H-EXTENSION OF RING

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(Received 20 February 1968; revised 23 July 1968)

A ring R is called an H-ring if for every $x \in R$ there exists an integer $n = n(x) > 1$ such that $x^n - x \in C$, where C is the center of R. I. N. Herstein proved that H-rings must be commutative [See 3 pp. 220—221]. We now introduce the following definition.

DEFINITION. R and R' are two rings, we say R is an H-extension of R' if R' is a subring of R and for any $x \in R$, there exists an integer $n > 1$ (depending on x) such that $x^n - x \in R'$.

In this paper we shall show how the Jacobson radical of R is related to that of R' (Theorem 1) and then we shall give some information about H-extension of a commutative one-sided ideal (Theorem 2). An example is also given at the end of section 2 to show in general we can not arrive at the sharper conclusion that an H-extension of commutative ideal is commutative.

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In this section, we denote R as an H-extension of a subring R' and $J(R)$, the Jacobson radical of the ring R. It is well known $J(R)$ can be characterized as the intersection of all primitive ideals of R or it is the set $\{x \in R | xR$ is a right quasi-regular right ideal of $R\}$. We shall prove the theorem 1 as follows, the proof was patterned after the argument of the paper of Armendariz [1].

LEMMA 1.1. (1). *For any $x \in R$, there exists an arbitrarily high n such that $x^n - x \in R'$.*

(2). *All nilpotent elements of R belong to R'.*

PROOF. (1) If this is false we have an integer m which is the largest m such that $x^m - x \in R'$. Let us choose another $n > 1$ which satisfies $(x^m)^n - x^m \in R'$, then $x^{mn} - x = (x^{mn} - x^m) + (x^m - x) \in R'$. This is contradictory to the maximality of m. (2) Let $x^m = 0$. Choose $N > m$ so that $x^N - x \in R'$, since $x^N = 0$, and we have $x \in R'$.

We now consider the n-square matrix ring Γ_n $(n > 1)$ over a ring Γ with unit element. If Γ_n is an H-extension of a subring B, then by

Lemma 1.1 B contains all nilpotent elements, in particular, the matrices $E_{ij}d(i \neq j,\ d \in \Gamma)$ and therefore the matrices $E_{ii}d = E_{ij}dE_{ji}$. So we have:

LEMMA 1.2. *If the n-square matrix ring* Γ_n $(n > 1)$ *is an H-extension of a subring B. Then* $\Gamma_n = B$.

LEMMA 1.3. *If R is a division ring, then R′ is also a division ring.*

PROOF. Let $0 \neq x \in R'$, then there exists an integer $n > 1$ such that $b = (x^{-1})^n - x^{-1} \in R'$. Multiplying b by x^n and x^{n-1} respectively, we see that 1 and x^{-1} belong to R'. So R' is a division ring.

Now let R be a primitive ring and $R' \neq 0$. By the theorems appearing in [3] chapter II, R can be considered as a dense subring of the ring of all linear transformations of a vector space V. If the dimension of V is one, R is a division ring. Then by Lemma 1.3 R' is also a division ring. This proves R' is a primitive ring. If the dimension of V is larger than one, then considering V as a right faithful module over R' we shall prove it is an irreducible module as follows: Let v_1 be a non-zero fixed element of V and v_2 any element of V. There exists a 2-dimensional vector subspace V_2 which contains v_1 and v_2. Let $U = \{x \in R | V_2 x \subseteqq V_2\}$, $K = \{x \in R | V_2 x = (0)\}$, $U_1 = U \cap R'$. Because R is dense, U/K is isomorphic to the full ring of linear transformations of V_2. Moreover, it is clear that U/K is still an H-extension of its subring $(U_1+K)/K$. So by Lemma 1.2 we have $U/K = (U_1+K)/K$. This assures there exists a linear transformation $x \in R'$ that sends v_1 to v_2. From this we see any element of V is the form $v_1 x$ for some $x \in R'$, in other words V is a cyclic R'-module with every non-zero element as a generator. This proves V is irreducible. So we have the following:

LEMMA 1.4. *If R is a primitive ring and* $R' \neq 0$, *then R′ is a primitive ring.*

REMARK. Some one may wonder in Lemma 1.4 R' is always equal to R. Here we give a primitive ring which is an H-extension of some proper subring. Let Z_p be the prime field of characteristic p and R be a ring of linear transformations of an infinite dimensional vector space M over Z_p. Here R is so chosen that the matrices of its elements have the form

$$\begin{bmatrix} A & & & & & & \\ & d & & & & & \\ & & d & & & 0 & \\ & & & d & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ & 0 & & & & & \cdot \end{bmatrix}$$

https://doi.org/10.1017/S1446788700007126 Published online by Cambridge University Press

and R' is the ring of the form

$$\begin{bmatrix} A & & & & & \\ & 0 & & & & \\ & & 0 & & & 0 \\ & & & 0 & & \\ & & & & \cdot & \\ & 0 & & & & \cdot \\ & & & & & \cdot \end{bmatrix}$$

where A is an arbitrary finite square matrix and d is any element of Z_p. Then for any $a \in R$, we have $a^p - a \in R'$. Moreover R is a primitive ring [See **3** p. **36** example **3**].

THEOREM 1. *If R is an H-extension of a subring R', then*

$$J(R) \cap R' = J(R').$$

PROOF. Let $x \in J(R) \cap R'$. We want to prove that any $y \in xR'$ has a right quasi inverse in R'. Since $y \in xR' \subsetneqq xR$, there is $z \in R$ such that

$$(*) \qquad y+z-yz = 0.$$

Now for some $n = n(z) > 1$, $z^n - z \in R'$. Then $y(z^n-z) = yz^n-z-y \in R'$. This implies $yz^n - z \in R'$. Multiply $(*)$ from right by z^{n-1} and we get $yz^{n-1} = yz^n - z^n = yz^n - z - (z^n - z) \in R'$. Again multiply y on the left and z^{n-2} on the right of $y = yz - z$ and we get $y^2 z^{n-2} \in R'$. Repeating the process $n-1$ times, we get

$$y^{n-1}z \in R', \; z = yz-y = y(yz-y)-y = \cdots = y^{n-1}z - y^{n-1} - \cdots y \in R'.$$

Consequently xR' is a right quasi-regular right ideal of R', so $x \in J(R')$.

The opposite inclusion can be proved as follows: If P is a primitive ideal of R, R/P is a primitive ring and an H-extension of $(R'+P)/P$. By Lemma 1.4 $(R'+P)/P \cong R'/(P \cap R')$ is a primitive ring, so $P \cap R'$ is a primitive ideal of R'. We have:

$$J(R) \cap R' = \Big(\bigcap_{P:\text{ primitive ideal of } R} P\Big) \cap R' = \cap\, (P \cap R') \supseteqq J(R').$$

COROLLARY. *R is semi-simple if and only if R' is semi-simple.*

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W. S. Martindale III defined an γ-ring as a ring R in which $w^{n(w)} - w$ belongs to the center C of R for every commutator w of R and proved in his paper [4] that every commutator of an γ-ring is contained in the center.

In this section we can obtain a parallel result about an H-extension of an one-sided ideal.

We first cite a theorem which is proved in Carl Faith's [2 p. 47] as follows. Let $\phi[X]$ be the polynomial ring over the field ϕ and $[\alpha_1, \cdots, \alpha_r, X]$ denote the subring of $\phi[X]$ generated by X and r fixed non-zero elements $\alpha_1, \cdots, \alpha_r$ in the field ϕ, and set:

$$(*) \quad N(\alpha_1, \cdots, \alpha_r) = \{X^n - X^{n+1}P(X) \mid P(X) \in [\alpha_1, \cdots, \alpha_r, X], n = 1, 2, \cdots\}.$$

THEOREM (Faith). *Let D be a division algebra over the field ϕ, and let Δ be a subalgebra such that to each $d \in D$ there corresponds non-zero elements $\alpha_1, \cdots, \alpha_r \in \phi$ (depending on d) such that for each $a \in \phi(d)$ there exists $f_a(X) \in N_d$ satisfying $f_a(a) \in \Delta$, where $N_d = N(\alpha_1, \cdots, \alpha_r)$ is a set of the type $(*)$. Then D is a field.*

If R is a division ring and an H-extension of a commutative subring R', by Lemma 1.2 R' is a division subring. So R' contains the prime field ϕ of R. We can consider R as a division algebra over ϕ and R' its subalgebra. Furthermore it is clear that every $x \in R$ satisfies the condition of the above theorem if we take all α_i are 1. So R is commutative.

LEMMA 2.1. *If R is a semi-simple H-extension of a commutative subring R', then R is commutative.*

PROOF. It is sufficient to prove this for a primitive ring, because R is a subdirect sum of primitive rings and the H-extension property is inherited by homomorphic images. In this case R ought to be a division ring, otherwise, by [3 p. 33 proposition 3] it contains a subring U which has a homomorphic image isomorphic to the complete matrix ring Γ_n $(n > 1)$ over a division ring Γ. As Γ_n is an H-extension of the homomorphic image U' of $U \cap R'$, by Lemma 1.2, we have $\Gamma_n = U'$. But U' is still commutative since it is the homomorphic image of the commutative ring $U \cap R'$. This is contradictory. So R is a division ring. Now by Faith's theorem we see R is commutative.

LEMMA 2.2. *If R is an H-extension of a commutative subring R', then every commutator $w = xy - yx$ of R belongs to $J(R)$.*

PROOF. $R/J(R)$ is an H-extension of its commutative subring $(R' + J(R))/J(R)$, where $(R' + J(R))/J(R)$ is isomorphic to $R'/(J(R) \cap R')$. By Theorem 1 $R'/(J(R) \cap R') = R'/J(R')$ which is semi-simple. So $R/J(R)$ is commutative by Lemma 2.1. The residue class of a commutator w modulo $J(R)$ is zero. This implies $w = xy - yx \in J(R)$.

LEMMA 2.3. *If R is an H-extension of a commutative right ideal I, then every commutator $w = xy - yx$ is nilpotent.*

https://doi.org/10.1017/S1446788700007126 Published online by Cambridge University Press

PROOF. By Zorn's Lemma we can find a maximal commutative subring R' of R, which contains I. Let $w = xy-yx$, $y \in R'$, there exists an integer $n = n(w) > 2$ such that $w^n - w \in I$, hence

$$(w^n-w)(xy-yx) = (w^n-w)xy-y(w^n-w)x = (w^n-w)xy-(w^n-w)xy = 0.$$

The quasi-regularity of w^{n-1} (by Lemma 2.2) forces: $w(xy-yx) = 0$, in other words $w^2 = 0$. These kinds of w belong to I by Lemma 1.1.

Now $J(R)$ shall be proved commutative as follows: If $a \in J(R)$, there exists an integer $m > 2$ such that $a^m - a \in I$. Then for any $y \in R'$

$$(a^m-a)(xy-yx) = (xy-yx)(a^m-a) = 0.$$

The quasi-regularity of a^{m-1} will yield $a(xy-yx) = 0$, $(xy-yx)a = 0$. Let $x = a$, then $a^2y = aya = ya^2$ for all $y \in R'$. Considering the subring R'' of R generated by R' and a^2 we get R'' is commutative containing R'. The maximal property of R' forces $R'' = R'$. So we have $a^2 \in R'$. If m is even, then $a^m - a \in R'$ implies $a \in R'$. If m is odd, $a^{m-1} \in R'$. The quasi-regularity of a^{m-1} and $a^m - a \in R'$ yield $a \in R'$. As a consequence we can see that $J(R)$ is contained in the commutative subring R'. So $J(R)$ is a commutative ideal.

Finally, by Lemma 2.2 w is contained in $J(R)$, we can conclude that:

$$\begin{aligned} w^3 &= w^2(xy-yx) = w^2xy-w(wy)x = w^2xy-(wy)(wx) \\ &= w^2xy-(wx)(wy) = w^2xy-((wx)w)y = w^2xy-w^2xy = 0. \end{aligned}$$

THEOREM 2. *If R is an H-extension of a commutative one sided ideal I, then every commutator w belongs to I.*

PROOF. By Lemma 2.3 and Lemma 1.1 we can see that w belongs to I.

REMARK. An example is given here to show that in general an H-extension of a commutative ideal is not necessarily commutative:

Let Z_2 be the prime field of characteristic 2 and R be the algebra over Z_2 generated by a, b satisfying

$$a^2 = a,\ ab = b^2 = 0,\ ba = b.$$

Then R is a non-commutative H-extension of its commutative ideal (o, b).

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