Proceedings of the Edinburgh Mathematical Society (1991) 34, 487–495 ©

ON LARGE DEVIATIONS IN HILBERT SPACE

by NIGEL J. CUTLAND

(Received 2nd March 1990)

Nonstandard methods and a flat integral representation are used to give a simple and intuitive proof of the large deviation principle for a Gaussian measure on a separable Hilbert space.

1980 *Mathematics subject classification* (1985 Revision): 03H05, 28C20, 46C10, 60F10.

Introduction and preliminaries

This brief note is to show how the ideas of [2] can be used to give a simple and intuitive nonstandard proof of the large deviation principle for a Gaussian measure on a separable Hilbert space. The general LDP for a Gaussian measure on a Banach space was established in [7] by a very complicated proof. Our technique [2] for Wiener measure was adapted in [3] to give an LDP for Lévy Brownian motion; a key part of that proof was a nonstandard version of Kolmogorov's continuity theorem used to identify nearstandard members of $C(\mathbb{R}^d, \mathbb{R})$. Here a similar idea is used to identify nearstandard members of l^2, and is the key to the proof of (4.4) below.

The author would like to thank Dona Strauss for a helpful conversation about this work.

Preliminaries. We assume knowledge of the basics of nonstandard analysis and the Loeb measure construction (see [1], [4] or [5] for example). For $x \in {}^*\mathbb{R}$ we write $x < \infty$ to mean that x is finite or negative infinite, and $x \geqq \infty$ means $x \nless \infty$; similarly with $x > -\infty$ and $x \leqq -\infty$. For $x \geqq \infty$ we set ${}^\circ x = \mathrm{st}(x) = \infty \in \bar{\mathbb{R}}$, the usual completion of $\mathbb{R}$. If ν is an internal measure, ν_L denotes the corresponding Loeb measure.

$\mathcal{N}(\mu, \sigma^2)$ denotes the distribution of a Gaussian random variable with mean μ and variance σ^2.

1. An elementary estimate

Lemma 1.1. *Suppose that $\theta_1, \ldots, \theta_n$ are independent random variables with $\theta_i \sim \mathcal{N}(0, \sigma_i^2)$, and let*

$$\theta^2 = \sum_{i=1}^{n} \theta_i^2, \quad \textit{with} \quad \theta \geqq 0 \textit{ iff } \prod_{i=1}^{n} \theta_i \geqq 0.$$

Then

$$E \exp(\theta) \leqq e^{\sigma^2/2}$$

where

$$\sigma^2 = \sum_{i=1}^{n} \sigma_i^2 = E(\theta^2).$$

Proof. Let $\xi = \theta_1 + \cdots + \theta_n \sim \mathcal{N}(0, \sigma^2)$; we know from classical theory that

$$E \exp(\xi) = e^{\sigma^2/2}.$$

We will see that for all k

$$E(\theta^k) \leqq E(\xi^k)$$

from which the result follows by dominated convergence, using the series for $\exp(\xi)$.

Note that θ is symmetric about 0, so for k odd,

$$E(\theta^k) = 0 = E(\xi^k).$$

If k is even, say $k = 2m$ then

$$\xi^k = (\xi^2)^m = \left(\theta_1^2 + \theta_2^2 + \cdots + \theta_n^2 + 2\sum_{i<j} \theta_i \theta_j\right)^m$$

$$= \theta^{2m} + \text{terms of the form } \prod_{i=1}^{n} \theta_i^{p_i}.$$

Now

$$E\left(\prod_{i=1}^{n} \theta_i^{p_i}\right) = \prod_{i=1}^{n} E(\theta_i^{p_i}) \geqq 0;$$

hence

$$E(\xi^k) \geqq E(\theta^k) \text{ as required.}$$ □

Corollary 1.2. *For* $a > 0$

https://doi.org/10.1017/S0013091500005241 Published online by Cambridge University Press

$$P(\theta \geqq a) \leqq \exp\left(-\frac{a^2}{2\sigma^2}\right).$$

Proof. This is proved in the same way as the corresponding estimate for normal θ: for any $\lambda > 0$, $E(e^{\lambda\theta}) \leqq e^{\lambda^2\sigma^2/2}$ (from Lemma 1.1) so

$$P(\theta \geqq a) = P(\lambda\theta \geqq \lambda a)$$
$$= P(e^{\lambda\theta} \geqq e^{\lambda a})$$
$$\leqq \exp(\tfrac{1}{2}\lambda^2\sigma^2 - \lambda a).$$

Now put $\lambda = a/\sigma^2$. □

2. Gaussian measures on a separable Hilbert space

The following facts are well known (see [6] for example).

Theorem 2.1. *Let $(\sigma_n^2)_{n=1,2,\dots}$ be a sequence of variances with $\sigma = \sum \sigma_n^2 < \infty$ and let μ_n be the probability $\mu = \Pi\mu_n$ on $\mathbb{R}^{\mathbb{N}}$, so that, writing $x = (x_{n)n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$, then under μ the variables $(x_n)_{n\in\mathbb{N}}$ are independent, $\mathcal{N}(0, \sigma_n^2)$. Then $\mu(l^2) = 1$.*

Proof.
$$E\left(\sum_{m=1}^{\infty} x_m^2\right) = \lim_{n\to\infty} E\left(\sum_{m<n} x_m^2\right) = \sum_{m=1}^{\infty} \sigma_m^2 < \infty.$$ □

Theorem 2.2. *If μ is a centred Gaussian measure on a separable Hilbert space H, there is an orthonormal basis $(e_n)_{n=1,2,\dots}$ for H and variances σ_n^2 with $\sum \sigma_n^2 < \infty$ such that the variables $x_n = (x, e_n)$ are independent $\mathcal{N}(0, \sigma_n^2)$.*

Proof. See [6].

2.3. Definitions.

(a) The *action functional* for the measure μ on l^2 given by Theorem 2.1 is

$$I(x) = \tfrac{1}{2}\sum \frac{x_n^2}{\sigma_n^2} \in \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}.$$

(b) The *Cameron-Martin subspace* is the space

$$H_0 = (x : I(x) < \infty\}$$

with inner product

https://doi.org/10.1017/S0013091500005241 Published online by Cambridge University Press

$$(x,y)_0=\sum \frac{x_n y_n}{\sigma_n^2}$$

and norm $|\cdot|_0$. The l^2 norm $|\cdot|$ is a measurable norm on H_0 in the sense of Gross (see [6]), and l^2 is the completion of H_0 with respect to $|\cdot|$.

3. Nonstandard representation of Gaussian measures on Hilbert space

The space l^2 is naturally represented in ${}^*\mathbb{R}^N$ for any fixed infinite $N\in{}^*\mathbb{N}$ as follows.

Definition 3.1.

(a) $X=(X_n)_{n\leqq N}$ is *nearstandard* if

$$\sum_{n\in\mathbb{N}} {}^\circ X_n^2 \approx \sum_{n\leqq N} X_n^2<\infty.$$

Write $X\in\text{ns}$ to mean X is nearstandard.

(b) For $X\in\text{ns}$ define ${}^\circ X=\text{st}(X)$ by

$${}^\circ X=({}^\circ X_n)_{n\in\mathbb{N}}\in l^2.$$

Remark 3.2.

(1) $X\in{}^*\mathbb{R}^N$ is nearstandard in the above sense if the sequence

$$\hat{X}_n=\begin{cases} X_n & n\leqq N\\ 0 & n>N, n\in{}^*\mathbb{N}\end{cases}$$

(which is in ${}^*l^2$) is nearstandard in the l^2 topology.

(2) An equivalent characterisation of $X\in\text{ns}$ is

$$\sum_{n\leqq N} X_n^2<\infty$$

and $\sum_{M\leqq n\leqq N} X_n^2\approx 0$ all infinite M.

Let Γ be the internal probability on ${}^*\mathbb{R}^N$ given by the variances $(\sigma_n^2)_{n\leqq N}$; i.e. $\Gamma=\prod_{n=1}^N {}^*\mu_n$. Then we have the 'flat integral' formula for *Borel $A\subseteq{}^*\mathbb{R}^N$:

$$\Gamma(A)=\kappa\int_A \exp\left(-\frac{1}{2}\sum_{n=1}^N \frac{X_n^2}{\sigma_n^2}\right)dX$$

https://doi.org/10.1017/S0013091500005241 Published online by Cambridge University Press

where $dX = {}^*$Lebesgue measure on ${}^*\mathbb{R}^N$ and $\kappa = \prod_{n=1}^N (2\pi\sigma_n^2)^{1/2}$.
We have:

Theorem 3.3. *Suppose that* $\sigma = \sum \sigma_n^2 < \infty$ *and* μ *is the probability on* l^2 *given by Theorem 2.2. Then*

(*a*) *X is nearstandard for* Γ_L – a.a. $X \in {}^*\mathbb{R}^N$

(*b*) $\mu(\cdot) = \Gamma_L(\mathrm{st}^{-1}(\cdot))$

Proof. (a) Since ${}^\circ X_n$ is $\mathcal{N}(0, \sigma_n^2)$ for finite n,

$$E\left(\sum_{n \in \mathbb{N}} {}^\circ X_n^2\right) = \lim_{n \to \infty} E\left(\sum_{m \leq n} {}^\circ X_m^2\right) = \sum_{n \in \mathbb{N}} \sigma_n^2 < \infty$$

and

$$E\left({}^\circ\left(\sum_{n \leq N} X_n^2\right) - \sum_{n \in \mathbb{N}} {}^\circ X_n^2\right) = \lim_{n \to \infty} E\left({}^\circ \sum_{m=n}^{N} X_m^2\right)$$

$$\leq \lim_{n \to \infty} {}^\circ \sum_{m=n}^{N} \sigma_m^2 = 0.$$

Hence, for a.a. X under Γ_L

$$\sum_{n \in \mathbb{N}} {}^\circ X_n^2 = {}^\circ\left(\sum_{n \leq N} X_n^2\right) < \infty.$$

(b) is obvious. □

Action. The counterpart for ${}^*\mathbb{R}^N$ of the action functional I is

$$J(X) = \tfrac{1}{2} \sum_{n=1}^{N} \frac{X_n^2}{\sigma_n^2}.$$

The connection with I is given by:

Lemma 3.3. (*a*) *If* $J(X)$ *is finite then* $X \in \mathrm{ns}$ *and*

$$I({}^\circ X) \leq {}^\circ J(X)$$

(*b*) *If* $X = {}^*x \upharpoonright N$ *for* $x \in l^2$, *then*

$$J(X) \approx I(x).$$

Proof. (a) For any m

$$\sum_{m \leq n \leq N} X_n^2 \leq \left(\sum_{m \leq n \leq N} \frac{X_n^2}{\sigma_n^2} \right) \left(\sum_{m \leq n \leq N} \sigma_n^2 \right) \leq 2J(X) \sum_{m \leq n \leq N} \sigma_n^2.$$

Put $m=1$ to obtain $\sum_{n \leq N} X_n^2$ finite, and putting $m=M$ infinite we have $\sum_{M \leq n \leq N} X_n^2 \approx 0$. Hence $X \in \mathrm{ns}$. The inequality follows from the fact that for finite n

$$\sum_{m \leq n} \frac{{}^\circ X_m^2}{\sigma_m^2} \leq {}^\circ 2J(X).$$

(b) In this case we have

$$2I(x) = \sum_{n=1}^{\infty} \frac{x_n^2}{\sigma_n^2} \leq 2{}^\circ J(X) \leq {}^\circ \sum_{n \in \mathbb{N}} \frac{x_n^2}{\sigma_n^2} = {}^\circ {}^* 2I(x) = 2I(x).$$

4. The large deviation principle

Let $\mu_\delta(A) = \mu(\delta^{-1}A)$ for $A \subseteq l^2$. The large deviation principle gives estimates for $\mu_\delta(A)$ as $\delta \to 0$ for A open or closed. It is proved for a general Gaussian measure on a Banach space in (7).

Theorem 4.1 (Open set). *If G is open, $G \subseteq l^2$, then*

$$\varliminf_{\delta \to 0} \delta^2 \log \mu_\delta(G) \geq -\inf I(G).$$

Proof. Let $z \in G$ with $I(z) < \infty$; it is sufficient to show that $\varliminf \delta^2 \log \mu_\delta(G) \geq -I(z)$. Pick $\beta > 0$ such that the set $A = \{x \in l^2 : |x - z| \leq \beta\} \subseteq G$ and let

$$B = \{X : |X - Z| < \beta\}$$

where $Z = {}^*z \upharpoonright N$. Clearly

$$B \cap \mathrm{ns} \subseteq \mathrm{st}^{-1}(A)$$

so for standard $\delta > 0$

$$\mu_\delta(G) = \mu(\delta^{-1}G) \geq \mu(\delta^{-1}A) = \Gamma_L(\delta^{-1}\mathrm{st}^{-1}A) \geq {}^\circ\Gamma(\delta^{-1}B).$$

Thus

$$\mu_\delta(G) \gtrsim \kappa \int_{\delta^{-1}B} \exp(-J(X))\, dX \quad \text{(definition of } \Gamma)$$

$$=\kappa\int_{C_\delta}\exp(-J(Y+\delta^{-1}Z))\,dY$$

(where $C_\delta=\{Y:|Y|<\delta^{-1}\beta\}$ and putting $Y=X-\delta^{-1}Z$)

$$=\int_{C_\delta}\exp\left(-\delta^{-2}J(Z)-\delta^{-1}\sum_{n\leqq N}Y_nZ_n/\sigma_n^2\right)d\Gamma(Y).$$

So (using Jensen's inequality)

$$\delta^2\log\mu_\delta(G)\gtrsim -J(Z)-\frac{\delta}{\Gamma(C_\delta)}\int_{C_\delta}\left(\sum_{n\leqq N}Y_nZ_n/\sigma_n^2\right)d\Gamma(Y)+\delta^2\log\Gamma(C_\delta).$$

Now $J(Z)\approx I(z)$, and for the other terms on the right observe that for $\delta\approx 0$, $C_\delta\supseteq\text{ns}$ and so $\Gamma(C_\delta)\approx 1$; finally

$$\left|\int\left(\sum_{n\leqq N}Y_nZ_n/\sigma_n^2\right)d\Gamma(Y)\right|^2\leqq E_\Gamma\left(\left(\sum_{n\leqq N}Y_nZ_n/\sigma_n^2\right)^2\right)=\sum_{n\leqq N}\frac{Z_n^2}{\sigma_n^2}\approx 2I(z)<\infty.$$

Hence $\underline{\lim}\ \delta^2\log\mu_\delta(G)\geqq -I(z)$, as required.

Theorem 4.2 (Closed Set). *If $F\subseteq l^2$ is closed, then*

$$\overline{\lim}\,\delta^2\log\mu_\delta(F)\leqq -\inf I(F).$$

Proof. Let $\gamma<\inf(I(F))$, it is sufficient to show that $\overline{\lim}\,\delta^2\log\mu_\delta(F)\leqq-\gamma$. Begin by observing that

$$^*F\cap\text{ns}\subseteq\{x\in {}^*l^2:J(x)\geqq\gamma\}$$
$$=D\quad\text{say,}$$

where $J(x)=J(x\restriction N)$ for $x\in {}^*l^2$ and ns here means ns($^*l^2$); this is because if $x\in {}^*F$ and $x\approx y\in l^2$ then $y\in F$ (closure) so $\gamma<I(y)=I({}^\circ x)\leqq {}^\circ J(x)$ by Lemma 3.3.

It is sufficient now to prove that

$$\overline{\lim_{\delta\to 0}}\,\delta^2\log{}^*\mu_\delta(D)\leqq-\gamma \tag{4.3}$$

$$\overline{\lim_{\delta\to 0}}\,\delta^2\log{}^*\mu_\delta({}^*F\backslash D)\leqq -R \tag{4.4}$$

for any finite R. The proof of (4.3) is almost identical to the proof of [2, Lemma 6.3] so we omit it.

Proof of 4.4. Pick an increasing sequence m_n such that $m_0=0$ and

$$\sum_{m_n<k} \sigma_k^2 \leqq \frac{1}{2^{n+1}} \quad \text{for } n \geqq 1.$$

Then

$$\sum_{m_{n-1}<k\leqq m_n} \sigma_k^2 \leqq \frac{1}{2^n} \qquad (n>1). \tag{4.5}$$

For $X \in {}^*l^2$ define

$$Y(X)=(Y_n)$$

where

$$Y_n = \sum_{m_{n-1}<k\leqq m_n} X_k^2.$$

Notice that by (4.5) and Corollary 1.2 (with $\theta^2 = Y_n$)

$${}^*\mu_\delta(Y_n \geqq 2^{-n/2}) \leqq 2\exp\left(-\frac{2^{n/2}}{2\delta^2}\right). \qquad (n>1). \tag{4.6}$$

Suppose we are given $X \in {}^*l^2$ such that $\sum_{n\leqq N} X_n^2 < \infty$ and $Y_n < 2^{-n/2}$ for all $n \geqq k$, for some finite k. Then X is nearstandard in ${}^*l^2$; so we have

$$\text{ns}^c \subseteq \bigcap_{k\in\mathbf{N}} \left(\left\{ \sum_{n\leqq N} X_n^2 \geqq k \right\} \cup \bigcup_{\substack{n\geqq k\\ n\in{}^*\mathbf{N}}} \{Y_n \geqq 2^{-n/2}\} \right).$$

Now ${}^*F\backslash D \subseteq \text{ns}^c$ and ${}^*F\backslash D$ is internal, so there is infinite K with

$${}^*F\backslash D \subseteq \left\{ \sum_{n\leqq N} X_n^2 \geqq K \right\} \cup \bigcup_{\substack{n\geqq K\\ n\in{}^*\mathbf{N}}} \{Y_n \geqq 2^{-n/2}\}.$$

Then, by Corollary 1.2 and 4.6

$$\begin{aligned} {}^*\mu_\delta({}^*F\backslash D) &\leqq 2\exp\left(-\frac{K}{2\delta^2\sigma^2}\right) + \sum_{n\geqq K} 2\exp\left(-\frac{2^{n/2}}{2\delta^2}\right) \\ &\lesssim 2\exp\left(\frac{-K}{2\delta^2\sigma^2}\right) + 2\exp\left(-\frac{2^{K/2}}{2\delta^2}\right) \end{aligned}$$

for finite δ, since the ratio of successive terms in the series is

$$\exp\left(-\frac{2^{n/2}(2^{1/2}-1)}{2\delta^2}\right)\approx 0 \qquad \text{for } n>K.$$

Hence

$$\delta^2 \log^* \mu_\delta(^*F\backslash D) \leqq -\infty$$

for finite δ, which establishes (4.4).

The proof of Theorem 4.2 is now complete.

REFERENCES

1. S. Albeverio, J. E. Fenstad, R. Høegh–Krohn and T. Lindstrøm, *Nonstandard Methods in Stochastic Analysis and Mathematical Physics* (Academic Press, New York 1986).

2. N. J. Cutland, Infinitesimals in action, *J. London Math. Soc.* **35** (1987), 202–216.

3. N. J. Cutland, An action functional for Lévy Brownian motion, *Acta Appl. Math.* **18** (1990), 261–281.

4. N. J. Cutland (Editor), *Nonstandard Analysis and its Applications* (Cambridge University Press, 1988).

5. A. E. Hurd and P. A. Loeb, *An Introduction to Nonstandard Real Analysis* (Academic Press, New York, 1985).

6. H.-H. Kuo, *Gaussian Measures in Banach Spaces* (Lecture Notes in Mathematics, Vol. **463,** Spinger-Verlag, Berlin, 1975).

7. D. W. Stroock, *An Introduction to the Theory of Large Deviations* (Springer-Verlag, New York, 1984).

Department of Pure Mathematics
University of Hull
HU6 7RX, England