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SOME MAPPINGS ASSOCIATED WITH THE PERMUTATION GROUPS.

BY
J. L. DAVISON

1. Introduction. Let S_n denote the permutation group on $\{1, 2, 3, \ldots, n\}$. Let $k \geq 1$ and $n \geq 1$ be integers. For $\sigma \in S_n$ we define ([1]) $m_{k,n}: S_n \to \mathbf{Z}$ by

$$m_{k,n}(\sigma) = \sum_{i=1}^{n} |\sigma(i) - i|^k.$$

Let $\rho_n \in S_n$ be the reverse permutation: that is,

$$\rho_n(i) = n + 1 - i \qquad (1 \leq i \leq n).$$

We define integers $f_{k,n}$ by setting $f_{k,n} = m_{k,n}(\rho_n)$. These integers play an important rôle in what follows, so we include a small table of values of the $f_{k,n}$.

k/n	1	2	3	4	5	6	7	8	9	10
1	0	2	4	8	12	18	24	32	40	50
2	0	2	8	20	40	70	112	168	240	330
3	0	2	16	56	144	306	576	992	1,600	2,450
4	0	2	32	164	544	1,414	3,136	6,216	11,328	19,338

Let $J_{k,n}$ denote the set of even integers in the interval $[0, f_{k,n}]$. It is proved in Theorem 1 that $m_{k,n}(S_n) \subseteq J_{k,n}$. In [1], it was further proved that $m_{2,n}(S_n) = J_{2,n}$ for $n \geq 4$. It is easy to see (Proposition 1) that $m_{1,n}(S_n) = J_{1,n}$ for all $n \geq 1$. Thus, it seems natural to ask whether $m_{k,n}(S_n) = J_{k,n}$ for each k (and large enough n). The main result of this paper is to prove:

THEOREM 2. *If $k \geq 3$ then for all $n \in N$, $f_{k,n} - 4 \notin m_{k,n}(S_n)$.*

Thus in general $m_{k,n}(S_n) \neq J_{k,n}$. In §3 we give a partial result on $m_{3,n}(S_n)$ and indicate a reasonable conjecture which we have been unable to prove as yet.

2. Main results

LEMMA 1. *If $\sigma \in S_n$, then $m_{k,n}(\sigma)$ is an even integer for any $k \geq 1$.*

Proof. Let $a_i = |\sigma(i) - i|$. Then $\sigma(i) - i = \pm a_i$
Let

$$I_1 = \{i : 1 \leq i \leq n : \sigma(i) - i = +a_i\}$$
$$I_2 = \{i : 1 \leq i \leq n : \sigma(i) - i = -a_i\}$$

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Then

$$\sum_{i=1}^{n} (\sigma(i)-i) = \sum_{i\in I_1} a_i - \sum_{i\in I_2} a_i$$

But

$$\sum_{i=1}^{n} \sigma(i) = \sum_{i=1}^{n} i \quad \text{so that} \quad \sum_{i\in I_1} a_i = \sum_{i\in I_2} a_i \tag{*}$$

Thus

$$\sum_{i=1}^{n} a_i = 2\sum_{i\in I_2} a_i$$

is even. So there is an even number of those integers a_i which are odd and hence for any $k \geq 1$ $m_{k,n}(\sigma) = \sum_{i=1}^{n} a_i^k$ is even.

REMARK. Equation (*) gives a condition on the permissible decompositions of an even integer into sums of kth powers. For example, the integer 24 cannot belong to $m_{3,9}(S_9)$ since the decomposition $24 = 2^3 + 2^3 + 2^3$ would imply that $\{2, 2, 2\}$ can be split into two subsets with equal sums, which clearly is not the case.

THEOREM 1. *For each pair of integers* $k, n : m_{k,n}(S_n) \subseteq J_{k,n}$

Proof. Let $\sigma \in S_n$, and suppose σ is not the reverse permutation. Then there exists an integer i, $1 \leq i \leq n-1$, such that $\sigma(i) < \sigma(i+1)$. Let τ be the transposition $(i\, i+1)$, and $\mu = \sigma \circ \tau$. Then we show that $m_{k,n}(\mu) > m_{k,n}(\sigma)$. Since ρ_n is obtainable from any σ by transpositions of such a kind this will show that $m_{k,n}(S_n) \subseteq J_{k,n}$.

To simplify the notation, let $x = \sigma(i) - i$ and $y = \sigma(i+1) - (i+1)$. Then $x \leq y$. The proof splits into two parts, depending on whether k is even or odd.

Suppose first that k is even. Then it is clear that

$$\begin{aligned} m_{k,n}(\mu) - m_{k,n}(\sigma) &= \{(x-1)^k + (y+1)^k\} - \{x^k + y^k\} \\ &= k(y^{k-1} - x^{k-1}) + \binom{k}{2}(y^{k-2} + x^{k-2}) + \cdots + k(y-x) + 2 \end{aligned}$$

Now since $y \geq x$ and k is even, every second term in this expansion, starting from the first, is non-negative. The other terms are clearly all non-negative also. Thus $m_{k,n}(\mu) \geq m_{k,n}(\sigma) + 2$. Note also that

$$m_{k,n}(\mu) - m_{k,n}(\sigma) = 2 \Leftrightarrow x = y = 0$$

and that otherwise

$$m_{k,n}(\mu) - m_{k,n}(\sigma) \geq \binom{k}{2} + 2 > 4 \quad \text{for} \quad k \geq 3.$$

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Consider now when k is odd. Three cases can arise

(i) $y \geq x \geq 1$ (ii) $x \leq y \leq -1$ (iii) $x \leq 0 \leq y$

We deal with these cases separately
(i) $y \geq x \geq 1$
Here

$$m_k(\mu) - m_k(\sigma) = \{(x-1)^k + (y+1)^k\} - \{x^k + y^k\}$$
$$= k(y^{k-1} - x^{k-1}) + \binom{k}{2}(y^{k-2} + x^{k-2}) + \cdots + k(y+x)$$

We see then that $m_{k,n}(\mu) \geq m_{k,n}(\sigma) + 2k$
(ii) $x \leq y \leq -1$. Let $u = -x$ and $v = -y$. Then $u \geq v \geq 1$ and

$$m_{k,n}(\mu) - m_{k,n}(\sigma) = \{(u+1)^k + (v-1)^k\} - \{u^k + v^k\}$$

which reduces to case (i).
(iii) $x \leq 0 \leq y$. Let $u = -x$.
Then

$$m_{k,n}(\mu) - m_{k,n}(\sigma) = \{(u+1)^k + (y+1)^k\} - \{u^k + y^k\}$$
$$= k(u^{k-1} + y^{k-1}) + \binom{k}{2}(u^{k-2} + y^{k-2}) + \cdots + 2.$$

So again, $m_{k,n}(\mu) \geq m_{k,n}(\sigma) + 2$, equality holding only if $x = y = 0$. Otherwise, we have

$$m_{k,n}(\mu) \geq m_{k,n}(\sigma) + 2^k > m_{k,n}(\sigma) + 4 \quad \text{for} \quad k \geq 3.$$

REMARK. The proof of the Theorem also shows that $m_{k,n}(\mu) = f_{k,n} \Leftrightarrow \mu = \rho_n$.

If $\tilde{\sigma} \in S_{n-2}$ we can construct a $\sigma \in S_n$ by defining

$$\sigma(1) = n; \ \sigma(n) = 1$$
$$\sigma(i) = \tilde{\sigma}(i-1) + 1 : 2 \leq i \leq n-1$$

Clearly,

$$m_{k,n}(\sigma) = 2(n-1)^k + m_{k,n-2}(\tilde{\sigma}) \tag{1}$$

PROPOSITION 1. *For each* $n \geq 1$; $m_{1,n} : S_n \to J_{1,n}$ *is surjective.*

Proof. It is clear by inspection that $m_{1,1}$; $m_{1,2}$ are surjective. Let $n \geq 3$. We proceed by induction. Assume $m_{1,n-1}(S_{n-1}) = J_{1,n-1}$ and $m_{1,n-2}(S_{n-2}) = J_{1,n-2}$. Then

$$m_{1,n}(S_n) \supset m_{1,n-1}(S_{n-1}) = J_{1,n-1} = [0, f_{1,n-1}].$$

Also, from equation (1)

$$m_{1,n}(S_n) \supset 2(n-1) + m_{1,n-2}(S_{n-2}) = [2(n-1), 2(n-1) + f_{1,n-2}]$$

But $2(n-1) + f_{1,n-2} = f_{1,n}$ and $f_{1,n-1} \geq 2(n-1)$ for $n \geq 3$. So that $m_{1,n}(S_n) = J_{1,n}$.

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LEMMA 2. *Let $k \geq 3$. If $\lambda \in S_n$ is such that $m_{k,n}(\lambda) = f_{k,n} - 2$ then n is even, say $n = 2m$, and $\lambda = \rho_n \circ \tau$, where τ is the transposition $(m\, m+1)$*

Proof. Suppose $\lambda \in S_n$ and $m_{k,n}(\lambda) = f_{k,n} - 2$. Since λ clearly does not equal ρ_n, the remarks included in the proof of Theorem 1 allow us to conclude that there is a transposition τ such that if $\mu = \lambda \circ \tau$ then $m_{k,n}(\mu) \geq m_{k,n}(\lambda) + 2 = f_{k,n}$. But we must have equality, so that $\mu = \rho_n$, and for some i, $\lambda(i) = i$; $\lambda(i+1) = i+1$. Thus $\lambda = \rho_n \circ \tau$ and it follows that i must equal m and $\tau = (m\, m+1)$.

THEOREM 2. *For $k \geq 3$ $f_{k,n} - 4 \notin m_{k,n}(S_n)$*

Proof. Suppose there exists a permutation $\sigma \in S_n$ such that $m_{k,n}(\sigma) = f_{k,n} - 4$. There exists a transposition τ_1 of the kind described in Theorem 1. If $\mu_1 = \sigma \circ \tau_1$ then $m_{k,n}(\mu_1) \geq m_{k,n}(\sigma) + 2 = f_{k,n} - 2$. Now equality must hold since we have seen that otherwise, $m_{k,n}(\mu_1) > m_{k,n}(\sigma) + 4 = f_{k,n}$ which cannot happen. But if $m_{k,n}(\mu_1) = f_{k,n} - 2$, then by Lemma 2, n is even, say $= 2m$, and

$$\mu_1 = \begin{pmatrix} 1 \ldots m-1 & m & m+1 & m+2 \ldots 2m \\ 2m \ldots m+2 & m & m+1 & m-1 \ldots 1 \end{pmatrix}$$

But $\sigma = \mu_1 \circ \tau_1$ and $\sigma(i) = i$, $\sigma(i+1) = i+1$ for some i. This clearly cannot happen so the Theorem is proved.

3. $m_{3,n}(S_n)$ and a Conjecture.

In this section we prove:

THEOREM 3. *$m_{3,n}(S_n) \supset [0, f_{3,n} - 112] \cap J_{3,n}$ for $n \geq 10$.*

Using the available computer facilities (IBM 360-40) $m_{3,n}(S_n)$ was tabulated for $n \leq 9$. This information, plus a few simple calculations is enough to show that $m_{3,10}(S_{10}) \supset [0, 2338] \cap J_{3,10}$; and $m_{3,11}(S_{11}) \supset [0, 3488] \cap J_{3,11}$. A few details on the computations are included for completeness.

From the tabulation it is known that $m_{3,9}(S_9) \supset [0, 1488] \cap J_{3,9} \sim \{24\}$, and it is clear that $24 \in m_{3,10}(S_{10})$. Thus $m_{3,10}(S_{10}) \supset [0, 1488] \cap J_{3,10}$. Also from equation (1) of §2, $m_{3,10}(S_{10}) \supset 2(9^3) + m_{3,8}(S_8) = 1458 + m_{3,8}(S_8)$. From the tabulation of $m_{3,8}(S_8)$ it is seen that of the numbers less than 922, only 24, 168, 734, and 898 are missing from the set $m_{3,8}(S_8)$. Now $1626 = 1458 + 168 \in m_{3,10}(S_{10})$ since

$$1626 = 9^3 + 8^3 + 7^3 + 2^3 + 2^3 + 2^3 + 2^3 + 2^3 + 1^3 + 1^3$$

and we put

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 4 & 5 & 2 & 7 & 8 & 6 & 9 & 1 & 3 \end{pmatrix}$$

$2192 = 1458 + 734 \in m_{3,10}(S_{10})$ since

$$2192 = 9^3 + 8^3 + 8^3 + 7^3 + 4^3 + 2^3 + 2^3 + 2^3 + 2^3$$

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and we put

$$\sigma=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\ 10 & 9 & 5 & 8 & 3 & 4 & 7 & 6 & 1 & 2\end{pmatrix}$$

Finally $2356=1458+898\in m_{3,10}(S_{10})$, since

$$2356=9^3+8^3+8^3+7^3+5^3+5^3+2^3+1^3+1^3$$

and we put

$$\sigma=\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\ 10 & 9 & 8 & 4 & 7 & 5 & 6 & 3 & 1 & 2\end{pmatrix}$$

Thus $m_{3,11}(S_{11})\supset[0,2378]\cap J_{3,11}$ and $m_{3,11}(S_{11})\supset 2000+m_{3,9}(S_9)$. But 24 is the only even integer less than 1488 which does not belong to $m_{3,9}(S_9)$, and from the first formula we see that $2024\in m_{3,11}(S_{11})$. So that our calculations are complete.

The Theorem is now proved by induction. We assume that for some $n\geq 11$,

$$m_{3,n}(S_n)\supset[0,f_{3,n}-112]\cap J_{3,n}$$

and

$$m_{3,n-1}(S_{n-1})\supset[0,f_{3,n-1}-112]\cap J_{3,n-1}.$$

Then

(1) $$m_{3,n+1}(S_{n+1})\supset m_{3,n}(S_n)\supset[0,f_{3,n}-112]\cap J_{3,n}$$

Also,

(2) $$m_{3,n+1}(S_{n+1})\supset 2n^3+m_{3,n-1}(S_{n-1})\supset[2n^3,2n^3+f_{3,n-1}-112]\cap J_{3,n+1}$$

Also, $f_{3,n+1}=2n^3+f_{3,n-1}$. Now, $f_{3,n}-112>2n^3$ for $n\geq 10$, so that combining (1) and (2) it follows that

$$m_{3,n+1}(S_{n+1})\supset[0,f_{3,n+1}-112]\cap J_{3,n+1}$$

and the Theorem is proved.

It seems reasonable to make the following conjecture. Let $k\geq 1$. Then there exist two integers α_k, n_k such that

$$m_{k,n}(S_n)\supset[0,f_{k,n}-\alpha_k]\cap J_{k,n}\quad\text{for}\quad n\geq n_k.$$

The conjecture would be proved if, for the given k, we could find an integer α_k such that the statement is true for two successive integers n, $n+1$ with $f_{k,n}>2n^k+\alpha_k$.

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Department of Mathematics
Laurentian University
Sudbury, Ontario.