ON CONTINUOUS ISOMORPHISMS OF TOPOLOGICAL GROUPS

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1. Let G be a locally compact connected group, and let $A(G)$ be the group of all continuous automorphisms of G. We shall introduce a natural topology into $A(G)$ as previously [1] (i.e. the topology of uniform convergence in the wider sense.) When the component of the identity of $A(G)$ coincides with the group of inner automorphisms, we shall call G *complete*. The purpose of this note is to prove the following theorem and give some applications of it.

THEOREM 1. *Let G be a locally connected complete group with compact center Z, and let H be a locally compact group. If φ is a continuous isomorphism which maps G into H, then φ is necessarily an open mapping and $\varphi(G)$ is closed in H.*

This theorem and the applications in § 2 form extensions, with a simplified way, of propositions which were previously shown by one of the authors.[2]

First we shall prove the following

LEMMA. *Let G be a locally compact connected and locally connected group and H a locally compact group. If φ is a continuous isomorphism which maps G on an everywhere dense subgroup in H, then $\varphi(G)$ is an invariant subgroup of H.*

Proof. In this proof U's and V's denote neighbourhoods of the identities of G and of H, respectively, whose closures are compact. Let us take an element h of H and an arbitrary neighbourhood V_1. For the boundary B of a neighbourhood U_1 there exists V_2 so that $\varphi(B) \cap V_2 = \phi$, where ϕ means the empty set. Now we can find V_3 and U_2 such that for all $k \in hV_1$, $k^{-1}V_3k \subset V_2$ and $\varphi(U_2) \subset V_3$. For an arbitrary element g of $\varphi^{-1}(hV_1 \cap \varphi(G))$ we have $\varphi(g^{-1}U_2g) \subset V_2$, and accordingly $\varphi(g^{-1}U_2g) \cap \varphi(B) = \phi$, that is $g^{-1}U_2g \cap B = \phi$.

Let C be the connected component of U_2 containing the identity. Then $g^{-1}Cg \cap B = \phi$ implies $g^{-1}Cg \subset U_1 \subset \overline{U}_1$, where $\overline{U}_1$ is the closure of U_1. Thus $\varphi(g)^{-1}\varphi(C)\varphi(g) \subset \varphi(\overline{U}_1)$ implies

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[1] See e.g. K. Nomizu and M. Gotô, "On the group of automorphisms of a topological group," forthcoming in Tôhoku Math. Journ.

[2] See M. Gotô, "Faithful representations of Lie groups I," Mathematica Japonicae, Vol. 1, No. 3, (1949). Referred to as F. R.

$$(*) \qquad h^{-1}\varphi(C)h \subset \varphi(\overline{U}_1).$$

On the other hand C generates G because C is open, and we obtain

$$h^{-1}\varphi(G)h \subset \varphi(G),$$

which proves our lemma.

Remark. When G is arcwise connected, we have an analoguous lemma by a similar argument as above. In this case we have only to pay attention to the fact that G is generated by the arcwise connected component of the identity in a neighbourhood.

Proof of the theorem. It is sufficient to prove the theorem for the case when $\varphi(G)$ is everywhere dense in H. According to the above lemma $\varphi(G)$ is invariant in H. Let h be an element of H. Let us consider the automorphism σ_h of G defined by $\sigma_h(x) = \varphi^{-1}(h^{-1}\varphi(x)h)$ for $x \in G$. The continuity of $\sigma_h(x)$ in x and h follows from (*) in the proof of the lemma because C is a neighbourhood of the identity. So the connectedness of H implies that σ_h is an inner automorphism of G because G is complete. That is, for a suitable $g \in G$, $\sigma_h(x) = g^{-1}xg$, and by operating φ on each side we have $h^{-1}\varphi(x)h = \varphi(g)^{-1}\varphi(x)\varphi(g)$, whence $(h\varphi(g)^{-1})\varphi(x) = \varphi(x)(h\varphi(g)^{-1})$ for every $x \in G$. Next let A be the centralizer of $\varphi(G)$ in H: $A = \{y;\ x^*y = yx^*;$ for all $x^* \in \varphi(G)\}$, which is clearly a closed invariant subgroup.

Then from the above fact it is easy to see that

$$H = \varphi(G)\cdot A.$$

Now by the assumption the center Z of G is compact and $\varphi(Z)$, which coincides with the intersection of $\varphi(G)$ and A, is also compact. Thus we have algebraically

$$H/\varphi(Z) = \varphi(G)/\varphi(Z) \times A/\varphi(Z),$$

where $\times$ means the direct product of groups. On the other hand the topological product group

$$L = G/Z \times A/\varphi(Z)$$

can be covered by countable compact sets since A is a subgroup of a connected group H. Hence the continuous isomorphism from L onto $H/\varphi(Z)$ obtained by extending the mapping φ and the identity mapping of $A/\varphi(Z)$, is necessarily open. Therefore $\varphi(G)/\varphi(Z)$ is closed in $H/\varphi(Z)$, whence $\varphi(G)$ is closed in H; this completes the proof.

2. *Applications to (L)-groups.*[3]

[3] For the definitions and the structures of (L)-groups, see K. Iwasawa, "On some types of topological groups," Ann. of Math., Vol. **50** (1949).

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THEOREM 2.[4] *Let G be a connected semi-simple (L)-group[5] with compact center. Then any continuous homomorphism of G into a locally compact group is open.*

Proof. We can readily prove that any factor group of G is complete and locally connected and has compact center.

THEOREM 3.[6] *Let G be a connected (L)-group, and $G = SR$ a "Levi decomposition" of G; R is the radical[7] of G, and S is a continuous isomorphic image of a connected semi-simple (L)-grnup S_1.[8] Assume that the center of S is compact. Let φ be a continuous isomorphism which maps G into a locally compact group H. Then $\varphi(G)$ is closed if $\varphi(R)$ is closed in H (and conversely.)*

Remark. It is to be noticed that the center of S_1 is also compact because of the connectedness of S_1, and hence $\varphi(S)$ is closed in H by Theorem 2.

Proof. Let $\overline{\varphi(G)}$ be the closure of $\varphi(G)$. The fact that G is locally a direct product of a closed local Lie group and a compact group[9] readily implies that $\varphi(G)$ is invariant in $\overline{\varphi(G)}$. Hence $\varphi(R)$ is also an invariant subgroup of $\overline{\varphi(G)}$. Now in the factor group $\overline{\varphi(G)}/\varphi(R)$, the subgroup $\varphi(G)/\varphi(R) = \varphi(S)\varphi(R)/\varphi(R)$ satisfies the assumptions in Theorem 2. Hence $\varphi(G)/\varphi(R)$ is closed in $\overline{\varphi(G)}/\varphi(R)$, i.e. $\varphi(G)$ is closed in H.

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4) See F. R. Lemma 4.

5) For the definitions etc. of semi-simple (L)-groups, see M .Gotô: "Linear representations of topological groups," forthcoming in Proc. Amer. Math. Soc.

6) See F. R. Theorem 2.

7) A locally compact group G contains the uniquely determined maximal connected solvable invariant subgroup R, which is closed in G. Following Iwasawa loc. cit., we shall call R the *radical* of G.

8) On decompositions of (L)-groups as such, see Y. Matsushima, "On the decomposition of an (L)-group," forthcoming in Journ. of Math. Soc. Japan.

9) See Iwasawa, loc. cit.