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SIGN CHANGES OF FOURIER COEFFICIENTS OF ENTIRE MODULAR INTEGRALS

YOUNGJU CHOIE

Department of Mathematics, Pohang Institute of Science and Technology and Pohang Mathematical Institute (PMI), Pohang 790-784, Korea
e-mail: yjc@postech.ac.kr

and WINFRIED KOHNEN

Mathematisches Institut der Universität, INF 288, D-69120 Heidelberg, Germany
e-mail: winfried@mathi.uni-heidelberg.de

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Abstract. Let f be a non-zero cusp form with real Fourier coefficients $a(n)\,(n \geq 1)$ of positive real weight k and a unitary multiplier system v on a subgroup $\Gamma \subset SL_2(\mathbb{R})$ that is finitely generated and of Fuchsian type of the first kind. Then, it is known that the sequence $(a(n))(n \geq 1)$ has infinitely many sign changes. In this short note, we generalise the above result to the case of entire modular integrals of non-positive integral weight k on the group $\Gamma_0^*(N)\,(N \in \mathbb{N})$ generated by the Hecke congruence subgroup $\Gamma_0(N)$ and the Fricke involution $W_N := \left(\begin{smallmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{smallmatrix}\right)$ provided that the associated period functions are polynomials.

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1. Introduction. Let f be a non-zero cusp form with real Fourier coefficients $a(n)\,(n \geq 1)$ of positive real weight k and a unitary multiplier system v on a subgroup $\Gamma \subset SL_2(\mathbf{R})$ that is finitely generated and of Fuchsian type of the first kind. Then it was shown in [**3**] that the sequence $(a(n))_{n\geq 1}$ has infinitely many sign changes. The proof uses analytical properties of the Hecke L-function attached to f.

In this short note we shall generalise the above result to the case of entire modular integrals of non-positive integral weight k on the group $\Gamma_0^*(N)\,(N \in \mathbf{N})$ generated by the Hecke congruence subgroup $\Gamma_0(N)$ and the Fricke involution $W_N := \left(\begin{smallmatrix} 0 & -1/\sqrt{N} \\ \sqrt{N} & 0 \end{smallmatrix}\right)$ [**2**], provided that the associated period functions are polynomials. The proof again uses the analytical properties of the Hecke L-function attached to F [**2**]. In addition, we will make use of an elementary trick first applied in [**4**] and exploit in a stronger way non-negativity of Fourier coefficients.

2. Statement of result and proof. Let $\mathcal{H}$ be a complex upper half-plane. If $k \in \mathbf{Z}$, $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbf{R})$ and if $f : \mathcal{H} \to \mathbf{C}$ is a function, we define the Petersson slash operator as usual by

$$(f|_k\gamma)(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)\ (z \in \mathcal{H}).$$

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Note that

$$(f|_k W_N)(z) = N^{-k/2} z^{-k} f\left(-\frac{1}{Nz}\right).$$

We shall prove.

THEOREM. *Let k be a non-positive integer. Let $F : \mathcal{H} \to \mathbf{C}$ be a non-zero holomorphic function periodic with period 1, with Fourier expansion*

$$F(z) = \sum_{n\geq 0} a(n) e^{2\pi i n z}$$

such that

$$a(n) \ll n^c$$

for some $c > 0$. We assume further that

$$(F|_k W_N)(z) = CF(z) + q_{W_N}(z) \ (z \in \mathcal{H}),$$

where $C \in \mathbf{C}$, $|C| = 1$ and $q_{W_N}(z)$ is a polynomial of degree $\leq -k$. Then if the $a(n)$ is real for all $n \geq 1$, the sequence $(a(n))_{n\geq 1}$ has infinitely many sign changes.

Proof. Put

$$L(F, s) := \sum_{n\geq 1} a(n) n^{-s} \ (\sigma := \Re(s) > c + 1).$$

Then according to Lemma 2 in [**2**] the completed function

$$L^*(F, s) := (2\pi)^{-s} \Gamma(s) L(F, s)$$

has meromorphic continuation to $\mathbf{C}$ with at most finitely many poles at certain integer points s. Observe that the poles arising from the constant terms of F and $F|_k W_N$ occur at $s = 0$ and $s = k$, respectively, and that $k \leq 0$. Since by hypothesis $q_{W_N}(z)$ is a polynomial of degree $\leq -k$, inspecting the proof of Lemma 2 in [**2**] in detail we find that $L^*(F, s)$ has no poles in $\sigma > 0$. Since $\Gamma(s)$ has its poles exactly at the points $s = 0, -1, -2, \ldots$, we conclude therefore that $L(F, s)$ is holomorphic everywhere and that there exists a non-negative integer M such that $L(F, \nu) = 0$ for $\nu = -M, -M - 1, -M - 2, \ldots$.

Now assume that $a(n) \geq 0$ for all but a finite number of n. Then according to Landau's classical theorem on Dirichlet series with non-negative coefficients, $L(F, s)$ either must have a singularity at the real point of its abscissa of convergence or must converge everywhere. From what we saw above, we thus conclude that $L(F, s)$ converges for all $s \in \mathbf{C}$ and that

$$\sum_{n\geq 1} a(n) n^{\nu} = 0 \ (\nu = M, M + 1, M + 2, \ldots). \tag{1}$$

We now argue in a similar way as in [**4**]. Recall that by hypothesis $a(m) \geq 0$ for all but a finite number of m. Let

$$a(m_1), a(m_2), \ldots, a(m_t) \ (m_1 < m_2 < \cdots < m_t; t \geq 0)$$

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be those coefficients that are strictly negative. Then (1) can be written as

$$\sum_{m\geq 1,\, m\neq m_1,\ldots,m_t} a(m)\left(\frac{m}{m_t}\right)^{\nu} = -a(m_1)\left(\frac{m_1}{m_t}\right)^{\nu} - \cdots - a(m_t).$$

Here the right-hand side has the limit $-a(m_t) \geq 0$ for $\nu \to \infty$. On the other hand, if on the left-hand side there exists $m > m_t$ with $a(m) > 0$, then the left-hand side will be arbitrarily large for arbitrarily large ν, a contradiction.

Hence, we find that $a(m) = 0$ for $m > m_t$. Then (1) means that

$$\sum_{m=1}^{m_t} a(m)m^{\nu} = 0 \qquad (\forall \nu \geq M). \tag{2}$$

Suppose that not all of the $a(m)$ $(1 \leq m \leq m_t)$ are zero and denote by $a(m_*)$ $(m_* \geq 1)$ the largest non-zero coefficient. Then from (2) we see that

$$\sum_{m=1}^{m_*-1} a(m)\left(\frac{m}{m_*}\right)^{\nu} + a(m_*) = 0,$$

hence for $\nu \to \infty$ we conclude that $a(m_*) = 0$, a contradiction.

This concludes the proof of the Theorem. □

REMARK. We note that our Theorem applies in the case where F is an entire modular integral on $\Gamma_0^*(N)$ of weight $k \in \mathbf{Z}$, $k \leq 0$ and with unitary multiplier system v, provided that the period functions are polynomials [**2**]. Recall that by definition such an F is a holomorphic complex-valued function on $\mathcal{H}$, such that

(1) $F|_k\gamma = \epsilon_\gamma F + q_\gamma \quad (\forall \gamma \in \Gamma_0(N))$,
where q_γ is a polynomial and $\epsilon_\gamma \in \mathbf{C}$, $|\epsilon_\gamma| = 1$,

(2) $F|_k W_N = CF + q_{W_N}$,
where $C \in \mathbf{C}$, $|C| = 1$ and q_{W_N} is a polynomial,

(3) F is holomorphic at the cusps.

For any smooth function g, the following holds (this is called 'Bol's identity' [**1**]): For a non-positive integer k and any $\gamma \in SL_2(\mathbf{R})$,

$$\frac{d^{-k+1}(g|_k\gamma)}{dz^{-k+1}} = \left.\frac{d^{-k+1}g}{dz^{-k+1}}\right|_{-k+2}\gamma.$$

If we take $g = F$ as an entire modular integral of weight k with $F(z) = \sum_{n\geq 1} a(n)q^n$, then
$\frac{d^{-k+1}F}{dz^{-k+1}}$ becomes a modular form of weight $-k+2$ with Fourier expansion of the form

$$(2\pi i)^{-k+1}\sum_{n>0} n^{-k+1}a(n)e^{2\pi inz}.$$

The theorem implies that the sequence $(a(n))_{n\geq 1}$ has infinitely many sign changes. In fact, since F is an entire modular integral, $\frac{d^{-k+1}F}{dz^{-k+1}}$ is a cusp form. Thus, Theorem 1 of [**3**] follows from the elementary argument given in the proof of the theorem here together with Landau's classical result given in [**5**]. Conversely, since F can be regarded as an

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Eichler integral of modular form, the theorem can be derived using Theorem 1 of [**3**] as well.

Finally, we remark that Theorem 2 of [**3**] can be combined with Bol's identity to derive a generalisation of our theorem in the above context to the case in which the condition that the coefficients are real is relaxed.

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