MODIFIED BOUNDARY VALUE PROBLEMS FOR A QUASI-LINEAR ELLIPTIC EQUATION

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1. Introduction. The quasi-linear elliptic partial differential equation to be studied here has the form

$$\Delta u = - F(P, u). \tag{1.1}$$

Here Δ is the Laplacian while $F(P, u)$ is a continuous function of a point P and the dependent variable u. We shall study the Dirichlet problem for (1.1) and will find that the usual formulation must be modified by the inclusion of a parameter in the data or the differential equation, together with a further numerical condition on the solution.

The negative sign on the right in (1.1) is included for convenience and also to emphasize that the behaviour of the right side will be the opposite of that usually studied. We shall generally take $F(P, u)$ to be a positive increasing function of u, these conditions being motivated by the following physical problem. Consider an equilibrium distribution of heat in a medium where the source density of heat generated depends on temperature u:

$$\rho = \rho(u) = F(P, u).$$

That ρ and hence $F(P, u)$ in (1.1) should be positive and increasing with u is a natural assumption.

The known results for quasi-linear equations such as (1.1) are, roughly speaking, of two kinds: local theorems, and in-the-large existence proofs for equations

$$\Delta u = + F(P, u) \tag{1.2}$$

where $F(P, u)$ is an increasing function of u. By local theorems are meant those in which some restriction of size is placed on the boundary values, the domain, or the non-linearity of the function $F(P, u)$. Among these we might include the case when $F(P, u)$ is bounded independently of u. The Dirichlet theorem and various other boundary value results have been proved in such circumstances. (**2**; **4**, vol. II, Ch. V; **6**, Ch. II).

On the other hand, global existence theorems for (1.2) have been found by many authors. (**3**; **6**, Ch. II). The possibility of this may be recognized if one

Received August 11, 1955. This paper was written at the 1955 Summer Research Institute of the Canadian Mathematical Congress. The author's thanks are due to the National Research Council of Canada for a fellowship held at this time. He is also indebted to Professor J. Leray for a stimulating discussion and helpful comments.

constructs the equation of variation of (1.2) with respect to an external parameter: this variational equation has the linear form

$$(1.3) \qquad \Delta v = F_u(P, u)\, v,$$

with positive coefficient $F_u(P, u)$. Such equations satisfy a maximum principle in the sense that the maximum absolute value of any solution is taken on the boundary. Thus a priori estimates can be found for the solutions of (1.3) and hence for those of (1.2).

These methods will not apply to (1.1). Even in the linear case, it is evident that the usual statement of the Dirichlet problem, namely the assertion that a solution having given boundary values exist, does not hold unless $F(P, u)$ is restricted in some way. Indeed, if λ is an eigenvalue, solutions of $\Delta u + \lambda u = 0$ have boundary values restricted by one or more conditions of orthogonality. This particular case will be relevant to Theorem III below; we shall later furnish a similar example which pertains to the main Theorem I and which shows that the conventional Dirichlet problem is not then always solvable.

This discussion suggests that we should frame boundary value problems for (1.1) in such a way that some *a priori* bound can be included in the statement of the problem. We will show that in a certain sense it is sufficient to bound the solution from above. In fact we assume that the actual maximum of the solution has a stated value. If, however, one additional numerical condition is assigned, it is evident that a corresponding degree of freedom should be allowed for the boundary values of the solution. This we shall permit by introducing a parameter t, of the nature of an eigenvalue parameter, into the boundary condition. Thus the main theorem asserts the existence of a solution with a stated maximum and with boundary values proportional to a given function.

We then establish some variations of this theorem, allowing the parameter to appear in various ways in the differential equation instead of the boundary condition. These solutions have an assigned maximum together with given boundary values. We conclude with a Neumann boundary value theorem for an equation similar to (1.1) but containing an additional linear term.

2. Preliminaries. Let V_N be a Riemannian manifold of dimension N with positive definite metric of class C^4 in a given coordinate network:

$$ds^2 = a_{ik}\, dx^i dx^k;$$

then the Laplace operator has the form

$$(2.1) \qquad \Delta u = \frac{1}{\sqrt{a}} \frac{\partial}{\partial x^i}\left(\sqrt{a}\, a^{ik} \frac{\partial u}{\partial x^k}\right).$$

where $a = |a_{ik}|$ and the associate tensor a^{ik} satisfies

$$a^{ik} a_{kj} = \delta^i{}_j.$$

We consider a compact domain D of V_N, having a boundary surface B of class C^2 in the above coordinate system. Points of D will be denoted by capitals

$P, Q, \ldots$ and points of B by lower case $p, q, \ldots$. We assume that $F(P, u)$ is a continuous function of P and u together; other conditions appropriate to each theorem will be stated separately. All functions and parameters used are real-valued.

The existence proofs which follow will be based on the Schauder-Leray theorem **(5)**, which we will state here. We work with the separable Banach space C, of continuous functions on the closure of the domain D, with the norm

$$||u|| = \max_{P \epsilon D} |u(P)| . \tag{2.2}$$

Let Ω be a bounded domain of C, with boundary Ω', and let $T_k[u]$ be an operator defined in $\Omega + \Omega'$ which satisfies the conditions

(a) $T_k[u]$ is jointly uniformly continuous in k and u, for $0 \leqslant k \leqslant 1$ and $u \in \bar{\Omega} = \Omega + \Omega'$.

(b) $T_k[u]$ is a compact or completely continuous operator, transforming bounded sets into compact sets **(1**, Ch. VI**)**. Suppose also that the equation

$$u = T_k[u] \tag{2.3}$$

has no solution on Ω' for $0 \leqslant k \leqslant 1$, and that for $k = 0$ it has a solution in Ω. Finally, let

$$v = u - T_0[u]$$

be a homeomorphism of C. Then the conclusion of the Schauder-Leray theorem is that *the equation*

$$u = T_1[u] \tag{2.4}$$

has at least one solution in Ω.

Separate choices for $T_k[u]$ and for Ω will be made in each of the following theorems. In each case condition (b) above is satisfied essentially because the integral operator with kernel the Green's function of D for $\Delta u = 0$ is completely continuous in the space C. We include a demonstration of this in the proof of Theorem I.

3. The modified Dirichlet problem. Let M be a positive number given in advance, and let $f(p)$ be a C^1 function on the boundary B which is positive:

$$0 < m_0 \leqslant f(p) \leqslant M_0 < \infty . \tag{3.1}$$

We also take $m_0 < 1$, which is always possible, for a reason which will appear later. Let $F(P, u)$ be a positive non-decreasing function of u.

THEOREM I. *There exists a solution of* (1.1) *with maximum value M and boundary values proportional to* $f(p)$.

The constant of proportionality being denoted by t, we have

$$u(p) = tf(p) \tag{3.2}$$

and also

$$\max_{P \epsilon D} u(P) = M \tag{3.3}$$

To establish the existence of such a solution, we begin by constructing the harmonic function $v_0(P)$ with boundary values $f(p)$. Thus

$$\Delta v_0(P) = 0, \quad v_0(p) = f(p), \tag{3.4}$$

and in view of the maximum principle for harmonic functions and (3.1) we have

$$m_0 \leqslant \min f(p) \leqslant v_0(P) \leqslant \max f(p) = M_0, \tag{3.5}$$

these inequalities holding for $P \in D + B$.

Now a solution of (1.1) with boundary values $tf(p)$ satisfies the integral equation

$$u(P) = \int_D G(P, Q)\, F(Q, u(Q))\, dV_Q + tv_0(P), \tag{3.6}$$

where $G(P, Q)$ is the harmonic Green's function of the domain D. We note that $G(P, Q)$ is non-negative **(2)**. Conversely, a solution of (3.6) is actually a solution of (1.1) with boundary values $tf(p)$, as may be verified by operating on (3.6) with the Laplacian, and noting that the integral on the right has vanishing boundary values. We observe that to satisfy the maximum condition (3.3) we must make an appropriate choice of t, which will in turn depend on $u(P)$, so that a fixed value for t cannot be determined at this stage.

We therefore define the non-linear functional

$$T_k[u](P) = \int_D G(P, Q)\, F(Q, ku(Q))\, dV_Q + t_k[u]\, v_0(P), \tag{3.7}$$

where $t = t_k[u]$ is so chosen that

$$\max_{P \epsilon D+B} T_k[u](P) = M. \tag{3.8}$$

Since $v_0(P)$ satisfies (3.5) we see that such a choice of t is always possible, since the right side of (3.7) is a strictly increasing function of t tending to $\pm\infty$ with t.

We now show that $t_k[u]$ is bounded, provided that $0 \leqslant k \leqslant 1$ and $u \leqslant K$, where K is a fixed constant. Let

$$A = \max_{P \epsilon D} \int_D G(P, Q)\, F(Q, K)\, dV_Q; \tag{3.9}$$

this number exists and is positive. Now if $A < M$ it would appear that $t_k[u]$ in (3.7) should be positive. However it is clear that

$$t_k[u] \leqslant M/M_0,$$

since $M_0 = \max v_0(P)$ and $T_k[u] \leqslant M$. This furnishes an upper bound for $t_k[u]$. If $M < A$, $t_k[u]$ may be negative; however

$$\int_D G(P, Q)\, F(Q, ku(Q))\, dV_Q \leqslant A,$$

since $ku \leqslant K$ for $0 \leqslant k \leqslant 1$ and F is a non-decreasing function of u. Thus the multiple of $v_0(P)$ required to reduce the maximum of $T_k[u]$ to M does not exceed $(A - M)/m_0$. We therefore have

$$-\frac{A - M}{m_0} \leqslant t_k[u] \leqslant \frac{M}{m_0}. \tag{3.10}$$

This shows that if u is bounded above, $t_k[u]$ is bounded below and in fact bounded. The lower bound depends on A and hence on the upper bound K of u.

Since the integral in (3.7) is non-negative, it follows that $T_k[u]$ is bounded below:

$$-\frac{A - M}{m_0} M_0 \leqslant T_k[u]. \tag{3.11}$$

Combining (3.8) and (3.11), we see that $T_k[u]$ is bounded in both directions.

To apply the Schauder-Leray theorem, we set

$$K = 2M/m_0 > M,$$

and let A be defined by (3.9) with this value of K. Then we choose Ω to be the connected domain of C defined by

$$\Omega: -\frac{2M}{m_0}|A - M| - \epsilon_0 < u(P) < \frac{2M}{m_0}, \qquad \epsilon_0 > 0. \tag{3.12}$$

The boundary Ω' consists of those functions u for which equality holds on either side for one or more points of $D + B$. Now $T_k[u]$ is defined on $\Omega + \Omega'$ and is continuous in both k and u. This is easily verified since $F(P, ku)$ is uniformly continuous in ku, and the integral

$$\int_D G(P, Q)\, dV_Q$$

is a continuous function of P, vanishing on B, and so is bounded. Thus the integral in (3.7) depends continuously on ku and so, therefore, does $t_k[u]$. Hence $T_k[u]$ is continuous in k and u together.

We now show that $T_k[u]$ is a compact operator in C. Let $\{u_n\}$ be a uniformly bounded sequence of continuous functions. From (3.8) and (3.11) we see that $T_k[u_n]$ is bounded, independently of n and P. We now show that the sequence $T_k[u_n](P)$ is equicontinuous in P by forming the difference

$$|T_k[u_n](P_2) - T_k[u_n](P_1)| \leqslant +\, t_k[u_n]|\, v_0(P_2) - v_0(P_1)| \tag{3.13}$$

$$+\int_D |G(P_2, Q) - G(P_1, Q)|\, F(Q, u_n(Q))\, dV_Q$$

Since u_n is bounded independently of n, so is $F(Q, u_n)$ and also $t_k[u_n]$: let F_0

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and t_0 be bounds for the absolute values of these sequences. Thus the preceding difference is less than

$$F_0 \int_D |G(P_2, Q) - G(P_1, Q)| \, dV_Q + t_0 |v_0(P_2) - v_0(P_1)|.$$

The second term here tends to zero as $P_2 \to P_1$, since $v_0(P)$ is continuous. To estimate the integral containing the Green's functions, we suppose that the distance $s(P, Q) < \delta$ and denote by S_η a geodesic sphere of radius η about P_1. For $P \neq Q$, $G(P, Q)$ is continuous, and we can therefore choose δ so small that for $Q \in D - S_\eta$, the difference

$$|G(P_2, Q) - G(P_1, Q)| < \epsilon_1.$$

We then write

$$\begin{aligned} \int_D |G(P_2, Q) - G(P_1, Q)| \, dV_Q \\ \leqslant \int_{D - S_\eta} |G(P_2, Q) - G(P_1, Q)| \, dV_Q \\ + \int_{S_\eta} \{G(P_1, Q) + G(P_2, Q)\} \, dV_Q \\ \leqslant \epsilon_1 \int_D dV_Q + \int_{\bar{S}} G(P_2, Q) \, dV_Q + \int_{S_\eta} G(P_1, Q) \, dV_Q. \end{aligned} \tag{3.14}$$

Here $\bar{S}$ is a sphere of radius 2η about P_1, which certainly contains S_η if $S(P_1 P_2) < \eta$. Since

$$G(P, Q) \sim \frac{1}{\omega_N (N - 2)} \, s(P, Q)^{-N+2}, \qquad P \to Q,$$

the integrals over small spheres converge like

$$\int_{S_\eta} G(P, Q) \, dV_Q \sim \frac{1}{2(N - 2)} \eta^2 \to 0, \qquad \eta \to 0,$$

uniformly with respect to P in D. Given $\epsilon > 0$, we choose η so small that the second and third terms on the right in (3.14) are each less than $\frac{1}{4}\epsilon$. We can then choose $\delta < \eta$ so small that the first term is less than $\frac{1}{4}\epsilon$. Also for $s(P_2, P_1)$ sufficiently small the second term on the right of (3.13) can be made less than $\frac{1}{4}\epsilon$. This shows, finally, that the sequence $T_k[u_n](P)$ is equicontinuous, uniformly for P in $D + B$. By Ascoli's theorem **(1)**, the sequence contains a uniformly convergent subsequence with a continuous limit. That is, $T_k[u]$ is a compact operator in C.

Next we demonstrate that for $0 \leqslant k \leqslant 1$, the equation

$$u(P) = T_k[u](P) \tag{3.15}$$

has no solution lying on the boundary Ω'. Since for any solution,

$$\max u = \max T_k[u] = M,$$

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we see from (3.12) and the condition $m_0 < 1$ that

$$u(P) \leqslant M < 2M/m_0 = K.$$

Since A was defined by (3.9) for this K, we see that if $t_k[u] \leqslant 0$, then

$$\begin{aligned} -\epsilon_0 - \frac{2M}{m_0}|A - M| < \frac{-M}{m_0}|A - M| &\leqslant M_0\, t_k[u] \\ &\leqslant t_k[u]\, v_0(P) \\ &\leqslant \int_D G(P, Q)\, F(Q, ku(Q))\, dV_Q + t_k[u]\, v_0(P) \\ &= T_k[u](P) = u(P). \end{aligned}$$

Hence the strict inequality on the left holds in (3.12) for any solution and so if $t_k[u] \leqslant 0$ no solution can lie in Ω'. If $t_k[u] \geqslant 0$, then $T_k[u] \geqslant 0$ and the same conclusion follows at once.

Now for $k = 0$ the equation (3.15) has a unique solution since the operator $T_0[u]$ is then independent of u. (Thus the mapping $v = u - T_0[u]$ is a homeomorphism). In fact the solution u for $k = 0$ is the solution of $\Delta u = -F(P, 0)$, with $\max u = M$ and $u(p) = tf(p)$.

From the Schauder-Leray theorem we may now conclude that (3.15) has a solution for each k, $0 \leqslant k \leqslant 1$. For $k = 1$, we observe that in view of (3.7), (3.15) becomes equivalent to the integral equation (3.6). Thus the solution $u(P)$ for $k = 1$ satisfies (1.1) and has boundary values $tf(p)$. From (3.8) and (3.15) it follows that its maximum value is M. This completes the proof of the theorem.

Two minor extensions of this result will be noted here. First, we can treat the case where $F(P, u)$ is only bounded below:

$$F(P, u) \geqslant -K_1$$

by taking as a new variable $\bar{u} = u + v$, with v the solution of $\Delta v = -K_1$ which vanishes on B. Second, we may replace the boundary values $tf(p)$ by a more general continuous function $f(p, t)$ which is strictly increasing with t and tends to $\pm\infty$ with t.

4. Qualitative behaviour of the boundary values. The theorem of the preceding section would be of comparatively small interest if it were possible to solve the conventional Dirichlet problem which concerns the existence of a solution with given boundary values. We show that this problem is not solvable for the class of non-linear equations here considered.

Let λ_1 be the lowest Dirichlet eigenvalue of D for $\Delta u + \lambda u = 0$, and let the corresponding eigenfunction be denoted by u_1. From (**4**, vol. I, ch. VI, §6) we see that u_1 is of one sign in D, say non-negative. Hence the outward normal derivative $\partial u_1/\partial n$ is non-positive, and also does not vanish identically.

Now let u be any solution of (1.1) with boundary values $t\,f(p)$, and let us suppose that

$$F(P, u) > \lambda_1 u \tag{4.1}$$

for all values of u. *Then the value of t is necessarily negative.*

This assertion follows readily from Green's formula, since

$$\begin{aligned} t\int_B f\,u_{1n} dS &= \int_B (u\,u_{1n} - u_n u_1)\,dS \\ &= \int_D (u\Delta u_1 - u_1 \Delta u)\,dV \\ &= \int_D u_1[F(P, u) - \lambda_1 u]\,dV. \end{aligned} \tag{4.2}$$

Since $u_1 \geqslant 0$ in D the integral on the right is positive and since $\int_B u_{1n}\,dS < 0$ we conclude that $t < 0$. If in (4.1) the equality sign is permitted we would find $t \leqslant 0$; the case

$$F(P, u) = \tfrac{1}{2}\lambda_1(u + |u|) \tag{4.3}$$

illustrates this possibility.

Thus, if (4.1) holds, (1.1) can not have any solutions with positive boundary values. This shows that the conventional Dirichlet problem for (1.1) is impossible. Since in the physical interpretation of heat generation one would expect $F(P, u)$ to be a rapidly increasing function of u as $u \to +\infty$, it seems worthwhile to find the closest analogue of the conventional Dirichlet theorem for such equations. Though Theorem I is not the only variant which might be considered, it has physical meaning since:

(a) the maximum temperature is prescribed.

(b) the distribution (or ratio) of temperatures on the boundary is prescribed, so that if the actual boundary value is known at one point, all other boundary values are determined.

We continue the qualitative discussion of the values of t. If (4.1) holds only for

$$u > u_0 > 0, \tag{4.4}$$

we have

$$t < \frac{u_0}{m_0}, \quad m_0 = \min_{p \in B} f(p), \tag{4.5}$$

since otherwise we should have $t\,f(p) \geqslant u_0$ and, the minimum value of a solution of (1.1) being assumed on the boundary, this would lead to

$$u > u_0 \qquad \text{in } D.$$

But then (4.1) and (4.2) show that $t < 0$, which is a contradiction.

If we regard t as a function of M for fixed $f(p)$, we can show that t is a

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continuous function of M. This follows from the Schauder-Leray theorem if we consider the functional

$$T_1[u](P) \equiv T_{1,M}[u](P)$$

in its dependence on M. We need only choose the domain Ω so that M is free to vary in a small interval and so that no solution of $u(P) = T_{1,M}[u](P)$ can cross the boundary of Ω. The reader will readily be able to supply the details here.

We now show that if

$$F_u(P, u) < \lambda_1, \qquad u < \bar{M}, \tag{4.6}$$

then t is a monotone strictly increasing function of M for $M \leqslant \bar{M}$. This will be established by finding a contradiction to the contrary assumption, which is that there exist M_1 and M_2, $M_1 < M_2 \leqslant \bar{M}$, such that $t_1 \geqslant t_2$. Let u_1 and u_2 be the respective solutions. Then $w = u_2 - u_1$ satisfies

$$\begin{aligned} \Delta w &= -F(P, u_2) + F(P, u_1) \\ &= -(u_2 - u_1)\, F_u(P, u_1 + \theta(P)(u_2 - u_1)) \\ &= -w\, F_u(P, u_3), \end{aligned}$$

say. Here u_3 is intermediate in value to u_1 and u_2, so $u_3 < \bar{M}$. Since

$$w = u_2 - u_1 = (t_2 - t_1) f(p) \leqslant 0 \qquad \text{on } B$$

and $w \geqslant M_2 - M > 0$ at the maximum of u_2, there exists a domain $D_1 \subseteq D$ wherein w is positive, and such that $w = 0$ on the boundary B_1 of D_1. Let λ' be the lowest Dirichlet eigenvalue of D_1; then (**4**, vol. I; ch. VI, §6) we have $\lambda_1 \leqslant \lambda'$ since $D_1 \subseteq D$. Let u_1' be the corresponding eigenfunction; we see as in (4.2) that

$$0 = \int_{D_1} u_1' w\, [F_u(P, u_3) - \lambda']\, dV,$$

and this is a contradiction since $u_1' \geqslant 0$, $w \geqslant 0$ and $F_u(P_2, u_3) < \lambda_1 \leqslant \lambda'$ in D_1, no one of the three factors vanishing in any open subset of D_1. This proves the results stated.

For example, if

$$F(P, u) = \begin{cases} 0, & u < 0, \\ u^n, & u \geqslant 0, \end{cases} \qquad n > 1,$$

we see that (4.1) holds for

$$u > u_0 = \lambda_1^{1/(n-1)}$$

and so an upper bound for t is known. For $M = 0$ the solution $u \equiv 0$ fulfills the conditions of Theorem I with $t = 0$. Since (4.6) holds for

$$u < \left(\frac{\lambda_1}{n}\right)^{1/(n-1)},$$

we see that t increases and is positive for

$$0 < M < \left(\frac{\lambda_1}{n}\right)^{1/(n-1)}.$$

The behaviour of t as $M \to \infty$ seems difficult to determine.

5. Related eigenvalue problems. The theorems of this section differ from the preceding result in that the parameter t appears in the differential equation instead of the boundary condition. They have therefore the character of eigenvalue problems, although the conditions to be fulfilled by the solution include the assigning of boundary values.

Let $F(P, u)$ be a continuous positive function, bounded away from zero:

(5.1) $$F(P, u) \geqslant \delta > 0,$$

and consider the problem of finding a solution of

(5.2) $$\Delta u = -tF(P, u)$$

with given boundary values $f(p)$ and a given maximum M. Let us assume that $f(p)$ is C^1 with maximum

(5.3) $$M_0 = \max_B f(p).$$

Then without loss of generality we may take

(5.4) $$M > M_0,$$

since in any case $M \geqslant M_0$ is necessary, while if $M = M_0$, we may take $t = 0$ (4.2) and find a harmonic solution of the problem.

Since a solution of the problem satisfies the integral equation

(5.5) $$u(P) = t\int_D G(P, Q)\, F(Q, u(Q))\, dV_Q + v_0(P),$$

where $v_0(P)$ is again harmonic with boundary values $f(p)$, we define the new operator

(5.6) $$T_k^{\,1}[u](P) = t_k^{\,1}[u]\int_D G(P, Q)\, F(Q, ku(Q))\, dV_Q + v_0(P),$$

with the choice of t governed by the condition

(5.7) $$\max_{P\epsilon D} T_k^{\,1}[u](P) = M.$$

To show that this choice is possible we note that the non-negative integral

(5.8) $$\int_D G(P, Q)\, dV_Q$$

has a maximum G_0 say for $P = P_0$ in D. Now for $t = 0$ the right side of (5.6) is less than M; consequently $t_k^{\,1}[u]$ must be positive. As t increases, so does the expression on the right in (5.6). However at $P = P_0$ we have

$$t\delta G_0 \leqslant t\delta\int_D G(P_0, Q)\, dV_Q$$

https://doi.org/10.4153/CJM-1956-024-5 Published online by Cambridge University Press

$$\leqslant t \int_D G(P_0, Q)\, F(Q, ku(Q))\, dV_Q$$
$$\leqslant M - v(P_0).$$

Let us denote by m_0 the minimum of $f(p)$, then by the maximum principle for harmonic functions

$$m_0 \leqslant v_0(P), \qquad P \in D,$$

and so we find

$$0 < t_k^1[u] \leqslant (M - |m_0|)\, \delta^{-1} G_0^{-1}. \tag{5.9}$$

Since $t_k^1[u]$ is positive, we have

$$m_0 \leqslant v_0(P) < T_k^1[u]$$

and therefore $T_k^1[u]$ has the bounds

$$m_0 < T_k^1[u] \leqslant M. \tag{5.10}$$

We now choose for Ω the connected region of C:

$$\Omega\colon m_0 - \epsilon < u < M + \epsilon, \tag{5.11}$$

and consider the equation

$$u = T_k^1[u], \tag{5.12}$$

for $0 \leqslant k \leqslant 1$. That $T_k^1[u]$ is jointly continuous in k and u is evident on inspection. To show that this operator is compact, we select from any bounded set of functions a subsequence $\{u_n\}$ such that $t_k^1[u_n]$ converges to a limit. This is possible on account of (5.9). A proof similar to that in the preceding sections shows that

$$\int_D G(P, Q)\, F(Q, u(Q))\, dV_Q$$

is compact, and the result follows if we consider the subsequence $\{u_n\}$.

For $0 \leqslant k \leqslant 1$, we see from (5.10) that (5.12) has no solutions on Ω', since this would contradict (5.11). For $k = 0$, $T_k^1[u](P)$ is independent of u and so (5.12) has a unique solution. The Schauder-Leray theorem now shows that for $k = 1$, (5.12) has a solution. Thus the integral equation (5.5) has a solution $u(P)$ with maximum M, and this establishes the result, which we state as follows.

THEOREM II. *There exists a solution for suitable t of*

$$\Delta u = -tF(P, u), \qquad F \geqslant \delta > 0,$$

with assigned boundary values $f(p) \leqslant M$ and maximum M.

The proof shows that the minimum value of the solution is attained on the boundary, and so is equal to m; however this could be deduced from the differential equation given that t is positive.

From our next theorem we insert the parameter t with the dependent variable

https://doi.org/10.4153/CJM-1956-024-5 Published online by Cambridge University Press

u in $F(P, u)$. This requires a different set of conditions to be satisfied by $F(P, u)$, namely

(5.13) $$F(P, 0) = 0$$

and

(5.14) $$F_u(P, u) \geqslant \delta > 0.$$

Thus we consider the differential equation

(5.15) $$\Delta u = - F(P, tu),$$

and look for a solution with maximum M and boundary values $f(p)$ where

(5.16) $$0 < m_0 \leqslant f(p) \leqslant M_0 < M.$$

The necessity of these restrictions will appear; meanwhile we remark that the case $M_0 = M$ can be solved for $t = 0$ with a harmonic solution.

The appropriate integral equation is now

(5.17) $$u(P) = \int_D G(P, Q)\, F(Q, tu(Q))\, dV_Q + v_0(P).$$

We shall supply the parameter k in front of the integral, but this leads to a minor difficulty which suggests the addition of a further term. We define

(5.18) $$T_k^2[u](P) = k \int_D G(P, Q)\, F(Q, tu(Q))\, dV_Q + C(1 - k)\, t + v_0(P),$$

where

$$2C = \delta m_0 G_0,$$

and G_0 is again the maximum value of the integral (5.8). For $0 \leqslant k \leqslant 1$ the right side of (5.18) is an increasing function of t, and we can choose $t = t_k^2[u]$ so that

(5.19) $$\max T_k^2[u] = M.$$

Since the first two terms in T_k^2 have the sign of t, and since $v_0(P) < M$, it is evident that $t_k^2[u]$ must be positive. Thus for $0 \leqslant k \leqslant 1$, $T_k^2[u]$ will have the lower bound m_0, since $m_0 \leqslant v_0(P)$. We therefore define the region Ω of function space C as

(5.20) $$\Omega\colon 0 < \tfrac{1}{2}m_0 < u(P) < K,$$

where K is a large positive constant as yet not fixed, but which exceeds M.

To show that T_k^2 is completely continuous in $\Omega + \Omega'$ we need a uniform bound for $t_k^2[u]$, $u \in \Omega + \Omega'$. To find this, we take the point P_0 where (5.8) has maximal value $G_0 > 0$, and note that for $u \in \Omega$, $F(P, u) \geqslant \tfrac{1}{2}\delta m_0$. Then

$$M \geqslant T_k^2[u] \geqslant \tfrac{1}{2}kt\delta m_0 G_0 + C(1 - k)\, t + m_0$$
$$= \tfrac{1}{2}\delta m_0 G_0 t + m_0$$

according to the definition of C in (4.18). Thus for $u \in \Omega$, we have

(5.21) $$0 \leqslant t_k^2[u] \leqslant 2\,\frac{M - m_0}{\delta m_0 G_0}.$$

The conclusion now follows quickly from the Leray-Schauder theorem. The equation

$$u(P) = T_k^2[u](P) \tag{5.22}$$

has no solutions on Ω' for $0 \leqslant k \leqslant 1$, since

$$\tfrac{1}{2}m_0 < m_0 \leqslant T_k^2[u] \leqslant M < K.$$

For $k = 0$, the operator T_k^2 is independent of u, so that a unique solution exists. Thus for $k = 1$ the conclusion follows that (5.22) has a solution. From (5.18) we see that (5.17) is then satisfied.

THEOREM III. *Let $F(P, u)$ satisfy* (5.13) *and* (5.14). *Then there exists for a suitable value of t a solution of*

$$\Delta u = -F(P, tu)$$

with assigned maximum $M > 0$ in $D + B$ and given boundary values $f(p) \leqslant M$ on B.

We note that $F(P, u) = \lambda_1 u$, where λ_1 is the lowest eigenvalue as in §4, yields a counterexample to the solvability of the conventional Dirichlet problem for this equation, since an orthogonality condition is necessary.

We conclude this section with a similar theorem for the equation

$$\Delta u = -F(P, u) - t\rho(P). \tag{5.23}$$

Again the solution is to have a given maximum M and boundary values $f(p) \leqslant M$. The detailed assumptions are as follows. We take for $F(P, u)$ the restrictions

$$F(P, u) \geqslant -F_0 \tag{5.24}$$

and

$$F_u(P, u) \geqslant 0, \tag{5.25}$$

while the coefficient of t on the right in (4.23) must satisfy

$$\rho(P) \geqslant \rho_0 > 0. \tag{5.26}$$

The integral equation of the problem is

$$u(P) = \int_D G(P, Q)\,[F(Q, u(Q)) + t\rho(Q)]\,dV_Q + v_0(P), \tag{5.27}$$

and so, defining

$$R(P) = \int_D G(P, Q)\,\rho(Q)\,dV_Q \geqslant 0, \tag{5.28}$$

we set

$$T_k^3[u](P) = k\int_D G(P, Q)\,F(Q, u(Q))\,dV_Q + tR(P) + v_0(P). \tag{5.29}$$

The choice of $t = t_k^3[u]$ is again governed by

(5.30) $$\max T_k^3[u] = M.$$

For the domain Ω we take

(5.31) $$\Omega: -K < u < M + \epsilon,$$

where K is a large positive constant. Now for $u \in \Omega$ we have from (5.24) and (5.25) a limitation for $F(P, u)$:

(5.32) $$|F(P, u)| < A.$$

Since $F(P, u)$ is bounded as $u \to -\infty$, A is independent of K.

We now obtain bounds for $t = t_k^3[u]$. Since $v_0(P) < M$, the first two terms together in (5.29) must be somewhere positive. Since $G(P, Q)$ is a non-negative kernel, this implies that the integrand

$$kF(Q, u(Q)) + t\rho(Q)$$

is somewhere positive. Hence at some point Q_1,

$$t\rho(Q_1) > -kF(Q_1, u(Q)) > -kA$$

and so

$$t > -kA/\rho_0.$$

This furnishes a lower bound for t. An upper bound may be found if we note that at the point P, where $R(P_1) = R_1$ is maximal, we have

$$tR_1 \leqslant M - k\int_D GF\,dV - v_0$$

$$\leqslant M + kG_0F_0 - m_0.$$

Thus

(5.33) $$-A/\rho_0 < t < (M + G_0F_0 - m_0)/R_1,$$

and these bounds are independent of K.

The necessary lower bound for $T_k^3[u]$ is obtained by taking lower bounds for each term. Thus

(5.34) $$T_k^3[u] \geqslant -F_0G_0 - \frac{A}{\rho_0}R_1 + m_0,$$

where m_0 is a lower bound for $f(p)$. This lower bound (4.34) is independent of K and so if we choose

$$K = 2\left(F_0G_0 + \frac{AR_1}{\rho_0} + |m_0|\right),$$

then the equation

(5.35) $$u = T_k^3[u]$$

will have no solutions on Ω' for $0 \leqslant k \leqslant 1$. For $k = 0$ there is a unique solution as before. For $k = 1$, there must accordingly exist a solution and from (5.29) we see that (5.27), is satisfied for a certain value of t. The maximum condition (5.30) also holds and the solution of the problem is thus completed.

THEOREM IV. *Let $F(P, u)$ satisfy* (5.24) *and* (5.25), *and let $\rho(P)$ satisfy* (5.26). *Then there exists for a suitable value of t a solution of*

$$\Delta u = -F(P, u) - t\rho(P),$$

with assigned maximum M in $D + B$ and given boundary values $f(P) < M$ on B.

The various conditions imposed on $F(P, u)$ in these theorems can be slightly relaxed in various ways. However it is to be noted that the conditions of Theorem III exclude all functions $F(P, u)$ satisfying the restrictions of the other theorems.

6. A modified Neumann problem. As an illustration of the way in which this method of proving existence theorems can be applied to other types of boundary condition, we include here a modified Neumann problem for the equation

$$\Delta u - \delta u = -F(P, u), \qquad \delta > 0, \tag{6.1}$$

where

$$F(P, u) \geqslant -F_0, \quad F_u(P, u) \geqslant 0. \tag{6.2}$$

The boundary condition shall be

$$\frac{\partial u}{\partial u} = g_0(p) + tg_1(p), \tag{6.3}$$

for some value of t. We take $g_0(p)$ and $g_1(p)$ to be C^1 with

$$g_1(p) > 0. \tag{6.4}$$

The usual maximum condition $\max u = M$ shall hold.

The Neumann function $N(P, Q)$ of the linear equation

$$\Delta u - \delta u = 0 \tag{6.5}$$

may be written as

$$N(P, Q) = G(P, Q) + K(P, Q), \tag{6.6}$$

where $G(P, Q)$ is the Green's function, and $K(P, Q)$ the Bergman kernel function, of (6.6). **(2)** We shall need the complete continuity in the space C of the operator with kernel $N(P, Q)$; this will be established by showing that the operators based on $G(P, Q)$ and $K(P, Q)$ are completely continuous. Indeed the proof for $G(P, Q)$ is the same as in §3. Now let us write down Green's first formula on D with argument functions $K(P, Q)$ and 1. Since $K(P, Q)$ is a solution of the differential equation, we get

$$\int_D [\nabla K \cdot \nabla 1 + \delta K \cdot 1]\, dV = \int_B 1 \frac{\partial K}{\partial n}\, dS.$$

The right hand expression is the solution of (6.5) with boundary values 1, and so is less than or equal to 1 in D. Thus we find

$$\int_D K(P, Q)\, dV \leqslant \delta^{-1};$$

this integral is uniformly bounded in $D + B$. We also note that $K(P, Q)$ is non-negative **(2)** in $D + B$. A calculation of the kind given in §3 now leads to the complete continuity of the operator based on $K(P, Q)$. Further details are here omitted.

The integral equation of the problem is

$$u(P) = \int_D N(P, Q)\, F(Q, u(Q))\, dV_Q + tv_0(P) + v_1(P), \tag{6.7}$$

where for $i = 0, 1$ we have

$$u_i(P) = \int_B N(P, q)\, g_i(q)\, dS_q. \tag{6.8}$$

Since $N(P, Q) \geqslant 0$ it follows from (6.4) that $v_1(P) > 0$, and we denote by v_1 and V_1 positive lower and upper bounds:

$$0 < v_1 \leqslant v_1(P) \leqslant V_1,$$

while similarly choosing bounds for $v_0(P)$:

$$v_0 \leqslant v_0(P) \leqslant V_0.$$

The operator T for this problem will now be defined as

$$\tilde{T}_k[u](P) = \int_D N(P, Q)\, F(Q, ku(Q))\, dV_Q + tv_0(P) + v_1(P), \tag{6.9}$$

while $t = \tilde{t}_k[u]$ is fixed by the condition

$$\max \tilde{T}_k[u] = M. \tag{6.10}$$

Setting

$$\Omega = \{u | -K < u < M + \epsilon\},$$

we find that for $u \in \Omega$, $F(P, u)$ satisfies an estimate

$$F(P, u) \leqslant A. \tag{6.11}$$

Then $t = \tilde{t}_k[u]$ has the bounds

$$\frac{|N_0 A + V_1 - M|}{v_0} \leqslant t \leqslant \frac{M - v_1 + N_0 F_0}{V_0}. \tag{6.12}$$

We therefore choose $-K$ less than $v_0^{-1}|N_0 A + V_1 - M|$, which is possible since this quantity is independent of K. The equation

$$u = \tilde{T}_k[u]$$

https://doi.org/10.4153/CJM-1956-024-5 Published online by Cambridge University Press

now has no solutions on Ω' for $0 \leqslant k \leqslant 1$; and a unique solution for $k = 0$. The result now follows as before.

THEOREM V. *There exists a solution of* (6.1) *which satisfies the boundary condition* (6.2) *for some t, and has maximum value M.*

As in Theorem I, the right side of (6.3) could be replaced by a more general increasing function of t. Corresponding results for the Dirichlet and Robin boundary conditions and this differential equation can be established along the same lines of proof.

In conclusion we note that the uniqueness of solutions in all of these results has not been established.

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