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A useful relationship in the conformal mapping of quadrilaterals

O.F. Hughes

A relationship is presented which greatly simplifies the application of the Schwarz-Christoffel formula in mapping a quadrilateral having one axis of symmetry. The relationship has been verified numerically, and it is hoped that its evident simplicity and usefulness will stimulate the development of an analytical proof.

A common application of the well-known Schwarz-Christoffel Theorem is the transformation of a complex plane z to the region around a polygon in the w plane. In the case of quadrilaterals with one axis of symmetry, as shown in Figure 1 (page 100), the transformation is

$$w = A\int_0^z \frac{\left(z-x_2\right)^{\alpha+\beta}}{\left(z-x_1\right)^{\alpha}\left(z-x_3\right)^{\beta}}\,dz + B\ . \tag{1.1}$$

Because of the symmetry we need consider only the upper half plane and thus in the w plane we shall be dealing with the triangle ABC . The geometry of the quadrilateral is specified completely by the angles $\alpha\pi$ and $\beta\pi$ and by the height h of this triangle. In the transformation the size, orientation and position of the quadrilateral are controlled by the complex constants A and B . The size requirement (that is, that the half width be equal to h) will be satisfied by an appropriate choice of A , and for purposes of this work we may arbitrarily choose $A = 1$ and regard this requirement as being satisfied. Also, it is convenient to choose a real value for B so that the axis of symmetry is the real axis of w . For

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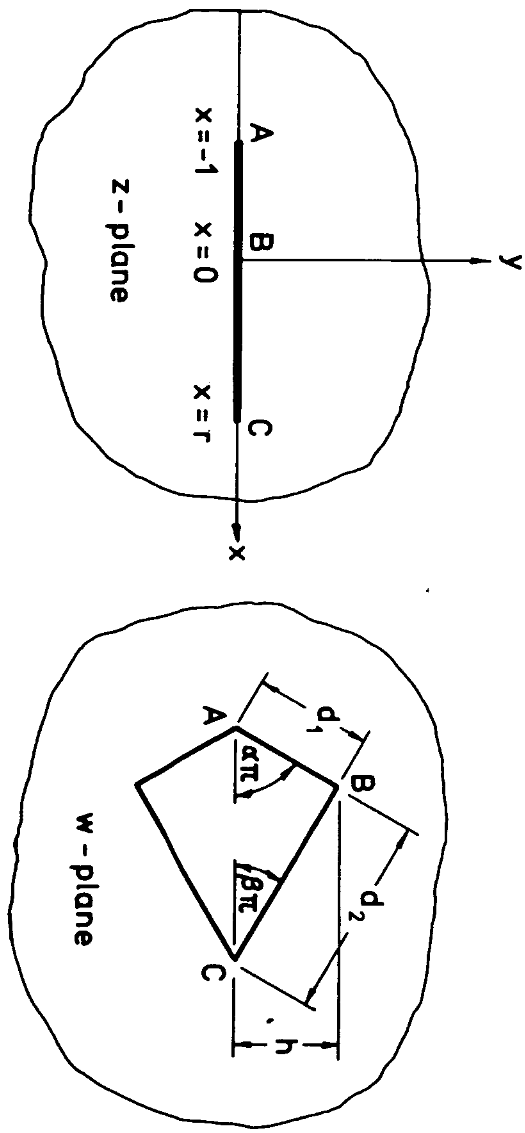

Figure 1.

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simplicity let us choose $B = 0$. Although for convenience we describe the above mapping in terms of the quadrilateral (or its upper half, the triangle ABC) it should be noted that the Schwarz-Christoffel formula maps the *interior* of a polygon onto a half plane, and that strictly speaking, the polygon being mapped in the present case is that portion of the upper half of the w plane *excluding* the triangle ABC . This polygon has four vertices, one of which is at infinity, and we therefore choose the corresponding image point in the z plane as $x_4 = \infty$. Of the other three image points, it is well-known that two of them can be chosen arbitrarily. (If $x = \infty$ had *not* been one of the vertex image points then three values of x_i could have been chosen arbitrarily.) For simplicity we choose $x_1 = -1$ and $x_2 = 0$. The location of the third image point is not arbitrary; there is some unique value, say $x_3 = r$, which it must have for the sides of the polygon to be mapped onto the real axis of z . The value of r depends on the specified geometry of the polygon, and some geometric relationship must be invoked in order for this constant to be calculated. In the present case the appropriate geometric relationship is the condition

$$d_1 \sin\alpha\pi = d_2 \sin\beta\pi \ , \tag{1.2}$$

which is seen from Figure 1 (opposite) to be the condition that the two inclined sides of the triangle ABC have the same height at C . The quantities d_1 and d_2 are given by

$$d_1 = \int_0^{-1} \frac{(-x)^{\alpha+\beta}}{(1+x)^{\alpha}(r-x)^{\beta}}\, dx \ , \tag{1.3}$$

$$d_2 = \int_0^{r} \frac{r^{\alpha+\beta}}{(1+x)^{\alpha}(r-x)^{\beta}}\, dx \ . \tag{1.4}$$

Substitution of these into (1.2) gives an implicit integral equation for r as a function of α and β . However, these integrals cannot be evaluated in terms of a finite number of elementary functions, and hence in satisfying condition (1.2) one must normally resort to numerical techniques. This in turn would normally preclude the possibility of obtaining r as an analytic function of α and β .

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However, during the course of an investigation of the flow around quadrilaterals [2] a series of numerical evaluations of (1.3) and (1.4) was made, and this study revealed that the relationship between r, α and β is simply

$$r = \alpha/\beta \; . \tag{1.5}$$

In spite of the extreme simplicity and great usefulness of this relationship, it does not appear to have been derived or even reported in the literature previously. It is obviously a very useful relationship in the mapping of quadrilaterals, since it obviates the need for arduous numerical integration of (1.3) and (1.4). Also, it provides new information about these integrals and it may lead to the development of other useful relationships. Hence, the purpose of this brief note is to call attention to the above relationship and to describe the numerical tests which have been performed to prove its validity. It is hoped that this will stimulate interest in the relationship and will encourage some mathematicians to undertake a more theoretical investigation so as to achieve an analytical proof.

We begin the demonstration by normalizing the integrals in (1.3) and (1.4) such that they each have 0 and 1 as limits of integration. This is achieved by substituting $x_1 = -x$ in (1.3) and $x_2 = x/r$ in (1.4). We note that since the limits are the same the subscripts may be dropped. The resulting expressions may be substituted into (1.2), giving

$$\sin\alpha\pi \int_0^1 \frac{x^{\alpha+\beta}}{(1-x)^{\alpha}(r+x)^{\beta}}\, dx = r\sin\beta\pi \int_0^1 \frac{x^{\alpha+\beta}}{(1/r+x)^{\alpha}(1-x)^{\beta}}\, dx \; . \tag{1.6}$$

Thus, we are faced with the task of showing that (1.5) completely satisfies and is equivalent to (1.6). The ideal procedure would be to begin with (1.6) and reduce it analytically to (1.5), and it is hoped that some subsequent contributor will achieve this. In the meantime a series of numerical tests has been performed which demonstrates that (1.5) completely satisfies (1.6). The method which was adopted was to substitute this relation into (1.6) and to show that the resulting expression is an identity; that is, that the left and right hand sides of (1.6) agree for all values of α and β .

Therefore, we now introduce (1.5) as an hypothesis and substitute it

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into (1.6). This yields, after some minor rearrangement,

$$\beta\sin\alpha\pi \int_0^1 \frac{x^{\alpha+\beta}}{(1-x)^{\alpha}(\alpha/\beta+x)^{\beta}}\,dx = \alpha\sin\beta\pi \int_0^1 \frac{x^{\alpha+\beta}}{(1-x)^{\beta}(\beta/\alpha+x)^{\alpha}}\,dx \ . \tag{1.7}$$

It will be seen that the left and right hand sides are "images" of each other, with α and β interchanged. That is, if we denote the functional relationship involved in (1.7) as $f_1(\alpha, \beta)$ we have

$$f_1(\alpha, \beta) \equiv \beta\sin\alpha\pi \int_0^1 \frac{x^{\alpha+\beta}}{(1-x)^{\alpha}(\alpha/\beta+x)^{\beta}}\,dx = f_1(\beta, \alpha) \ . \tag{1.8}$$

Thus, if it could be proved that $f_1(\alpha, \beta)$ is invariant under an interchange of arguments, this would constitute sufficient proof of (1.5).

The integral in (1.8) is found to be

$$(\beta/\alpha)^{\beta}B(1-\alpha,\ 1+\alpha+\beta)F(\beta,\ 1+\alpha+\beta;\ \beta+2;\ -\beta/\alpha) \ ,$$

in which B and F denote the beta function (Euler's integral of the first kind) and the hypergeometric function respectively. These may each be expressed in terms of the gamma function, whereupon (1.8) becomes

$$f_1(\alpha, \beta) = \beta(\beta/\alpha)^{\beta}\sin\alpha\pi\,\frac{\Gamma(1-\alpha)}{\Gamma(\beta)} \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n)\Gamma(1+\alpha+\beta+n)}{\Gamma(\beta+2+n)n!}(-\beta/\alpha)^{n} = f_1(\beta,\alpha) \ .$$

By using the reflection property $\Gamma(1-x)\Gamma(x) = \pi\csc x\pi$, this may be reduced to

$$f_2(\alpha, \beta) \equiv \beta(\beta/\alpha)^{\beta} \sum_{n=0}^{\infty} \frac{\Gamma(\beta+n)\Gamma(1+\alpha+\beta+n)}{\Gamma(\beta+2+n)n!}(-\beta/\alpha)^{n} = f_2(\beta, \alpha) \tag{1.9}$$

in which f_2 is used because the functional relationship on each side is now different from that in (1.8).

The function $f_2(\alpha, \beta)$ has been evaluated by computer using double precision arithmetic (approximately 14 significant figures). Hastings' [1] 7th order polynomial approximation was used for the Γ function. Calculations were performed for values of α and β between 0 and 1 in steps of 1/16 . The results for all of the 210 cases (plus the trivial cases in which $\alpha = \beta$) verified without exception that

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$f_2(\alpha, \beta) = f_2(\beta, \alpha)$, thus confirming the hypothesis expressed in (1.5). Although merely numerical, this test is considered to be a rigorous demonstration of the validity of (1.5). This may in turn provide the basis for other interesting or useful relationships. For instance, it may be shown that (1.9) implies

$$(\beta/\alpha)^{\alpha+\beta} = \frac{1+\beta}{1+\alpha}\left[\frac{\sum_{n=0}^{\infty} \frac{(\alpha)_n(\alpha+\beta+1)_n}{(\alpha+2)_n n!}(-\alpha/\beta)^n}{\sum_{n=0}^{\infty} \frac{(\beta)_n(\alpha+\beta+1)_n}{(\beta+2)_n n!}(-\beta/\alpha)^n}\right]$$

where $(\alpha)_n = \alpha(\alpha+1)(\alpha+2) \ldots (\alpha+n-1)$ and $(\alpha)_0 = 1$.

The relationship presented herein greatly simplifies the conformal mapping of quadrilaterals since it converts the Schwarz-Christoffel mapping function (1.1) from an integral equation to an explicit integral expressed directly in terms of the known mapping parameters, α and β . It is interesting to note that this relationship holds true for all values of α and β less than unity, whereas the Schwarz-Christoffel formula is only valid for $\alpha+\beta < 1$.

References

[1] Cecil Hastings, Jr and Jeanne T. Hayward, James P. Wong, Jr, *Approximations for digital computers* (Princeton University Press, Princeton, New Jersey, 1955).

[2] O.F. Hughes, *Wedge penetration of a free surface* (School of Mechanical and Industrial Engineering, Report 1971/NA/2. University of New South Wales, Kensington, February 1971).

School of Mechanical and Industrial Engineering,
University of New South Wales,
Kensington,
New South Wales.

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