Canad. Math. Bull. Vol. 19 (2), 1976

HAMILTONIAN CYCLES IN SQUARES OF VERTEX-UNICYCLIC GRAPHS

BY

HERBERT FLEISCHNER AND ARTHUR M. HOBBS(1)

In this paper we determine necessary and sufficient conditions for the square of a vertex–unicyclic graph to be Hamiltonian. The conditions are simple and easily checked. Further, we show that the square of a vertex-unicyclic graph is Hamiltonian if and only if it is vertex-pancyclic.

We use the terminology and notation of [1]. The *square* G^2 of a graph G is the graph with $V(G^2)=V(G)$ in which two vertices are adjacent if and only if their distance in G is one or two. A graph G is *Hamiltonian* if there is a cycle in G, called a *Hamiltonian cycle*, which includes all of the vertices of G. A graph G is *vertex-pancyclic* if, for every vertex v of G, there are cycles of every length from 3 through $|V(G)|$ in G which include v. A *cactus* is a graph in which each edge is in at most one cycle, and a graph is *vertex-unicyclic* if each vertex is in at most one cycle. Every vertex-unicyclic graph is a cactus. For any i, $V_i(G)$ is the set of vertices of G of degree i.

Given a vertex v of a graph G, a *v-fragment* of G is any maximal connected subgraph of G in which v is not a cut vertex. Clearly, if G is connected and if v is not a cut vertex of G, the only v-fragment is G itself, while if G is connected and v is a cut vertex of G, then there are as many v-fragments of G as there are components of $G-v$, and containment specifies a one-to-one relation between the components of $G-v$ and the v-fragments of G. (Note: v-fragments are a specialization of the J-components of Tutte [4]). If v is in a cycle C of G, the $\bar{C}$, v-fragments of G are those v-fragments of G which do not include C.

Let S be a v-fragment of a graph G. We say S is a *short* fragment of G if and only if there is a Hamiltonian path p in S^2-v whose first and last vertices are both adjacent in S to v; we call p a *short path for S*. S is a *long* fragment of G if and only if S is not a short fragment and there is a Hamiltonian path q in S^2-v whose first vertex is adjacent in S to v and whose last vertex is at distance 2 in S from v; we call q a *long path for S*. If a v-fragment is neither

Received by the editors April 16, 1975.

(1) Partial support for this work was received from the National Science Foundation Grant No. GP-43696.

short nor long, we say it is *insufferable*. In [3], the following three theorems were proved:

THEOREM A. *Let G be a connected cactus with cycle $C=x_0, x_1, \ldots, x_n, x_0$. Then G^2 is Hamiltonian if and only if*

(1) *no $\bar{C}, x_i$-fragment of G is insufferable, for each $i \in \{0, 1, \ldots, n\}$;*

(2) *no more than two $\bar{C}, x_i$-fragments of G are long, for each $i \in \{0, 1, \ldots, n\}$; and*

(3) *if two distinct $\bar{C}, x_i$-fragments and two distinct $\bar{C}, x_j$-fragments of G are long, then each non-trivial trail in C beginning on x_i and ending on x_j includes a vertex whose degree in G is 2(x_i and x_j may be the same vertex).*

THEOREM B. *Suppose v-fragment S of a graph G is a tree. Then*

(1) *S is short if and only if $|V(S)|=2$; and*

(2) *S is long if and only if $|V(S)|\geq 3$ and $S-(V_1(G)\cap V(S))$ is a path.*

THEOREM C. *Let G be a graph with exactly one cycle $C=x_0, x_1, \ldots, x_n, x_0$. Suppose G is connected, every vertex x of C meets at most two long $\bar{C}, x$-fragments and no insufferable fragments, and suppose any path in C which joins two vertices, both of which meet two long fragments, includes a vertex whose degree in G is two. Then G^2 is Hamiltonian.*

Using these theorems and a few further definitions, we can now give a characterization of vertex-unicyclic graphs whose squares are Hamiltonan. Given sequences $a=r_1, \ldots, r_i$, $b=s_1, s_2, \ldots, s_j$, and $c=t_1, t_2, \ldots, t_k$, (a), (b), (c) *is the sequence* $r_1, r_2, \ldots, r_i$, $s_1, \ldots, s_j$, $t_1, \ldots, t_k$. We denote the sequence $r_i, r_{i-1}, \ldots, r_2, r_1$ by a^{-1}. Further, we say that a, b, and c are *sections* of the sequence (a), (b), (c). A bridge of a graph is an *endbridge* if it meets a vertex of degree 1.

THEOREM 1. *Let G be a vertex-unicyclic graph with at least three vertices. Then G^2 is Hamiltonian if and only if*

(1) *G is connected;*

(2) *no vertex of G meets more than two non-end bridges of G; and*

(3) *if v is on a cycle C of G and if v meets two non-end bridges of G, and if P is a trail in C from v to any vertex of G meeting two non-end bridges of G (P may join v to v), then P includes a vertex whose degree in G is two.*

Proof. The necessity of (1) is obvious and the necessity of (2) was shown in [2]. Since any x-fragment which includes a non-end bridge of G incident with x is either long or insufferable, the necessity of (3) follows from the third part of Theorem A.

We prove the sufficiency of these conditions by induction. If G has no cycles, then the theorem is a consequence of Theorem B. If G has just one cycle, then

https://doi.org/10.4153/CMB-1976-025-3 Published online by Cambridge University Press

the theorem follows immediately from Theorem C. Suppose the theorem is true for any vertex-unicyclic graph with k cycles and suppose G has $k+1$ cycles, $k+1 \geq 2$. Then G has a bridge x_1x_2 such that each component G_i' of $G-x_1x_2$ containing x_i has a cycle. Form G_i from G_i' by adding a path $P_i = x_i, y_i, z_i$ of length two to G_i' with one end at x_i, such that $\{y_i, z_i\} \cap V(G) = \emptyset$. Since x_1x_2 is a non-end bridge of G, both graphs G_i satisfy the conditions of this theorem if G does, and each has no more than k cycles. Thus G_i^2 contains a Hamiltonian cycle h_i. Since z_i has degree one in G_i there is a vertex w_i of G_i' adjacent to x_i in G such that the sequence w_i, y_i, z_i, x_i is a section of h_i or h_i^{-1} (or of a rotation of one of these). We may suppose $h_i = w_i, y_i, z_i, x_i, (p_i), w_i$ for some sequence p_i. But w_1x_2 and w_2x_1 are edges of G^2. Thus $x_1, (p_1), w_1, x_2, (p_2), w_2, x_1$ is a Hamiltonian cycle in G^2. This proves the theorem.

Because of the relatively simple structure of a vertex-unicyclic graph, Theorem 1 can be improved very easily, as follows:

THEOREM 2. *Let G be a vertex-unicyclic graph. Then G^2 is Hamiltonian if and only if G^2 is vertex-pancyclic.*

Proof. The result is immediate from the definitions if G^2 is vertex-pancyclic. Suppose G^2 is Hamiltonian and suppose v is a vertex of G. Since the theorem is immediate from Theorem B if G is a tree, we may suppose the theorem holds for all vertex-unicyclic graphs smaller than G. By Theorem 1, no vertex of G meets more than two non-end bridges of G, and given any two vertices on a cycle, both of which meet two non-end bridges of G, and a trail joining the vertices, the trail includes a vertex whose degree in G is 2. If G has a vertex x of degree 1 other than v, these properties are also satisfied by $G-x$, which is also connected and vertex-unicyclic. Thus $(G-x)^2$ is vertex-pancyclic, so G^2 has cycles of every length from 3 through $|V(G)|$ which include v.

Suppose G has no vertices of degree one other than (possibly) v. If G is a cycle, we can remove any edge from G to obtain a tree whose square is vertex-pancyclic. Otherwise, G includes a cycle which does not include v. Among all cut vertices x' of G and all x'-fragments F' not including v, choose a cut vertex x and x-fragment F which includes a smallest number of cycles. Because G is vertex-unicyclic, if F includes more than one cycle, it includes a bridge which meets a cut vertex y' for which a y'-fragment contained in F includes fewer cycles than F. Thus F includes just one cycle C. Let y be the only vertex of C which meets an edge not in C. Then y has degree three in G. Let z be a vertex of C adjacent to y. Then $G-z$ is clearly a graph which satisfies the conditions of Theorem 1, includes v, and has fewer vertices than G. Thus $(G-z)^2$ includes cycles of all lengths from 3 through $|V(G-z)|$ and containing v, so G^2 is vertex-pancyclic.

References

1. M. Behzad and G. Chartrand, *Introduction to the Theory of Graphs*, Allyn and Bacon, Inc. (1972).

2. H. Fleischner, *Über Hamiltonsche Linien im Quadrat kubischer und pseudokubischer Graphen*, Mathematische Nachrichten **49** (1971), 163–171.

3. A. M. Hobbs, *Hamiltonian Squares of Cacti*, J. Comb. Theory (to appear).

4. W. T. Tutte, *Connectivity in Graphs*, U. of Toronto Press, (1966).

Institut für Informationsverarbeitung
Wien, Austria

Department of Mathematics
Texas A & M University
College Station, TX77843
U.S.A.