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# UNRULY HILBERT DOMAINS

BY
J. L. MOTT AND M. ZAFRULLAH

ABSTRACT. We give a simple construction of non-Noetherian Hilbert domains whose maximal ideals are all finitely generated. Such domains we call unruly Hilbert domains.

A commutative ring $R$ is called a *Hilbert ring* if every prime ideal of $R$ is an intersection of maximal ideals of $R$. In a Hilbert domain, because the ideal (0) is an intersection of maximal ideals, one could hope that a Hilbert domain is Noetherian if all the maximal ideals are finitely generated. Geramita mentioned this possibility to Gilmer and Heinzer who, in [4], produced an example of a two dimensional non-Noetherian Hilbert domain in which all maximal ideals are principal. Gilmer and Heinzer also pointed out in [4] other constructions that produce non-Noetherian Hilbert domains of arbitrary dimension ($>1$) with principal maximal ideals. The aim of this note is to indicate a much simpler construction that seems to work even better than the Gilmer-Heinzer construction. Those non-Noetherian Hilbert domains whose maximal ideals are all finitely generated we shall call *unruly* Hilbert domains.

We shall use a special case of the generalized $D+M$-construction of [1] to construct examples of unruly Hilbert domains of various descriptions, including ones of the same description as the example in [4]. Let $L$ be a field and let $x$ be an indeterminate over $L$. For a subring $D$ of $L$ the set of polynomials:

$$\left\{ a_0 + \sum_{i=1}^{n} a_i x^i \mid a_0 \in D \quad \text{and} \quad a_i \in L \right\} = D + xL[x]$$

is a subring of the principal ideal domain $(PID), L[x]$. Since $R = D + xL[x]$ is a direct sum of the maximal ideal $xL[x]$ of $L[x]$ and a subring of $L$, $R$ comes under the more general title of $D+M$-constructions. We show, using the following results from [3], that if $D$ is a Hilbert domain, then so is $D + xL[x]$.

We include in Lemmas 1 and 2 and Theorems 3 and 4 the statements of the relevant results from [3] because [3] contains some printing errors. We recall, by way of explaining the notation used in [3], that $T$ is an integral domain expressible as a direct sum of a subfield $L$ and a maximal ideal $M$. Moreover, we let $R = D + M$,

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where $D$ is a subring of $L$. (In the context of our present intended application, we expect to apply Theorem 3 to the case where $T = L[x], M = xL[x]$, and $R = D+xL[x]$).

LEMMA 1. ([3, *Lemma* 1.1]). *Let $I$ be an ideal of $R = D + M$. Then the following are equivalent.*

(1) $I \cap D \neq (0)$

(2) $I \supset M$

(3) $IT = T$.

*Furthermore if any of these hold, then $I = I \cap D + M = (I \cap D)R$.*

LEMMA 2. ([3, *Lemma* 1.2]). *Let $P$ be a prime ideal of $R$. Then $P$ is comparable to $M$ if and only if $P$ contains no elements of the form $1 + m$, where $m \in M$.*

THEOREM 3. ([3, *Theorem* 1.3]). *Each prime ideal of $R = D + M$ is either the contraction of a prime ideal of $T$ or is of the form $P_0 + M$, where $P_0$ is a prime ideal of $D$. The map, $Q \longmapsto Q \cap R$ from Spec($T$) to Spec($R$), is a one-to-one order preserving correspondence between the prime ideals of $T$ and the prime ideals $P$ of $R$ such that $P \cap D = (0)$. The maximal ideals of $R$ are the contractions of maximal ideals of $T$ distinct from $M$, and the ideals $P_0 + M$, where $P_0$ is maximal in $D$.*

THEOREM 4. ([3, *Lemma* 1.5]). *Let $u \in 1 + M = \{1 + m | m \in M\}$.*

(1) *$u$ is irreducible in $R$ if and only if $u$ is irreducible in $T$.*

(2) *If $u$ is prime in $T$, it is also prime in $R$.*

(3) *If $T$ is a PID and $u$ is irreducible, then $u$ generates a height one maximal ideal of $R$.*

In our present context, (3) is especially useful because $T$ is the principal ideal domain $L[x]$.

With this review, we proceed to prove the following simple result.

THEOREM 5. *Let $D$ be a Hilbert domain and let $L$ be a field containing $D$. Then $R = D + xL[x]$ is a Hilbert domain.*

PROOF.The proof consists in checking that each prime ideal is indeed an intersection of maximal ideals.

(1) *The prime ideal* (0). The primes of the form $(1 + xf_\alpha(x))R$ are all maximal by (3) of Theorem 4. Noting that a polynomial can only be of finite degree, we conclude that $\bigcap_\alpha (1 + xf_\alpha(x)) = (0)$.

(2) *The prime ideals $P$ of $R$ such that $P \cap D = P_0 \neq (0)$.* These prime ideals have the form $P = P_0 + xL[x]$ where $P_0$ is a prime ideal of $D$. Since $D$ is a Hilbert domain, there exist maximal ideals $\{M_\alpha\}_{\alpha \in I}$ of $D$ such that $P_0 = \bigcap_{\alpha \in I} M_\alpha$. But then obviously $P = \cap_{\alpha \in I}(M_\alpha + xL[x])$.

(3) *The prime ideals $P$ such that $P \cap D = (0)$.* If $P \neq xL[x]$ then, combining Theorems 3 and 4, we conclude that $P = (1+xf(x))R$. On the other hand, if $P = xL[x]$,

then let $\{N_\alpha\}_{\alpha\in I}$ be the set of maximal ideals of $D$ such that $\bigcap_{\alpha\in I} N_\alpha = (0)$. Then $xL[x] = \bigcap_{\alpha\in I}(N_\alpha + xL[x],)$ and this completes the proof.

COROLLARY 6. *If $D$ is a PID such that Spec$(D)$ is infinite and if $L$ is a proper extension of the quotient field $K$ of $D$, then $D + xL[x]$* **is a non-Noetherian, non-Bezout, Hilbert domain of Krull dimension 2, in which every maximal ideal is principal.**

PROOF.The only maximal ideals to be checked are those primes $P$ such that $P \cap D \neq (0)$. But then $P \cap D$ is a principal prime of $D$ and Lemma 1 applies. To see that $D + xL[x]$ is non-Bezout, take $\alpha \in L \setminus K$ and verify that $(x)R \cap (\alpha x)R$ is not principal. To see that $D + xL[x]$ is non-Noetherian, note that for any non-zero, non-unit $d \in D$; $d^n$ divides $x$ for all $n$ and hence the chain of principal ideals $(x)R \subseteq (x/d)R \subseteq \cdots \subseteq (x/d^n)R \subseteq \cdots$ has infinite length.

COROLLARY 7. *If $D$ is a PID such that Spec$(D)$ is infinite and if $K$ is the quotient field of $D$, then $D + K[x]$* **is a two dimensional, non-Noetherian, Bezout, Hilbert domain in which every maximal ideal is principal.**

PROOF.All that is needed is the observation that $D + xK[x]$ is Bezout. But this is proved in [2, Corollary 4.13].

REMARKS 8.

(1) According to [3, Corollary 1.4], $\dim(D + xL[x]) = 1 + \dim D$.

(2) The $D + xL[x]$ construction can be used for constructing Hilbert domains of other descriptions. Let us list a few examples:

(a) Without any change in the proof of Theorem 5 we can show that if $D = K$ is a field, then $K + xL[x]$ is a Hilbert domain of dimension $1 (= \dim K + 1)$.

(b) If $D$ is a Hilbert domain with the $n$-generator property for maximal ideals (that is, every maximal ideal can be generated by at most $n$ elements), then $D + xL[x]$ is again a Hilbert domain with $n$-generator property for maximal ideals.

(3) The $D + xL[x]$ construction can be used recursively to construct an endless chain of unruly Hilbert domains with the Krull dimension increasing by one at each step in the chain. The idea is this: Let $K_0$ be a field, $L_0$ an extension field of $K_0$, and $\{x_i\}$ an infinite collection of variables. Let $L_1$ be a field extension of $L_0(x_1), L_2$ a field extension of $L_1(x_2)$ and so on where $L_n$ is a field extension of $L_{n-1}(x_n)$ for each integer $n \geqq 1$. If $D$ is a Hilbert domain with quotient field $K_0$, we can form the rings $R_n$ where $R_1 = D + x_1L_0[x_1], R_2 = R_1 + x_2L_1[x_2]$ and in general, for $n \geqq 1$, if $R_{n-1}$ is constructed then,

$$R_n = R_{n-1} + x_nL_{n-1}[x_n].$$

The maximal ideals of $R_n$ either come from maximal ideals of $R_{n-1}$ or have the form $(1 + x_nf(x_n))R_n$ where $x_nf(x_n) \in x_nL_{n-1}[x_n]\setminus\{0\}$. Because the new maximal ideals that are created at each step of the process are principal (and stay principal at each successive step), we can start with a Hilbert PID and obain an $n$-dimensional, non-Noetherian Hilbert domain $R_n$ where each maximal ideal of $R_n$ is principal. This $R_n$

will be Bezout if for each $0 < i \leqq n, L_i = L_{i-1}(x_i)$, and $R_n$ will be non-Bezout if at some step $0 < i \leqq n; L_i$ is a proper extension of $L_{i-1}(x_i)$.

(4) Several statements similar to (3) can be made by starting with specal kinds of Hilbert domain, using special field extensions $L_i$, and applying corollaries 6 and 7. For example, we can start with a Hilbert domain with the $n$-generator property for maximal ideals and build a chain of unruly Hilbert domains with the $n$-generator property for maximal ideals. (We thank Budh Nashier for a suggestion amounting to this statement).

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*Department of Mathematics*
*Florida State University*
*Tallahassee, FL 32306*