))}|y)\varphi \\ &= (x|_{(\mathbf{r}(x)\diamond\mathbf{d}(y))\mathbf{r}(x)})\varphi \cdot ({}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond\mathbf{d}(y))}|y)\varphi \quad \text{(by (PF3))} \\ &= x\varphi|_{((\mathbf{r}(x)\diamond\mathbf{d}(y))\mathbf{r}(x))\varphi} \cdot {}_{(\mathbf{d}(y)(\mathbf{r}(x)\diamond\mathbf{d}(y)))\varphi}|y\varphi \quad \text{(by (PF4))} \\ &= x\varphi|_{((\mathbf{r}(x)\varphi)\diamond(\mathbf{d}(y)\varphi))(\mathbf{r}(x)\varphi)} \cdot {}_{(\mathbf{d}(y)\varphi)((\mathbf{r}(x)\varphi)\diamond(\mathbf{d}(y)\varphi)}|y\varphi \quad \text{(by (PF1))} \\ &= x\varphi|_{(\mathbf{r}(x\varphi)\diamond\mathbf{d}(y\varphi))\mathbf{r}(x\varphi)} \cdot {}_{\mathbf{d}(y\varphi)(\mathbf{r}(x\varphi)\diamond\mathbf{d}(y\varphi))}|y\varphi \quad \text{(by (PF2))} \\ &= (x\varphi) \otimes (y\varphi). \end{aligned}$$

This implies that $(x \otimes y)(\varphi\mathcal{S}) = (x(\varphi\mathcal{S})) \otimes (y(\varphi\mathcal{S}))$. Hence $\varphi\mathcal{S}$ preserves '$\otimes$'. Secondly,

$$x^{\clubsuit}(\varphi\mathcal{S}) = x^{\clubsuit}\varphi = (\mathbf{d}(x))\varphi = \mathbf{d}(x\varphi) = (x\varphi)^{\clubsuit} = (x(\varphi\mathcal{S}))^{\clubsuit}$$

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

by (PF2), which shows that $\varphi\mathcal{S}$ preserves '♣'. Dually, $\varphi\mathcal{S}$ also preserves '♠'. Finally, since φ preserves '$\otimes$' by the above discussion, it follows that

$$\begin{aligned}\widehat{x}\varphi = (\mathbf{d}^\circ(x) \otimes x \otimes \mathbf{r}^\circ(x))\varphi &= (\mathbf{d}^\circ(x))\varphi \otimes (x\varphi) \otimes (\mathbf{r}^\circ(x))\varphi \\ &= (\mathbf{d}(x)\varphi)^\circ \otimes (x\varphi) \otimes (\mathbf{r}(x)\varphi)^\circ \quad \text{(by (PF1) and (2.5))} \\ &= \mathbf{d}^\circ(x\varphi) \otimes (x\varphi) \otimes \mathbf{r}^\circ(x\varphi) \quad \text{(by (PF2))} \\ &= \widehat{x\varphi},\end{aligned}$$

which implies that $\widehat{x}(\varphi\mathcal{S}) = \widehat{x(\varphi\mathcal{S})}$ and so $\varphi\mathcal{S}$ preserves '$\frown$'. Thus $\varphi\mathcal{S}$ is a (2,1,1,1)-morphism. The final part of the lemma is clear. □

5. Correspondence

In the previous section, we started with an inductive pseudocategory over an admissible quadruple and constructed a pseudo-Ehresmann semigroup. Our present aim is to prove a converse to this result and thus provide a correspondence between the class of inductive pseudocategories over admissible quadruples and the class of pseudo-Ehresmann semigroups.

Let $\mathbf{S} = (S, \cdot, +, *, -)$ be a pseudo-Ehresmann semigroup and let $Q_{\mathbf{S}} = (I_{\mathbf{S}}, \Lambda_{\mathbf{S}}, E^\circ_{\mathbf{S}}, \diamond_{\mathbf{S}})$ be the admissible quadruple of $\mathbf{S}$, where

$$g^\circ = g^*, \quad f^\circ = f^+ \quad \text{and} \quad f \diamond_{\mathbf{S}} g = fg \text{ for all } g \in I_{\mathbf{S}} \text{ and } f \in \Lambda_{\mathbf{S}},$$

by Lemma 2.13. We shall define a pseudocategory $\mathbf{S}C = (S, \cdot, \mathbf{d}, \mathbf{r}, Q_{\mathbf{S}})$. Obviously, S contains the underlying set $I_{\mathbf{S}} \cup \Lambda_{\mathbf{S}}$ of $Q_{\mathbf{S}}$. Define a partial binary operation '$\cdot$' on S,

$$x \cdot y = \begin{cases} xy & \text{if } \overline{x}^* = x^*y^+ = \overline{y}^+, \\ \text{undefined} & \text{otherwise,} \end{cases}$$

where xy is the product of x and y in $\mathbf{S}$. Define

$$\mathbf{d} : S \to I_{\mathbf{S}}, x \mapsto x^+, \quad \mathbf{r} : S \to \Lambda_{\mathbf{S}}, x \mapsto x^*.$$

Then $\mathbf{d}(e) = e^+ = e$, $\mathbf{r}(e) = e^* = e^\circ$ and $\mathbf{r}(f) = f^* = f$, $\mathbf{d}(f) = f^+ = f^\circ$ for all $e \in I_{\mathbf{S}}$ and $f \in \Lambda_{\mathbf{S}}$ by Lemma 2.9(a). Clearly, for $x, y \in S$, by Lemma 2.13, $x \cdot y$ is defined if and only if

$$(x^*)^\circ = \overline{x}^* = x^*y^+ = \overline{y}^+ = (y^+)^\circ$$

if and only if

$$\mathbf{r}^\circ(x) = \mathbf{r}(x) \diamond_{\mathbf{S}} \mathbf{d}(y) = \mathbf{d}^\circ(y).$$

If this is the case, by Lemma 2.9(c), (d) and the identity (1)′ in Table 1,

$$\mathbf{d}(xy) = (xy)^+ = x^+(\overline{x}x^*y^+)^+ = x^+(\overline{x}\,\overline{x}^*)^+ = x^+\overline{x}^+ = x^+ = \mathbf{d}(x).$$

Dually, $\mathbf{r}(xy) = (xy)^* = y^* = \mathbf{r}(y)$. This implies that $\mathbf{d}(x \cdot y) = \mathbf{d}(x)$ and $\mathbf{r}(x \cdot y) = \mathbf{r}(y)$. Thus (PC1) holds. Let $x, y, z \in S$ and $x \cdot y$ and $y \cdot z$ be defined. Since $\mathbf{r}(x \cdot y) = \mathbf{r}(y)$ and

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

$\mathbf{d}(y \cdot z) = \mathbf{d}(y)$, $(x \cdot y) \cdot z$ and $x \cdot (y \cdot z)$ are also defined and so $(x \cdot y) \cdot z = x \cdot (y \cdot z) = xyz$. Finally, it is easy to see that $\mathbf{d}(x) \cdot x$ and $x \cdot \mathbf{r}(x)$ are defined and

$$\mathbf{d}(x) \cdot x = x^+ x = x = xx^* = x \cdot \mathbf{r}(x)$$

for all $x \in S$. Thus (PC2) and (PC3) are valid. From the above discussion, we have the following result.

Lemma 5.1. *Let* $\mathbf{S} = (S, \cdot, +, *, -)$ *be a pseudo-Ehresmann semigroup. Then* $\mathbf{S}C = (S, \cdot, \mathbf{d}, \mathbf{r}, Q_{\mathbf{S}})$ *is a pseudocategory, where*

$$\mathbf{d}(x) = x^+, \quad \mathbf{r}(x) = x^*, \quad e^\circ = e^*, \quad f^\circ = f^+, \quad f \diamond_{\mathbf{S}} e = fe$$

for all $e \in I_{\mathbf{S}}$ *and* $f \in \Lambda_{\mathbf{S}}$.

We build on the above to show that $\mathbf{S}C = (S, \cdot, \mathbf{d}, \mathbf{r}, Q_{\mathbf{S}})$ may be equipped with restrictions and co-restrictions, under which it becomes an inductive pseudocategory. For $x \in S$, $e \in I_{\mathbf{S}}$, $f \in \Lambda_{\mathbf{S}}$ with $e \leq_{I_{\mathbf{S}}} \mathbf{d}(x)$ and $f \leq_{\Lambda_{\mathbf{S}}} \mathbf{r}(x)$, we define the restriction and co-restriction of x to e and f as

$$_{e}|x = ex, \quad x|_{f} = xf.$$

Lemma 5.2. *Let* $\mathbf{S} = (S, \cdot, +, *, -)$ *be a pseudo-Ehresmann semigroup. Then* $\mathbf{S}C = (S, \cdot, \mathbf{d}, \mathbf{r}, Q_{\mathbf{S}})$ *is an inductive pseudocategory with the above definitions of restriction and co-restriction.*

Proof. It remains to show that the conditions in Definition 3.5 are all satisfied. By symmetry, we only need to check the conditions (IPC1)–(IPC5).

(IPC1) Let $x \in S$ and $e \in I_{\mathbf{S}}$ with $e \leq_{I_{\mathbf{S}}} \mathbf{d}(x)$. Since $\mathbf{d}(x) = x^+$, we have $ex^+ = e$ and

$$\mathbf{d}(_{e}|x) = \mathbf{d}(ex) = (ex)^+ = (ex^+)^+ = ex^+ = e$$

by the identities (4) and (3) in Table 1. Moreover, $_{\mathbf{d}(x)}|x = _{x^+}|x = x^+x = x$. On the other hand, by the identities (3)′, (4)′ and (1)′,

$$\mathbf{r}(_{e}|x)\mathbf{r}(x) = (ex)^* x^* = ((ex)^* x^*)^* = (exx^*)^* = (ex)^* = \mathbf{r}(_{e}|x).$$

Since $\Lambda_{\mathbf{S}}$ is a right normal band by Lemma 2.9(e),

$$\mathbf{r}(x)\mathbf{r}(_{e}|x) = x^*(ex)^* = (ex)^* x^* (ex)^* = ((ex)^* x^*)(ex)^* = (ex)^*(ex)^* = (ex)^*.$$

This implies that $\mathbf{r}(_{e}|x) = (ex)^* \leq_{\Lambda_{\mathbf{S}}} x^* = \mathbf{r}(x)$.

(IPC2) If $e, g \in I_{\mathbf{S}}$ and $x \in S$ with $e \leq_{I_{\mathbf{S}}} g \leq_{I_{\mathbf{S}}} \mathbf{d}(x)$, then

$$_{e}|(_{g}|x) = e(_{g}|x) = e(gx) = egx = ex = _{e}|x.$$

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

(IPC3) If $e \in I_{\mathbf{S}}, x, y \in S, e \leq_{I_{\mathbf{S}}} \mathbf{d}(x) = x^+$ and $x \cdot y$ is defined, then ${}_e|(x \cdot y) = {}_e|(xy) = exy$ and

$$\begin{aligned}({}_e|x) \cdot_{\mathbf{d}(y)(\mathbf{r}({}_e|x) \diamond_{\mathbf{S}} \mathbf{d}(y))} |y &= ({}_e|x) \cdot_{y^+((ex)^* y^+)} |y \\ &= ex(y^+(ex)^* y^+ y) = exy^+(ex)^* y^+ y \\ &= ex(ex)^* y^+(ex)^* y^+ y \quad \text{(by the identity (1)}' \text{ in Table 1)} \\ &= ex(ex)^* y^+ y \quad \text{(since } (ex)^* y^+ \in E^\circ_{\mathbf{S}} \text{ by Lemma 2.9(b))} \\ &= exy \quad \text{(by the identities (1) and (1)}' \text{ in Table 1).}\end{aligned}$$

(IPC4) Let $e, g \in I_{\mathbf{S}}, u \in E^\circ_{\mathbf{S}}$ with $e^\circ u = u$. Then $e|_u = eu$ and

$${}_{eg}|e = ege = eg, \quad eg \cdot ge = (eg)(ge) = egge = ege = eg,$$

as $I_{\mathbf{S}}$ is a left normal band. Dually, for $f, h \in \Lambda_{\mathbf{S}}, u \in E^\circ_{\mathbf{S}}$ with $uf^\circ = u$,

$${}_u|f = uf, \quad f|_{hf} = hf, \quad hf \cdot fh = fh.$$

Moreover, if $f \in \Lambda_{\mathbf{S}}, g \in I_{\mathbf{S}}$ and $f \cdot g$ is defined in $\mathbf{S}C$, then $f \cdot g = fg = f \diamond_{\mathbf{S}} g$.

(IPC5) Let $e, g \in I_{\mathbf{S}}, f \in \Lambda_{\mathbf{S}}, x \in S$ and

$$p = x|_{(\mathbf{r}(x) \diamond_{\mathbf{S}} g)\mathbf{r}(x)}, \quad q = {}_{\mathbf{d}(x)(f \diamond_{\mathbf{S}} \mathbf{d}(x))}|x.$$

Then

$$\begin{aligned}&{}_{\mathbf{d}(p)(f \diamond_{\mathbf{S}} \mathbf{d}(p))}|p \\ &= (xx^*gx^*)^+ f (xx^*gx^*)^+ xx^*gx^* = (xx^*gx^*)^+ fxx^*gx^* \quad \text{(using (1) in Table 1)} \\ &= (xx^*g(x^*)^+)^+ fxx^*gx^* = (xx^*g\overline{x}^*)^+ fxx^*gx^* \quad \text{(using (4) and (5)}' \text{ in Table 1)} \\ &= (x\overline{x}^*(x^*g))^+ fxx^*gx^* \quad (x^*g, \overline{x}^* \in E^\circ_{\mathbf{S}} \text{ by Lemma 2.9(b), } E^\circ_{\mathbf{S}} \text{ is a semilattice)} \\ &= (xg)^+ fxgx^* \quad (\overline{x}^* \mathcal{R} x^*, xx^* = x) \\ &= x^+(xg)^+ fx^+ xgx^* = x^+(xg)^+(fx^+)xgx^* \\ &\quad \text{(since } x^+(xg)^+ = (xg)^+ \quad \text{by Lemma 2.9(c), } x^+x = x) \\ &= x^+(fx^+)(xg)^+ xgx^* \; (fx^+ \in E^\circ_{\mathbf{S}} \quad \text{by Lemma 2.9(b), } I_{\mathbf{S}} \text{ is left normal)} \\ &= x^+(fx^+)xgx^* = x^+ fxgx^* \quad \text{(by (1) in Table 1).}\end{aligned}$$

Similarly, we can also obtain that $q|_{(\mathbf{r}(q) \diamond_{\mathbf{S}} g)\mathbf{r}(q)} = x^+ fxgx^*$, as required. □

We now proceed to establish an isomorphism between the category **IPC** of inductive pseudocategories over admissible quadruples and the category **PES** of pseudo-Ehresmann semigroups. The next lemma demonstrates that a (2,1,1,1)-morphism between two pseudo-Ehresmann semigroups gives rise to a pseudofunctor.

Lemma 5.3. *Let* $\mathbf{S}_1 = (S_1, \cdot, +, *, -), \mathbf{S}_2 = (S_2, \cdot, +, *, -)$ *and* $\mathbf{S}_3 = (S_3, \cdot, +, *, -)$ *be three pseudo-Ehresmann semigroups and let* θ *a (2,1,1,1)-morphism from* $\mathbf{S}_1$ *to* $\mathbf{S}_2$. *Then the mapping* $\theta C : S_1 \to S_2$ *given by the rule that* $x(\theta C) = x\theta$ *provides a pseudofunctor from* $\mathbf{S}_1 C$ *to* $\mathbf{S}_2 C$. *Further, if* $\theta_1 : \mathbf{S}_1 \to \mathbf{S}_2$ *and* $\theta_2 : \mathbf{S}_2 \to \mathbf{S}_3$ *are (2,1,1,1)-morphisms, then* $(\theta_1\theta_2)C = (\theta_1 C)(\theta_2 C)$.

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

PROOF. Firstly, θ induces a morphism from $Q_{\mathbf{S}_1}$ to $Q_{\mathbf{S}_2}$ by Lemma 2.14. Secondly, for $x \in S$,

$$\mathbf{d}(x\theta) = (x\theta)^+ = x^+\theta = (\mathbf{d}(x))\theta \quad \text{and} \quad \mathbf{r}(x\theta) = (x\theta)^* = x^*\theta = \mathbf{r}(x)\theta,$$

as θ is a (2,1,1,1)-morphism. Thirdly, if $x, y \in S_1$ and $x \cdot y$ is defined in $\mathbf{S}_1\mathcal{C}$, then $x \cdot y = xy$ and so

$$(x \cdot y)\theta = (xy)\theta = (x\theta)(y\theta) = (x\theta) \cdot (y\theta)$$

in $\mathbf{S}_2\mathcal{C}$. Finally, if $x \in S_1, e \in I_{\mathbf{S}_1}, f \in \Lambda_{\mathbf{S}_1}$ and $e \leq_{I_{\mathbf{S}_1}} \mathbf{d}(x) = x^+, f \leq_{\Lambda_{\mathbf{S}_1}} \mathbf{r}(x) = x^*$, then

$$({}_e|x)\theta = (ex)\theta = (e\theta)(x\theta) = {}_{e\theta}|x\theta, \quad (x|_f)\theta = (xf)\theta = (x\theta)(f\theta) = x\theta|_{f\theta},$$

as required. The final assertion is clear. □

LEMMA 5.4. *Let $\mathbf{S} = (S, \cdot, +, *, -)$ be a pseudo-Ehresmann semigroup whose admissible quadruple is $Q_{\mathbf{S}} = (I_{\mathbf{S}}, \Lambda_{\mathbf{S}}, E^\circ_{\mathbf{S}}, \diamond_{\mathbf{S}})$. Then $(\mathbf{S}\mathcal{C})\mathcal{S} = \mathbf{S}$.*

PROOF. By Lemmas 5.1 and 5.2, $\mathbf{S}\mathcal{C} = (S, \cdot, \mathbf{d}, \mathbf{r}, Q_{\mathbf{S}})$ is an inductive pseudocategory and

$$\mathbf{d}(x) = x^+, \quad \mathbf{r}(x) = x^*, \quad e^\circ = e^*, \quad f^\circ = f^+, \quad f \diamond_{\mathbf{S}} e = fe \tag{5.1}$$

for all $e \in I_{\mathbf{S}}$ and $f \in \Lambda_{\mathbf{S}}$. For all $x, y \in S$, $x \cdot y = xy$ if and only if

$$\overline{x}^* = (x^*)^+ = \mathbf{r}^\circ(x) = \mathbf{r}(x) \diamond_{\mathbf{S}} \mathbf{d}(y) = \mathbf{d}^\circ(y) = (y^+)^* = \overline{y}^+, \tag{5.2}$$

where xy is the product of x and y in $\mathbf{S}$. Moreover, for $x \in S, e \in I_{\mathbf{S}}, f \in \Lambda_{\mathbf{S}}$ with $e \leq_{I_{\mathbf{S}}} \mathbf{d}(x)$ and $f \leq_{\Lambda_{\mathbf{S}}} \mathbf{r}(x)$, we have ${}_e|x = ex, x|_f = xf$.

We now construct a pseudo-Ehresmann semigroup $(\mathbf{S}\mathcal{C})\mathcal{S} = (S, \otimes, ♣, ♠, \frown)$ by defining the pseudoproduct

$$x \otimes y = x|_{(\mathbf{r}(x)\diamond_{\mathbf{S}}\mathbf{d}(y))\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond_{\mathbf{S}}\mathbf{d}(y))}|y$$

on S. Observe that

$$\begin{aligned} x \otimes y &= x|_{(\mathbf{r}(x)\diamond_{\mathbf{S}}\mathbf{d}(y))\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond_{\mathbf{S}}\mathbf{d}(y))}|y \\ &= x|_{x^*y^+x^*} \cdot {}_{y^+x^*y^+}|y \\ &= xx^*y^+x^*y^+x^*y^+y \\ &= xx^*y^+y \quad (x^*y^+ \in E^\circ_{\mathbf{S}} \text{ by Lemma 2.9(b)}) \\ &= xy \quad (\text{by (1) and (1)}' \text{ in Table 1}) \end{aligned} \tag{5.3}$$

so the binary operations in $\mathbf{S}$ and $(\mathbf{S}\mathcal{C})\mathcal{S}$ are the same. In view of (4.10), (5.1), (5.2) and (5.3),

$$x^♣ = \mathbf{d}(x) = x^+, \quad x^♠ = \mathbf{r}(x) = x^*, \quad \widehat{x} = \mathbf{d}^\circ(x) \otimes x \otimes \mathbf{r}^\circ(x) = \overline{x}^+ x \overline{x}^* = \overline{x}$$

by the identity (8) in Table 1. Thus, $(\mathbf{S}\mathcal{C})\mathcal{S} = \mathbf{S}$. □

LEMMA 5.5. *Let $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, Q)$ be an inductive pseudocategory over an admissible quadruple $Q = (I, \Lambda, E^\circ, \diamond)$. Then $(\mathbf{P}\mathcal{S})\mathcal{C} = \mathbf{P}$.*

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

Proof. By Theorem 4.4, we can construct the pseudo-Ehresmann semigroup $\mathbf{P}\mathcal{S} = (P, \otimes, \clubsuit, \spadesuit, \frown)$ by defining the pseudoproduct

$$x \otimes y = x|_{(\mathbf{r}(x)\diamond\mathbf{d}(y))\mathbf{r}(x)} \cdot {}_{\mathbf{d}(y)(\mathbf{r}(x)\diamond\mathbf{d}(y))}|y$$

and

$$x^{\clubsuit} = \mathbf{d}(x), \quad x^{\spadesuit} = \mathbf{r}(x), \quad \widehat{x} = \mathbf{d}^{\circ}(x) \otimes x \otimes \mathbf{r}^{\circ}(x), \tag{5.4}$$

for all $x, y \in P$. Moreover, by Lemma 2.13 we have the admissible quadruple $Q_{\mathbf{P}\mathcal{S}} = (I_{\mathbf{P}\mathcal{S}}, \Lambda_{\mathbf{P}\mathcal{S}}, E^{\circ}_{\mathbf{P}\mathcal{S}}, \diamond_{\mathbf{P}\mathcal{S}})$ with

$$u^{\circ} = u^{\spadesuit}, v^{\circ} = v^{\clubsuit}, v \diamond_{\mathbf{P}\mathcal{S}} u = v \otimes u$$

for all $u \in I_{\mathbf{P}\mathcal{S}}$ and $v \in \Lambda_{\mathbf{P}\mathcal{S}}$. We first show that $Q = Q_{\mathbf{P}\mathcal{S}}$. In fact, by Proposition 3.3 and (5.4),

$$I_{\mathbf{P}\mathcal{S}} = \{x^{\clubsuit} \mid x \in P\} = \{\mathbf{d}(x) \mid x \in P\} = I,$$
$$\Lambda_{\mathbf{P}\mathcal{S}} = \{x^{\spadesuit} \mid x \in P\} = \{\mathbf{r}(x) \mid x \in P\} = \Lambda.$$

Let $e, g \in I = I_{\mathbf{P}\mathcal{S}}$ and $f, h \in \Lambda = \Lambda_{\mathbf{P}\mathcal{S}}$. By Lemma 4.3, the product eg of e and g in I (respectively, fh of f and h in Λ) is equal to the product $e \otimes g$ of e and g in $I_{\mathbf{P}\mathcal{S}}$ (respectively, $f \otimes h$ of f and h in $\Lambda_{\mathbf{P}\mathcal{S}}$). Moreover, by Definition 3.2, (4.13) and (5.4), for all $f \in \Lambda = \Lambda_{\mathbf{P}\mathcal{S}}$ and $g \in I = I_{\mathbf{P}\mathcal{S}}$,

$$f \diamond g = \mathbf{r}(f) \diamond \mathbf{d}(g) = f^{\spadesuit} \otimes g^{\clubsuit} = \mathbf{r}(f) \otimes \mathbf{d}(g) = f \otimes g = f \diamond_{\mathbf{P}\mathcal{S}} g.$$

Thus $Q = Q_{\mathbf{P}\mathcal{S}}$. By Lemmas 5.1 and 5.2, we can construct an inductive pseudocategory $(\mathbf{P}\mathcal{S})C = (P, \odot, \mathbf{d}_1, \mathbf{r}_1, Q_{\mathbf{P}\mathcal{S}})$ by setting

$$\mathbf{d}_1(x) = x^{\clubsuit}, \mathbf{r}_1(x) = x^{\spadesuit} \tag{5.5}$$

for all $x \in P$. Moreover, for all $u \in I_{\mathbf{P}\mathcal{S}}$ and $v \in \Lambda_{\mathbf{P}\mathcal{S}}$ with $u \leq_{I_{\mathbf{P}\mathcal{S}}} \mathbf{d}_1(x)$ and $v \leq_{\Lambda_{\mathbf{P}\mathcal{S}}} \mathbf{r}_1(x)$, we have ${}_u|x$ (in $(\mathbf{P}\mathcal{S})C$)$= u \otimes x$ and $x|_v$(in $(\mathbf{P}\mathcal{S})C$) $= x \otimes v$. In view of (5.4) and (5.5), we have $\mathbf{d}_1(x) = x^{\clubsuit} = \mathbf{d}(x)$ and $\mathbf{r}_1(x) = x^{\spadesuit} = \mathbf{r}(x)$ for all $x \in P$. This implies that $\mathbf{d} = \mathbf{d}_1$ and $\mathbf{r} = \mathbf{r}_1$. Therefore for all $x, y \in P$, $x \odot y$ is defined in $(\mathbf{P}\mathcal{S})C$ if and only if $x \cdot y$ is defined in $\mathbf{P}$. In this case, $x \odot y = x \otimes y = x \cdot y$ by Lemma 4.2. Furthermore,

$${}_e|x \text{ is defined in} (\mathbf{P}\mathcal{S})C \text{ if and only if } {}_e|x \text{ is defined in } \mathbf{P}$$
$$\text{if and only if } e \leq_I \mathbf{d}(x) \text{ if and only if } e\mathbf{d}(x) = \mathbf{d}(x)e = e$$

for all $e \in I$ and $x \in P$. If this is the case, $e^{\circ}\mathbf{d}^{\circ}(x) = \mathbf{d}^{\circ}(x)e^{\circ} = e^{\circ}$ by (3.1) and

$$\begin{aligned}
{}_e|x \text{ (in } (\mathbf{P}\mathcal{S})C) = e \otimes x &= e|_{(\mathbf{r}(e)\diamond\mathbf{d}(x))\mathbf{r}(e)} \cdot {}_{\mathbf{d}(x)(\mathbf{r}(e)\diamond\mathbf{d}(x))}|x \\
&= e|_{e^{\circ}\mathbf{d}(x)e^{\circ}} \cdot {}_{\mathbf{d}(x)e^{\circ}\mathbf{d}(x)}|x \quad \text{(by Definition 3.2 and (2.4))} \\
&= e|_{e^{\circ}\mathbf{d}^{\circ}(x)e^{\circ}} \cdot {}_{\mathbf{d}(x)e\mathbf{d}(x)}|x \quad \text{(by (3.1))} \\
&= e|_{e^{\circ}} \cdot {}_e|x = (ee^{\circ}) \cdot {}_e|x \quad \text{(by (IPC4))} \\
&= e \cdot {}_e|x \quad \text{(by } e\mathcal{L}e^{\circ}) \\
&= \mathbf{d}({}_e|x) \cdot {}_e|x \quad \text{(by (IPC1))} \\
&= {}_e|x \text{ (in } \mathbf{P}) \quad \text{(by (PC3))}.
\end{aligned}$$

Dually, $x|_f$ (in $(\mathbf{P}\mathcal{S})C$) $= x|_f$ (in $\mathbf{P}$) for all $f \in \Lambda$ with $f \leq_{\Lambda} \mathbf{r}(x)$. Thus, $(\mathbf{P}\mathcal{S})C = \mathbf{P}$. □

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

Using Lemmas 5.3 and 4.5, we can show the following result easily.

LEMMA 5.6. *Let* $\theta : \mathbf{S}_1 \to \mathbf{S}_2$ *be a (2,1,1,1)-morphism of pseudo-Ehresmann semigroups and let* $\varphi : \mathbf{P}_1 \to \mathbf{P}_2$ *be a pseudofunctor of inductive pseudocategories. Then* $(\theta\mathcal{C})\mathcal{S} = \theta$ *and* $(\varphi\mathcal{S})\mathcal{C} = \varphi$.

Theorem 4.4 and Lemma 4.5 show that $\mathcal{S} : \mathbf{IPC} \to \mathbf{PES}$ is a functor, and Lemmas 5.2 and 5.3 show that $\mathcal{C} : \mathbf{PES} \to \mathbf{IPC}$ is a functor in the sense of Definition 2.17. Moreover, Lemmas 5.4, 5.5 and 5.6 give that $\mathcal{S}$ and $\mathcal{C}$ are mutually inverse. Hence we deduce our main result of this paper.

THEOREM 5.7. *The category* **PES** *of pseudo-Ehresmann semigroups and (2,1,1,1)-morphisms is isomorphic to the category* **IPC** *of inductive pseudocategories over admissible quadruples and pseudofunctors.*

6. Two special cases

This section concentrates on two special kinds of pseudo-Ehresmann semigroups. We firstly concern the class of regular semigroups with a multiplicative inverse transversal. We say an inductive pseudocategory $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, Q)$ over an admissible quadruple Q is an *inductive pseudogroupoid* if there exists $x^\circ \in P$ such that $x \cdot x^\circ$ and $x^\circ \cdot x$ are defined in $\mathbf{P}$ and $x \cdot x^\circ = \mathbf{d}(x)$, $x^\circ \cdot x = \mathbf{r}(x)$ for any $x \in P$. Using this notion, we have the following corollary.

COROLLARY 6.1. *The category of regular semigroups with a multiplicative inverse transversal as (2,1,1,1)-algebras and (2,1,1,1)-morphisms is isomorphic to the category of inductive pseudogroupoids and pseudofunctors.*

PROOF. Let S be a regular semigroup with a multiplicative inverse transversal S° and let x° be the unique inverse of x in S° for any $x \in S$. Then we have the induced pseudo-Ehresmann semigroup $\mathbf{S} = (S, \cdot, +, *, -)$ by setting $x^+ = xx^\circ$, $x^* = x^\circ x$, $\overline{x} = x^{\circ\circ}$. In the pseudocategory $\mathbf{S}\mathcal{C}$ constructed in section 5, we have $\overline{x}^* = (x^{\circ\circ})^* = (x^{\circ\circ})^\circ x^{\circ\circ} = x^\circ x^{\circ\circ}$ and $x^*(x^\circ)^+ = x^\circ x x^\circ x^{\circ\circ} = x^\circ x^{\circ\circ}$, $\overline{x^\circ}^+ = ((x^\circ)^{\circ\circ})^+ = (x^\circ)^+ = x^\circ x^{\circ\circ}$. This implies that $\overline{x}^* = x^*(x^\circ)^+ = \overline{x^\circ}^+$ and hence $x \cdot x^\circ$ is defined in $\mathbf{S}\mathcal{C}$ and $x \cdot x^\circ = xx^\circ = x^+ = \mathbf{d}(x)$. Dually, $x^\circ \cdot x$ is also defined in $\mathbf{S}\mathcal{C}$ and $x^\circ \cdot x = x^\circ x = x^* = \mathbf{r}(x)$. Thus, $\mathbf{S}\mathcal{C}$ is an inductive pseudogroupoid.

Conversely, let $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, Q)$ be an inductive pseudogroupoid. From Theorem 4.4, we have the pseudo-Ehresmann semigroup $\mathbf{P}\mathcal{S} = (P, \otimes, \clubsuit, \spadesuit, \frown)$. Since $\mathbf{P}$ is an inductive pseudogroupoid, there exists $x^\circ \in P$ such that $x \cdot x^\circ$ and $x^\circ \cdot x$ are defined in $\mathbf{P}$ and $x \cdot x^\circ = \mathbf{d}(x)$, $x^\circ \cdot x = \mathbf{r}(x)$ for any $x \in P$. This implies that $x \otimes x^\circ = x \cdot x^\circ = \mathbf{d}(x) = x^\clubsuit$ and $x^\circ \otimes x = x^\circ \cdot x = \mathbf{r}(x) = x^\spadesuit$ by Lemma 4.2 and Theorem 4.4. In view of Proposition 2.12, $\widehat{P} = \{\widehat{x} \mid x \in P\}$ is a multiplicative inverse transversal of the semigroup $(P, \cdot)$ and induces the pseudo-Ehresmann semigroup $\mathbf{P}\mathcal{S}$. □

On the other hand, for Ehresmann semigroups, we have the following corollary.

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

Corollary 6.2. *The category of Ehresmann semigroups as (2,1,1)-algebras and (2,1,1)-morphisms is isomorphic to the category of inductive pseudocategories over semilattices and pseudofunctors.*

Proof. Let $(S, \cdot, +, *)$ be an Ehresmann semigroup. Then by Example 2.6 we have the pseudo-Ehresmann semigroup $\mathbf{S} = (S, \cdot, +, *, -)$ by setting $\overline{x} = x$ for all $x \in S$. In this case, $I_{\mathbf{S}} = \{x^+ \mid x \in S\} = \{x^* \mid x \in S\} = \Lambda_{\mathbf{S}}$ is a semilattice by Lemma 2.1. So $Q_{\mathbf{S}} = (I_{\mathbf{S}}, \Lambda_{\mathbf{S}}, E^{\circ}_{\mathbf{S}}, \diamond_{\mathbf{S}})$ is just the semilattice $I_{\mathbf{S}}$ (or $\Lambda_{\mathbf{S}}$). Thus $\mathbf{S}\mathcal{C}$ constructed in Section 5 is an inductive pseudocategory over the semilattice $I_{\mathbf{S}}$. Conversely, let $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, E)$ be an inductive pseudocategory over a semilattice E. Then we have the pseudo-Ehresmann semigroup $\mathbf{P}\mathcal{S} = (P, \otimes, \clubsuit, \spadesuit, \frown)$ by Theorem 4.4. Since E is a semilattice, we have $\mathbf{d}^{\circ}(x) = \mathbf{d}(x)$ and $\mathbf{r}^{\circ}(x) = \mathbf{r}(x)$ for all $x \in P$. This implies that $\widehat{x} = \mathbf{d}^{\circ}(x) \otimes x \otimes \mathbf{r}^{\circ}(x) = \mathbf{d}(x) \otimes x \otimes \mathbf{r}(x) = x$ for all $x \in P$ by (4.11) and its dual. In view of Proposition 2.10, $(P, \otimes, \clubsuit, \spadesuit)$ is an Ehresmann semigroup. □

In the remainder of this section, we shall explore the connection between *Ehresmann categories* and *strongly ordered functors* introduced in Lawson [20] and pseudocategories over semilattices and pseudofunctors considered in this paper. To this aim, we need some notions taken from Lawson [20]. Recall that $(P, \cdot, \mathbf{d}, \mathbf{r}, \leq)$ is called an *ordered category*, if the following conditions hold.

(OC1) $(P, \cdot, \mathbf{d}, \mathbf{r})$ is a small category in the sense of Definition 2.19 and $(P, \leq)$ is a poset.

(OC2) $x \leq y$ implies that $\mathbf{r}(x) \leq \mathbf{r}(y)$ and $\mathbf{d}(x) \leq \mathbf{d}(y)$.

(OC3) If $x_1 \leq y_1$ and $x_2 \leq y_2$, and $x_1 \cdot x_2$ and $y_1 \cdot y_2$ are defined, then $x_1 \cdot x_2 \leq y_1 \cdot y_2$.

(OC4) If $\mathbf{r}(x) = \mathbf{r}(y)$ and $\mathbf{d}(x) = \mathbf{d}(y)$, $x \leq y$, then $x = y$.

Also from Lawson [20], an *Ehresmann category* $(P, \cdot, \mathbf{d}, \mathbf{r}, \leq_l, \leq_r)$ is a small category $(P, \cdot, \mathbf{d}, \mathbf{r})$ in the sense of Definition 2.19 with set of local identities

$$P_o = \{\mathbf{d}(x) \mid x \in P\} = \{\mathbf{r}(x) \mid x \in P\},$$

equipped with two partial orders '$\leq_l$' and '$\leq_r$' such that the following conditions, and the duals (E1)′ and (E5)′ of (E1) and (E5) hold.

(E1) $(P, \cdot, \mathbf{d}, \mathbf{r}, \leq_l)$ is an ordered category and, for all $x \in P$ and $e \in P_o$ with $e \leq_l \mathbf{d}(x)$, there exists a unique element ${}_e|x \in P$ called the *restriction* of x to e such that ${}_e|x \leq_l x$ and $\mathbf{d}({}_e|x) = e$.

(E2) If $e, g \in P_o$, then $e \leq_l g$ if and only if $e \leq_r g$.

(E3) $(P_o, \leq)$ is a meet semilattice, where $\leq = \leq_l = \leq_r$ on P_o and $e \wedge f$ is the meet of e and f in P_o.

(E4) $\leq_l \circ \leq_r = \leq_r \circ \leq_l$.

(E5) If $x \leq_l y$ and $e \in P_o$, then $x|_{\mathbf{r}(x)\wedge e} \leq_l y|_{\mathbf{r}(y)\wedge e}$.

We note that Lawson [20] interchanges the symbols $\mathbf{d}$ and $\mathbf{r}$, the symbols $\leq_l$ and $\leq_r$ and the notions of restriction and co-restriction from the conventions of this paper.

The following lemma shows that an inductive pseudocategory over a semilattice can produce an Ehresmann category.

Lemma 6.3. *If* $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, E)$ *is an inductive pseudocategory over the semilattice* E, *then* $(P, \cdot, \mathbf{d}, \mathbf{r} \leq_l, \leq_r)$ *forms an Ehresmann category, where* $\leq_l$ *and* $\leq_r$ *are defined as in the statements before Lemma 3.7.*

Proof. From Remark 3.4, $\mathbf{P} = (P, \cdot, \mathbf{d}, \mathbf{r}, E)$ is a small category in the sense of Definition 2.19 and the set of local identities of $\mathbf{P}$ is E. By Lemma 3.7, $\leq_l$ is a partial order on P. This gives (OC1). We next assert that

$$\text{the restriction of } \leq_l \text{ to } E \text{ is equal to } \leq_E . \tag{6.1}$$

In fact, if $e, g \in E$ and $e \leq_l g$, then $e =_u |g$ for some $u \in E$ with $u \leq_E \mathbf{d}(g) = g$. By (IC4) of Remark 3.6, $e =_u |g = u$, this implies that $e = u \leq_E g$. Conversely, if $e \leq_E g$, then we have $e =_e |g$ again by (IC4), which gives $e \leq_l g$.

Now let $x \leq_l y$. Then $x =_g |y$ for some $g \in E$ with $g \leq_E \mathbf{d}(y)$. This implies that $\mathbf{d}(x) = g \leq_E \mathbf{d}(y)$ and $\mathbf{r}(x) \leq_E \mathbf{r}(y)$ by (IC1) and so $\mathbf{d}(x) \leq_l \mathbf{d}(y)$ and $\mathbf{r}(x) \leq_l \mathbf{r}(y)$ by (6.1). This shows (OC2).

To see (OC3), let $x_1 \leq y_1$ and $x_2 \leq y_2$, and $x_1 \cdot x_2$ and $y_1 \cdot y_2$ be defined in $\mathbf{P}$. Then $x_1 =_e |y_1, x_2 =_g |y_2$ for some $e, g \in E$ with $e \leq_E \mathbf{d}(y_1)$, $g \leq_E \mathbf{d}(y_2)$ and $\mathbf{r}(x_1) = \mathbf{d}(x_2), \mathbf{r}(y_1) = \mathbf{d}(y_2)$. This implies that

$$\mathbf{r}({}_e|y_1) = \mathbf{r}(x_1) = \mathbf{d}(x_2) = \mathbf{d}({}_g|y_2) = g$$

by (IC1). Thus

$$\begin{aligned} x_1 \cdot x_2 = ({}_e|y_1) \cdot ({}_g|y_2) &= ({}_e|y_1) \cdot ({}_{g\mathbf{d}(y_2)}|y_2) \quad (\text{by } g \leq_E \mathbf{d}(y_2)) \\ &= ({}_e|y_1) \cdot ({}_{\mathbf{r}({}_e|y_1)\mathbf{d}(y_2)}|y_2) =_e |(y_1 \cdot y_2) \quad (\text{by (IC3)}), \end{aligned}$$

which shows that $x_1 \cdot x_2 \leq y_1 \cdot y_2$. Thus (OC3) holds.

If $\mathbf{r}(x) = \mathbf{r}(y)$ and $\mathbf{d}(x) = \mathbf{d}(y), x \leq_l y$, then $x =_e |y$ for some $e \in E$. This gives $\mathbf{d}(y) = \mathbf{d}(x) = \mathbf{d}({}_e|y) = e$ by (IC1) and so $x =_{\mathbf{d}(y)} |y = y$ by (IC1) again. Therefore (OC4) is true.

Suppose that $x \in P, e \in E$ and $e \leq_l \mathbf{d}(x)$. Then $e \leq_E \mathbf{d}(x)$ by (6.1). By (IC1), there exists ${}_e|x \in P$ such that $\mathbf{d}({}_e|x) = e$. Clearly, ${}_e|x \leq_l x$ by the definition of '$\leq_l$'. If $y \in P$ satisfying $y \leq_l x$ and $\mathbf{d}(y) = e$, then $y =_g |x$ for $g \in E$ and so $e = \mathbf{d}(y) = \mathbf{d}({}_g|x) = g$ by (IC1). This implies that $y =_e |x$.

The above discussion proves that (E1) is true. Items (E2) and (E3) are direct consequences of the fact that the set of local identities of $\mathbf{P}$ is E, and of (6.1) and its dual. Now let $x, y \in P$ and $(x, y) \in \leq_l \circ \leq_r$. Then $x \leq_l z \leq_r y$ for some $z \in P$. This yields that $x =_e |z$ and $z = y|_f$ for some $e, f \in E$ with $e \leq_E \mathbf{d}(z)$ and $f \leq_E \mathbf{r}(y)$. So $\mathbf{r}(y)f = f, y|_f = y|_{\mathbf{r}(y)f}$ and

$$e \leq_E \mathbf{d}(z) = \mathbf{d}(y|_f) = \mathbf{d}(y|_{\mathbf{r}(y)f}).$$

Thus

$$x =_e |z =_e |(y|_f) =_{e\mathbf{d}(y|_{\mathbf{r}(y)f})} |(y|_{\mathbf{r}(y)f}) = ({}_{e\mathbf{d}(y)}|y)|_{\mathbf{r}({}_{e\mathbf{d}(y)}|y)f}$$

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

by (IC5). This implies that $x \leq_r ({}_{e\mathbf{d}(y)}|y) \leq_l y$. Consequently, $\leq_l \circ \leq_r \subseteq \leq_r \circ \leq_l$. With the dual, we obtain (E4).

If $x \leq_l y$ and $e \in E$, then $x =_g |y$ for some $g \in E$ with $g \leq_E \mathbf{d}(y)$. This implies that $x =_g |y =_{g\mathbf{d}(y)} |y$ and so

$$x|_{\mathbf{r}(x)e} = ({}_{g\mathbf{d}(y)}|y)|_{\mathbf{r}({}_{g\mathbf{d}(y)}|y)e} = {}_{g\mathbf{d}(y|_{\mathbf{r}(y)e})}|(y|_{\mathbf{r}(y)e})$$

by (IC5), which implies that $x|_{\mathbf{r}(x)e} \leq_l y|_{\mathbf{r}(y)e}$. Thus (E5) is proved. □

The following lemma gives a converse of Lemma 6.3.

Lemma 6.4. *Every Ehresmann category $(P, \cdot, \mathbf{d}, \mathbf{r}, \leq_l, \leq_r)$ with semilattice of local identities P_o can be regarded as a pseudocategory over the semilattice P_o.*

Proof. Let $(P, \cdot, \mathbf{d}, \mathbf{r}, \leq_l, \leq_r)$ be an Ehresmann category with semilattice of local identities $P_o = \{\mathbf{d}(x) \mid x \in P\} = \{\mathbf{r}(x) \mid x \in P\}$. Since $(P, \cdot, \mathbf{d}, \mathbf{r})$ is a small category in the sense of Definition 2.19, we have $\mathbf{d}(x), \mathbf{r}(x) \in P_o$ and $\mathbf{d}(e) = \mathbf{r}(e) = e$ for all $e \in P_o$ and hence we can easily show that $(P, \cdot, \mathbf{d}, \mathbf{r}, P_o)$ is a pseudocategory over the semilattice P_o in the sense of Definition 3.2. We only need to show that the conditions in Remark 3.6 hold. We denote the restriction of $\leq_l$ (or $\leq_r$) to P_o by $\leq$ and use uv to denote the meet of u and v in P_o under the order $\leq$.

By (E1), for all $e \in P_o$ and $x \in P$ with $e \leq \mathbf{d}(x)$, there exists a unique element ${}_e|x \in P$ such that ${}_e|x \leq_l x$ and $\mathbf{d}({}_e|x) = e$. In view of (OC2), $\mathbf{r}({}_e|x) \leq_l \mathbf{r}(x)$, that is, $\mathbf{r}({}_e|x) \leq \mathbf{r}(x)$. Moreover, since $x \leq_l x$ and $\mathbf{d}(x) = \mathbf{d}(x)$, by the uniqueness of the restriction of x to e, ${}_e|x = x$ if $e = \mathbf{d}(x)$. This proves (IC1).

For (IC2), if $e, g \in P_o$ and $x \in P$ with $e \leq g \leq \mathbf{d}(x)$, in view of (E1), ${}_e|({}_g|x) \leq_l {}_g|x \leq_l x$ and $\mathbf{d}({}_e|({}_g|x)) = e$. By the uniqueness of the restriction of x to e, ${}_e|({}_g|x) =_e |x$.

To see (IC3), let $e \in P_o, x, y \in P, e \leq \mathbf{d}(x)$ and $x \cdot y$ be defined. Then $\mathbf{r}(x) = \mathbf{d}(y)$ and ${}_e|(x \cdot y) \leq_l x \cdot y, \mathbf{d}({}_e|(x \cdot y)) = e$ by (E1). On the other hand, since ${}_e|x \leq_l x$ and ${}_{\mathbf{r}({}_e|x)\mathbf{d}(y)}|y \leq_l y$, it follows that ${}_e|x \cdot {}_{\mathbf{r}({}_e|x)\mathbf{d}(y)}|y \leq_l x \cdot y$ by (OC3) and

$$\mathbf{d}({}_e|x \cdot {}_{\mathbf{r}({}_e|x)\mathbf{d}(y)}|y) = \mathbf{d}({}_e|x) = e.$$

By the uniqueness of the restriction of $x \cdot y$ to e, ${}_e|(x \cdot y) =_e |x \cdot {}_{\mathbf{r}({}_e|x)\mathbf{d}(y)}|y$, as required.

If $e, u \in P_o$ and $u \leq e$, then $u \leq_l e$ and $\mathbf{d}(u) = u$. By the uniqueness of the restriction of e to u, we have ${}_u|e = u$. Dually, $e|_u = u$. This gives (IC4).

Finally, we consider (IC5). If $f, g \in P_o$ and $x \in P$, then by [20, Theorem 4.20], we have $(f \overline{\otimes} x) \overline{\otimes} g = f \overline{\otimes} (x \overline{\otimes}) g$, where

$$y \overline{\otimes} z = y|_{\mathbf{r}(y)\mathbf{d}(z)} \cdot {}_{\mathbf{r}(y)\mathbf{d}(z)}|z$$

for all $y, z \in P$. Hence

$$\begin{aligned}
(f \overline{\otimes} x) \overline{\otimes} g &= (f|_{\mathbf{r}(f)\mathbf{d}(x)} \cdot {}_{\mathbf{r}(f)\mathbf{d}(x)}|x)|_{\mathbf{r}({}_{\mathbf{r}(f)\mathbf{d}(x)}|x)\mathbf{d}(g)} \cdot {}_{\mathbf{r}({}_{\mathbf{r}(f)\mathbf{d}(x)}|x)\mathbf{d}(g)}|g \\
&= (f|_{f\mathbf{d}(x)} \cdot {}_{f\mathbf{d}(x)}|x)|_{\mathbf{r}({}_{f\mathbf{d}(x)}|x)g} \cdot {}_{\mathbf{r}({}_{f\mathbf{d}(x)}|x)g}|g \quad (\text{since } \mathbf{r}(f) = f, \mathbf{d}(g) = g) \\
&= (f\mathbf{d}(x) \cdot {}_{f\mathbf{d}(x)}|x)|_{\mathbf{r}({}_{f\mathbf{d}(x)}|x)g} \cdot {}_{\mathbf{r}({}_{f\mathbf{d}(x)}|x)g}|g \quad (\text{by (IC4)}) \\
&= ({}_{f\mathbf{d}(x)}|x)|_{\mathbf{r}({}_{f\mathbf{d}(x)}|x)g} \cdot {}_{\mathbf{r}({}_{f\mathbf{d}(x)}|x)g}|g \quad (\text{by Definition 2.19(iii) and (IC1)}) \\
&= ({}_{f\mathbf{d}(x)}|x)|_{\mathbf{r}({}_{f\mathbf{d}(x)}|x)g} \cdot \mathbf{r}({}_{f\mathbf{d}(x)}|x)g \quad (\text{by (IC4)}) \\
&= ({}_{f\mathbf{d}(x)}|x)|_{\mathbf{r}({}_{f\mathbf{d}(x)}|x)g} \quad (\text{by Definition 2.19(iii) and (IC1)}')
\end{aligned}$$

https://doi.org/10.1017/S1446788717000349 Published online by Cambridge University Press

and, similarly, $f\overline{\otimes}(x\overline{\otimes})g = {}_{f\mathbf{d}(x|_{\mathbf{r}(x)g})}|(x|_{\mathbf{r}(x)g})$. So

$$ {}_{f\mathbf{d}(x|_{\mathbf{r}(x)g})}|(x|_{\mathbf{r}(x)g}) = ({}_{f\mathbf{d}(x)}|x)|_{\mathbf{r}({}_{f\mathbf{d}(x)}|x)g}, $$

which gives (IC5). □

Let $(C, \cdot, \mathbf{d}, \mathbf{r}, \leq_l, \leq_r)$ and $(D, \cdot, \mathbf{d}, \mathbf{r}, \leq_l, \leq_r)$ be Ehresmann categories with semilattices C_o and D_o of local identities, respectively. From Lawson [20, Lemma 4.22] and Corollary 2.21, a *strongly ordered functor* is just a mapping φ from C to D which satisfies the two conditions in Corollary 2.21 and preserves $\leq_l, \leq_r$ and the binary operation of the semilattices of local identities. Hence φ induces a morphism from the semilattices C_o to D_o. As shown in [20, Lemma 4.23], a strongly ordered functor φ preserves restrictions and co-restrictions. Thus φ is a pseudofunctor from the pseudocategory $(C, \cdot, \mathbf{d}, \mathbf{r}, C_o)$ to the pseudocategory $(D, \cdot, \mathbf{d}, \mathbf{r}, D_o)$ in the sense of Definition 3.8.

On the other hand, let ψ be a pseudofunctor from the pseudocategory $(P_1, \cdot, \mathbf{d}, \mathbf{r}, E_1)$ to the pseudocategory $(P_2, \cdot, \mathbf{d}, \mathbf{r}, E_2)$ in the sense of Definition 3.8, where E_1 and E_2 are semilattices. Then ψ is a functor from $(P_1, \cdot, \mathbf{d}, \mathbf{r}, \leq_l, \leq_r)$ to $(P_2, \cdot, \mathbf{d}, \mathbf{r}, \leq_l, \leq_r)$ by Corollary 2.21 and (PF2) and (PF3), where $\leq_l$ and $\leq_r$ are defined as in the statements before Lemma 3.7, and the restriction of $\leq_l$ on P_1 to E_1 is equal to $\leq_{E_1}$and the restriction of $\leq_l$ on P_2 to E_2 is equal to $\leq_{E_2}$ by (6.1). Moreover, condition (PF1) in Definition 3.8 gives that ψ preserves the binary operation of the semilattices E_1 and E_2 of local identities. Suppose now that $x, y \in P_1$ with $x \leq_l y$. Then $x = {}_e|y$ for some $e \in E_1$, by (PF4), $x\psi = {}_{e\psi}|y\psi$ so that $x\psi \leq_l y\psi$. Dually, ψ preserves $\leq_r$, so that ψ is a strongly ordered functor from the Ehresmann category $(P_1, \cdot, \mathbf{d}, \mathbf{r}, \leq_l, \leq_r)$ to the Ehresmann category $(P_2, \cdot, \mathbf{d}, \mathbf{r}, , \leq_l, \leq_r)$.

Observe that Lawson's admissible morphisms between Ehresmann semigroups are just (2,1,1)-morphisms between Ehresmann semigroups. Therefore, Corollary 6.2, Lemmas 6.3 and 6.4 and the comments above now give us Lawson's result (Theorem 1.2 in this paper).

Corollary 6.5 [20, Theorem 4.24]. *The category of Ehresmann semigroups and admissible morphisms is isomorphic to the category of Ehresmann categories and strongly ordered functors.*

Acknowledgements

The author expresses his profound gratitude to Professor Marcel Jackson and the anonymous referees for the valuable comments and suggestions which improved greatly the presentation of this article.

References

[1] S. M. Armstrong, 'Structure of concordant semigroups', *J. Algebra* **118** (1988), 205–260.
[2] T. S. Blyth, 'Inverse transversals: A guided tour', in: *Proc. Int. Conf.*, Braga, Portugal, 18–23 June 1999 (eds. P. Smith, E. Giraldes and P. Martins) (World Scientific Publishing, River Edge, NJ, 2000), 26–43.

[3] T. S. Blyth and M. H. Almeida Santos, 'E-special ordered regular semigroups', *Comm. Algebra* **43** (2015), 3294–3312.
[4] T. S. Blyth and M. H. Almeida Santos, 'A classification of inverse transversals', *Comm. Algebra* **29** (2001), 611–624.
[5] T. S. Blyth and R. McFadden, 'Regular semigroups with a multiplicative inverse transversal', *Proc. Roy. Soc. Edinburgh Sect. A* **92** (1982), 253–270.
[6] M. J. J. Brancoa, G. M. S. Gomes and V. Gould, 'Ehresmann monoids', *J. Algebra* **443** (2015), 349–382.
[7] A. El-Qallali, 'Abundant semigroups with a multiplicative type A transversal', *Semigroup Forum* **47** (1993), 327–340.
[8] J. B. Fountain, G. M. S. Gomes and V. Gould, 'A Munn type representation for a class of E-semiadequate semigroups', *J. Algebra* **218** (1999), 693–714.
[9] G. M. S. Gomes and V. Gould, 'Fundamental Ehresmann semigroups', *Semigroup Forum* **63** (2001), 11–33.
[10] V. Gould, 'Notes on restriction semigroups and related structures; formerly (weakly) left E-ample semigroups', http://www-users.york.ac.uk/ varg1/restriction.pdf (2010).
[11] V. Gould, 'Restriction and Ehresmann semigroups', in: *Proc. Int. Conf.,* Gadjah Mada University, Indonesia, 7–10 October 2010 (eds. W. Hemakul, S. Wahyuni and P. W. Sy) (World Scientific Publishing, Hackensack, NJ, 2012), 265–288.
[12] V. Gould and Y. H. Wang, 'Beyond orthodox semigroups', *J. Algebra* **368** (2012), 209–230.
[13] X. J. Guo, 'Abundant semigroups with a multiplicative adequate transversal', *Acta Math. Sin. (Engl. Ser.)* **18** (2002), 229–244.
[14] C. Hollings, 'From right PP monoids to restriction semigroups: a survey', *Eur. J. Pure Appl. Math.* **2** (2009), 21–57.
[15] C. Hollings, 'The Ehresmann–Schein–Nambooripad theorem and its successors', *Eur. J. Pure Appl. Math.* **5** (2012), 414–450.
[16] C. Hollings, 'Three approaches to inverse semigroups', *Eur. J. Pure Appl. Math.* **8** (2015), 294–323.
[17] J. M. Howie, *An Introduction to Semigroup Theory* (Academic Press, London, 1976).
[18] P. R. Jones, 'A common framework for restriction semigroups and regular $*$-semigroups', *J. Pure Appl. Algebra* **216** (2012), 618–632.
[19] P. R. Jones, 'Almost perfect restriction semigroups', *J. Algebra* **445** (2016), 193–220.
[20] M. V. Lawson, 'Semigroups and ordered categories I: The reduced case', *J. Algebra* **141** (1991), 422–462.
[21] M. V. Lawson, *Inverse Semigroups, the Theory of Partial Symmetries* (World Scientific, Singapore, 1998).
[22] J. Meakin, 'The structure mappings on a regular semigroups', *Proc. Edinb. Math. Soc. (2)* **21** (1978), 135–142.
[23] K. S. S. Nambooripad, 'Structure of regular semigroups', *Mem. Amer. Math. Soc.* **22**(224) (1979), 1–103.
[24] X. L. Tang and Z. Gu, 'Words on free bands with inverse transversals', *Semigroup Forum* **91** (2015), 101–116.
[25] Y. H. Wang, 'Beyond regular semigroups', *Semigroup Forum* **92** (2016), 414–448.
[26] S. F. Wang, 'A Munn type representation of abundant semigroups with a multiplicative ample transversal', *Period. Math. Hungar.* **73** (2016), 43–61.

SHOUFENG WANG, Department of Mathematics,
Yunnan Normal University,
Kunming, Yunnan, 650500, PR China
e-mail: wsf1004@163.com