COMPOSITIO MATHEMATICA

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Compositio Math. **147** (2011), 1151–1178.

doi:10.1112/S0010437X10005257

FOUNDATION COMPOSITIO MATHEMATICA

The London Mathematical Society

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Abstract

We prove vanishing of the μ-invariant of the p-adic Katz L-function in N. M. Katz [*p-adic L-functions for CM fields*, Invent. Math. **49** (1978), 199–297].

Contents

1. Introduction

Let F be a totally real number field with discriminant D_F and M be a totally imaginary quadratic extension (a CM field) over F. For a p-adic CM type (M,Σ), Katz [Kat78a] constructed a p-adic Hecke L-function as a p-adic bounded measure φ supported on the ray class group Z of M modulo $\mathfrak{C}p^\infty$ (see also [HT93]). Splitting $Z = \Gamma \times \Delta$ for the maximal finite subgroup Δ and fixing a branch character ψ_0 of Δ, we project the measure φ to its ψ_0-branch φ_{ψ_0} defined on Γ. A main result of this paper is the following.

Theorem I. *Let $p > 2$ be a prime unramified in $F/\mathbb{Q}$. Suppose the following condition.*

(S) *The prime-to-p part of the conductor of the reduction modulo $\mathfrak{m}_W$ of the branch character ψ_0 is a product of primes of M split over the maximal totally real subfield F of M.*

Then the Iwasawa μ-invariant of φ_{ψ_0} vanishes.

Actually, in this paper, we prove a stronger result than the above theorem. In order to state precisely this result, we recall some details about Katz p-adic measure. We fix a rational prime $p > 2$ and take, as the base ring, a finite extension W of the Witt ring $W(\mathbb{F})$ of a fixed algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Write $\mathfrak{m}_W$ for the maximal ideal of W. Fix an algebraic closure $\overline{\mathbb{Q}}_p$ (respectively $\overline{\mathbb{Q}}$) of $\mathbb{Q}_p$ (respectively $\mathbb{Q}$) and write $\mathbb{C}_p$ for the p-adic completion of $\overline{\mathbb{Q}}_p$. We regard W as contained in $\mathbb{C}_p$. We fix two embeddings: $i_\infty : \overline{\mathbb{Q}} \to \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \to \mathbb{C}_p$ and denote by c the complex conjugation induced by i_∞. We suppose the following condition.

(ord) *Every prime factor of p in F splits in M.*

Received 1 May 2009, accepted in final form 7 September 2010, published online 24 March 2011.
2010 Mathematics Subject Classification 11R23, 11F33, 11F41, 11F60, 11F67, 11G18 (primary).
Keywords: Iwasawa invariant, Hecke L-function, Hilbert modular variety, Eisenstein series, Shimura variety.
The author is partly supported by the following NSF grants: DMS 0753991 and DMS 0854949.

Then, we can choose a set of embeddings Σ of M into $\overline{\mathbb{Q}}$ such that the following two conditions hold.

(cm1) The disjoint union $\Sigma \sqcup \Sigma c$ *is the set of all embeddings of* M *into* $\overline{\mathbb{Q}}$.

(cm2) *The* p*-adic place induced by any element of* Σ *composed with* i_p *is distinct from any of those induced by elements in* Σc.

The set Σ satisfying (cm1)–(cm2) is called a p-adic CM-type. Under (ord), we can find a p-adic CM-type, and we fix one such Σ. We write Σ_p for the set of p-adic places (hence of prime ideals of M over p) induced by the embedding $i_p \circ \sigma$ for $\sigma \in \Sigma$. Let $\lambda : M_{\mathbb{A}}^\times / M^\times \to \mathbb{C}^\times$ be a type A_0 Hecke character (of conductor $\mathfrak{C}p^\infty$ for $\mathfrak{C}$ prime to p). Then λ has values in $\overline{\mathbb{Q}}$ on the finite part $M_{\mathbb{A}^{(\infty)}}^\times$ of $M_{\mathbb{A}}^\times$. For the ray-class group Z modulo $\mathfrak{C}p^\infty$ of M, write $\widehat{\lambda} : Z \to \overline{\mathbb{Q}}_p^\times$ for the p-adic avatar of λ. Write O (respectively $\mathfrak{O}$) for the integer ring of F (respectively M). Finally, we choose an element $\delta \in M$ such that the following conditions hold.

(d1) The identity $\delta^c = -\delta$ and $i_\infty(\mathrm{Im}(\delta^\sigma)) > 0$ for all $\sigma \in \Sigma$.

(d2) The alternating form $\langle x, y\rangle = \mathrm{Tr}_{M/F}(xy^c/2\delta)$ induces an isomorphism $\mathfrak{O} \wedge \mathfrak{O} \cong \mathfrak{c}^{-1}\mathfrak{d}^{-1}$ for a fractional F-ideal $\mathfrak{c}$ prime to $p\mathfrak{C}\mathfrak{C}^c$, where $\mathfrak{d}$ is the different of $F/\mathbb{Q}$.

The alternating form in (d2) induces a polarization on the abelian scheme $A(\mathfrak{O})$ defined over $\mathcal{W} = i_p^{-1}(W)$ with complex multiplication by $\mathfrak{O}$ (and $\mathfrak{c}$ is the polarization ideal in (M2)). A choice of Néron differential on $A(\mathfrak{O})_{/\mathcal{W}}$ produces its complex period and p-adic period $(\Omega_\infty, \Omega_p) \in (\mathbb{C}^\times)^\Sigma \times (W^\times)^\Sigma$. Put $\Omega_?^\kappa = \prod_\sigma \Omega_?^{\kappa_\sigma}$ for $? = p, \infty$ and $\pi^\kappa = \pi^{\sum_\sigma \kappa_\sigma}$ for $\sum_{\sigma\in\kappa} \kappa_\sigma \sigma \in \mathbb{Z}[\Sigma]$. Katz constructed in [Kat78a] (see also [HT93] where the case $\mathfrak{C} \neq 1$ is treated) a measure φ on the ray-class group Z modulo $\mathfrak{C}p^\infty$ characterized by the following formula:

$$\frac{\int_Z \widehat{\lambda}\, d\varphi}{\Omega_p^{k\Sigma+2\kappa}} = \frac{c(\lambda)\pi^\kappa E(\lambda) L(0,\lambda)}{\sqrt{|D_F|}\,\mathrm{Im}(\delta)^\kappa \Omega_\infty^{k\Sigma+2\kappa}} \times \prod_{\mathfrak{L}|\mathfrak{C}} (1 - \lambda(\mathfrak{L})) \tag{1.1}$$

for all Hecke characters λ modulo $\mathfrak{C}p^\infty$ primitive at every prime factor of $\mathfrak{C}$ split over F. Here the infinity type of λ is $k\Sigma + \kappa(1-c)$ for an integer k and $\kappa = \sum_{\sigma\in\Sigma} \kappa_\sigma \sigma$ with integers κ_σ satisfying either $k > 0$ and $\kappa_\sigma \geqslant 0$ or $k \leqslant 1$ and $\kappa_\sigma \geqslant 1 - k$, $c(\lambda) \neq 0$ is a simple algebraic constant involving the root number of λ and the value of its Γ-factor, and $E(\lambda)$ is the standard modifying Euler p-factor. We refer to the introduction of [Hid10] for the factors $c(\lambda)$ and $E(\lambda)$.

Let Δ be the maximal torsion subgroup of Z. A character $\psi_0 : \Delta \to W^\times$ is called a *branch* character. We fix a splitting $Z = \Delta \times \Gamma$ for a $\mathbb{Z}_p$-free subgroup Γ so that ψ_0 and any function ϕ on Γ can be considered to be functions on Z via pull-back by the projections: $Z \twoheadrightarrow \Delta$ and $Z \twoheadrightarrow \Gamma$. The ψ_0-branch φ_{ψ_0} of the measure φ is defined on Γ and is given by $\int_\Gamma \phi\, d\varphi_{\psi_0} = \int_Z \psi_0 \phi\, d\varphi$. Let $\widetilde{\psi}_0$ be the Teichmüller lift of ψ_0 mod $\mathfrak{m}_W$. By (S), the support of the conductor of $\widetilde{\psi}_0$ does not contain any inert or ramified prime over F. Taking the branch of $\widetilde{\psi}_0$, by the result of [Hid10], the anticyclotomic μ-invariant with branch character $\widetilde{\psi}_0$ vanishes (and hence the full μ also vanishes) unless the following three conditions are met.

(M1) The quadratic extension M/F is unramified at every finite place.

(M2) The identity $\left(\frac{M/F}{\mathfrak{c}}\right) = -1$ for the quadratic residue symbol $\left(\frac{M/F}{}\right)$.

(M3) The map $\mathfrak{a} \mapsto (\psi_0(\mathfrak{a}) N_{F/\mathbb{Q}}(\mathfrak{a}) \bmod \mathfrak{m}_W)$ is the character $\left(\frac{M/F}{}\right)$ of M/F.

We therefore assume (M1) and (M3) to study the vanishing of the μ-invariant for $\widetilde{\psi}_0$ of the full p-adic Hecke L-function. By (M1) and (M3), $\widetilde{\psi}_0$ has p-power conductor. By the interpolation

https://doi.org/10.1112/S0010437X10005257 Published online by Cambridge University Press

formula (1.1), the μ-invariant of the branch ψ_0 and that of $\widetilde{\psi}_0$ differ by the μ-invariant of the Euler factor at primes appearing in the prime-to-p conductor $\mathfrak{C}$ of ψ_0. Since the Frobenius map for any prime outside p has infinite order over the cyclotomic $\mathbb{Z}_p$-extension, the μ-invariant of the Euler factor vanishes, and we get the vanishing of μ for the original ψ_0 (assuming (S)). So we assume that $\psi_0 = \widetilde{\psi}_0$ and that the prime-to-p conductor of ψ_0 is 1 (i.e., $\mathfrak{C} = 1$) and that M/F is in the Hilbert class field of F. In particular, the p-adic L-function is the one originally constructed in [Kat78a].

Since Γ is isomorphic to Z/Δ, $\mathrm{Gal}(M/F)$ acts on Γ naturally. We write π^- for the projection of Γ onto $\Gamma^- = \Gamma/\Gamma^{\mathrm{Gal}(M/F)}$, on which the generator $c \in \mathrm{Gal}(M/F)$ acts by $-1 : x \mapsto x^c = x^{-1}$. Pick a character ψ of Z with $\widehat{\psi}|_\Delta = \psi_0$, and we write $\varphi_\psi^- = \pi_*^-(\widehat{\psi}\varphi_{\psi_0})$:

$$\int_{\Gamma^-} \phi \, d\varphi_\psi^- = \int_\Gamma (\phi \circ \pi^-)\widehat{\psi}(\gamma) \, d\varphi_{\psi_0}(\gamma).$$

We have $\mu(\varphi_\psi^-) \geqslant \mu(\varphi_{\psi_0})$; so, $0 \leqslant \mu(\varphi_{\psi_0}) \leqslant \liminf_{\psi:\psi|_\Delta=\psi_0} \mu(\varphi_\psi^-)$. Then Theorem I follows from the following result.

THEOREM II. *Suppose that the branch character ψ_0 modulo p has prime-to-p conductor* 1. *Then* $\liminf_\psi \mu(\varphi_\psi^-) = 0$, *where ψ runs over all arithmetic characters of Z with $\widehat{\psi}|_\Delta = \psi_0$.*

The full μ-invariant $\mu(\varphi_{\psi_0})$ is expected to vanish without any condition (cf. [Gil91, Conjecture]), and we proved this under the condition (S). The anticyclotomic μ-invariant $\mu(\varphi_{\psi_0}^-)$ for $\psi_0 : \Delta \to W^\times$ of conductor $\mathfrak{C}$ is positive in some exceptional cases for the following two different reasons.

(i) We have a functional equation $\mu(\varphi_{\psi_0}^-) = \epsilon \cdot \mu(\varphi_{\psi_0^*}^-)$ for the involution sending $\psi(z)$ to $\psi^*(z^{-c})N(z)^{-1}$ for the p-adic norm map N (see [Hid10] above (V)). We never have $\psi_0 = \psi_0^*$ but we could have $\psi_0^* \equiv \psi_0 \bmod \mathfrak{m}_W$. If this happens, $\epsilon \equiv -1 \bmod \mathfrak{m}_W$ forces $\mu(\varphi_{\psi_0}^-) > 0$.

(ii) Let $\overline{\psi}_0 := (\psi_0 \bmod \mathfrak{m}_W)$, and write $\mathfrak{C}(\overline{\psi}_0)$ for the conductor of $\overline{\psi}_0$. Suppose that there exist some primes $\mathfrak{L}|\mathfrak{C}/\mathfrak{C}(\overline{\psi}_0)$ such that $\mathfrak{L} \nmid \mathfrak{C}(\overline{\psi}_0)$ with $N(\mathfrak{L}) \equiv 1 \bmod p$ and that ψ_0 has order divisible by p. For the Teichmüller lift $\widetilde{\psi}_0$ of $\overline{\psi}_0$, φ_{ψ_0} is then congruent modulo $\mathfrak{m}_W$ to the multiple of $\varphi_{\widetilde{\psi}_0}$ by the product of Euler $\mathfrak{L}$-factors $1 - \widetilde{\psi}_0(\mathfrak{L})[\mathfrak{L}]_\Gamma$ over such primes $\mathfrak{L}|\mathfrak{C}$. Here $[\mathfrak{L}]_\Gamma$ is the projection of $\mathfrak{L}$ to Γ. For the full μ-invariant, this does not matter as the Euler factor is prime to p, however it matters for $\mu(\varphi_{\psi_0}^-)$ if Γ^--projection $[\mathfrak{L}]^-$ is trivial (i.e., $\mathfrak{L}$ is ramified or inert in $/F$) and $\widetilde{\psi}_0(\mathfrak{L}) = 1$. Thus in this exceptional case, we have positive $\mu(\varphi_{\psi_0}^-)$.

These exceptional cases should be the only cases where we have $\mu(\varphi_{\psi_0}^-) > 0$ (i.e., we need to modify slightly Conjecture in [Gil91] to include the above first exceptional case which is not mentioned in [Gil91]). We note that the first case is equivalent to (M1)–(M3) (assuming that $\mathfrak{C}$ is a product of primes split over F; see [Hid10, Lemma 5.2]).

We specify in Corollary 5.3 an explicit construction of a subset Ψ of arithmetic characters of Z with $\widehat{\psi}|_\Delta = \psi_0$ such that $\liminf_{\psi\in\Psi} \mu(\varphi_\psi^-) = 0$. Though our idea is the same as the one exploited in [Hid10], the proof in this paper is somehow simpler as we assume that $\mathfrak{C} = 1$ and that M/F is unramified; so, we recall some details of the argument (as we believe that this paper is actually a good introduction to the technical and lengthy article [Hid10]).

https://doi.org/10.1112/S0010437X10005257 Published online by Cambridge University Press

2. Serre–Tate deformation space

We recall, without proofs, deformation theory of ordinary abelian schemes.

2.1 Deformation space of an abelian variety

Let R be a pro-Artinian local ring with residue field $\mathbb{F}$ (so R is canonically a $W(\mathbb{F})$-algebra). Write $\mathrm{CL}_{/R}$ for the category of complete local R-algebras with residue field $\mathbb{F}$. We write $\mathcal{O}_S$ for an object of $\mathrm{CL}_{/R}$ with $S = \mathrm{Spf}(\mathcal{O}_S)$. We fix an *ordinary* abelian variety $A_{0/\mathbb{F}}$. Consider the following deformation functor $\widehat{\mathcal{P}} : \mathrm{CL}_{/R} \to \mathrm{SETS}$:

$$\widehat{\mathcal{P}}_{A_0}(\mathcal{O}_S) = [(A_{/S}, \iota_A) \mid A_{/S} \text{ is an abelian scheme and } \iota_A : A \otimes_{\mathcal{O}_S} \mathbb{F} \cong A_0].$$

Here '[ ]' indicates the set '{ }/$\cong$' of isomorphism classes of the objects inside the straight brackets, and $f : (A, \iota_A)_{/S} \cong (A', \iota_{A'})_{/S}$ if $f : A \to A'$ is an isomorphism of abelian schemes with $\iota_{A'} \circ f_0 = \iota_A$. We write $TA[p^\infty]^{\mathrm{et}}$ for the Tate module of the maximal étale quotient of $A[p^\infty]$ and $A^t_{/R}$ for $\mathrm{Pic}^0_{A/R}$. The functor $\widehat{\mathcal{P}}_{A_0}$ is representable by the formal torus (see [Kat78b] and [Hid10, § 2.3])

$$\mathrm{Hom}_{\mathbb{Z}_p}(TA_0[p^\infty]^{\mathrm{et}} \times TA_0^t[p^\infty]^{\mathrm{et}}, \widehat{\mathbb{G}}_m(S)), \tag{2.1}$$

and each deformation $(A_{/S}, \iota_A) \in \widehat{\mathcal{P}}_{A_0}(\mathcal{O}_S)$ gives rise to the Serre–Tate coordinate

$$t_{A/S} : TA_0[p^\infty]^{\mathrm{et}} \times TA_0^t[p^\infty]^{\mathrm{et}} \to \widehat{\mathbb{G}}_m(S).$$

2.2 Abelian variety with real multiplication

We consider the following fiber category $\mathcal{A}_F$ of abelian schemes $A_{/S}$ over the category of $\mathbb{Z}_{(p)}$-schemes. Here $\mathbb{Z}_{(p)} = \mathbb{Q} \cap \mathbb{Z}_p$ inside $\mathbb{Q}_p$. An object of $\mathcal{A}_F$ is the triple $(A_{/S}, \theta : O \hookrightarrow \mathrm{End}(A_{/S}), \lambda)$, where the following four conditions hold.

(rm1) The map $\theta = \theta_A$ is an embedding of algebras taking identity to identity.

(rm2) The symbol λ is an O-linear symmetric polarization $\lambda : A \to A^t$ with $p \nmid \deg(\lambda)$.

(rm3) The image of θ_A is stable under the Rosati involution induced by λ.

(rm4) As $O \otimes_{\mathbb{Z}} \mathcal{O}_S$-modules, we have $\mathrm{Lie}(A) \cong O \otimes_{\mathbb{Z}} \mathcal{O}_S$ locally under the Zariski topology of S.

We call such a triple (A, θ, λ) satisfying the above four conditions (rm1)–(rm4) an *abelian variety with real multiplication* (abbreviated as AVRM).

A morphism $\phi : (A, \theta, \lambda)_{/S} \to (A', \theta', \lambda')_{/S}$ in the category $\mathcal{A}_F$ is an O-linear morphism $\phi : A_{/S} \to A'_{/S}$ of abelian schemes over S with $\lambda = \phi^t \circ \lambda' \circ \phi$. See [Hid04, § 4.1.1] for technical details of $\mathcal{A}_F$.

Take an ordinary abelian scheme $(A_0, \theta_0, \lambda_0)$ defined over $\mathbb{F}$. We fix a polarization $\lambda_0 : A_0 \to A_0^t$ of degree prime to p. We consider the following subfunctor of $\widehat{\mathcal{P}}_{A_0}$ defined from $\mathrm{CL}_{/W(\mathbb{F})}$ into SETS:

$$\widehat{\mathcal{P}}_{A_0,\theta_0,\lambda_0}(R) = [(A_{/R}, \iota_A, \theta, \lambda) \in \mathcal{A}_F \mid (A, \iota_A) \in \widehat{\mathcal{P}}_{A_0}(R), \lambda \text{ and } \theta \text{ induce } \lambda_0 \text{ and } \theta_0].$$

Here we call $f : (A, \lambda_A, \iota_A) \to (B, \lambda_B, \iota_B)$ an isomorphism if $f : (A, \iota_A) \cong (B, \iota_B)$ and $f^t \circ \lambda_B \circ f = \lambda_A$. We identify $TA_0[p^\infty]^{\mathrm{et}}$ and $TA_0^t[p^\infty]^{\mathrm{et}}$ by λ_0. Then by (2.1) (cf. [Hid10, § 2.4]), for $O_p = O \otimes_{\mathbb{Z}} \mathbb{Z}_p$,

$$\widehat{\mathcal{P}}_{A_0,\theta_0,\lambda_0}(R) \cong \mathrm{Hom}_{\mathbb{Z}_p}(TA_0[p^\infty]^{\mathrm{et}} \otimes_{O_p} TA_0[p^\infty]^{\mathrm{et}}, \widehat{\mathbb{G}}_m(R)).$$

https://doi.org/10.1112/S0010437X10005257 Published online by Cambridge University Press

Since we have $TA_0[p^\infty]^{\mathrm{et}} \cong O_p$ as O-modules [Hid04, Proposition 4.1], we get the following proposition.

PROPOSITION 2.1. *Suppose that O is unramified at p. Let $S = \mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} = \mathrm{Spec}(\mathbb{Z}[O])$ for the group algebra $\mathbb{Z}[O]$. Then identifying $TA_0[p^\infty]^{\mathrm{et}}$ with O_p, the functor $\widehat{\mathcal{P}}_{A_0,\theta,\lambda_0}$ is represented by the formal scheme $\widehat{S}_{/W}$, where $\widehat{S}$ is the formal completion of S along the identity section of $\mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}(\mathbb{F})$.*

3. Hilbert modular Shimura varieties

Let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}(2))$. We recall reciprocity laws for the Hilbert modular Shimura variety described in [Hid10, § 3] to the extent we need without much proof. We write $h_{\mathbf{0}} : \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G_{/\mathbb{R}}$ for the homomorphism of real algebraic groups sending $a + b\sqrt{-1}$ to $\left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$. We write $\mathfrak{X}$ for the conjugacy class of $h_{\mathbf{0}}$ under $G(\mathbb{R})$. The group $G(\mathbb{R})$ acts on $\mathfrak{X}$ from the left by conjugation. The identity connected component $\mathfrak{X}^+$ containing $\mathbf{0} = h_{\mathbf{0}}$ is isomorphic to the product $\mathfrak{Z} = \mathfrak{H}^I$ of copies of the upper half complex plane $\mathfrak{H}$ indexed by embeddings I of F into $\mathbb{R}$ by $g(\mathbf{0}) \mapsto g(\mathbf{i})$ for $\mathbf{i} = (\sqrt{-1}, \ldots, \sqrt{-1})$. Thus $\mathfrak{X}$ is a finite union of the Hermitian symmetric domain isomorphic to $\mathfrak{Z}$; indeed, $\mathfrak{X} \cong (\mathbb{C} - \mathbb{R})^I$ (which we identify). The pair $(G, \mathfrak{X})$ satisfies Deligne's axiom for Shimura varieties in [Del79, 2.1.1]. The $\mathbb{C}$-points of the Shimura variety with right $G(\mathbb{A}^{(\infty)})$-action is given by

$$\mathrm{Sh}(G, \mathfrak{X})(\mathbb{C}) = \varprojlim_K G(\mathbb{Q})\backslash(\mathfrak{X} \times G(\mathbb{A}^{(\infty)}))/K = G(\mathbb{Q})\backslash(\mathfrak{X} \times G(\mathbb{A}^{(\infty)}))/\overline{Z(\mathbb{Q})}, \tag{3.1}$$

where $(\gamma, u) \in G(\mathbb{Q}) \times K$ acts on $(z, g) \in \mathfrak{X} \times G(\mathbb{A}^{(\infty)})$ by $\gamma(z, g)u = (\gamma(z), \gamma g u)$, $\overline{Z(\mathbb{Q})}$ is the closure of the center $Z(\mathbb{Q})$ in $G(\mathbb{A}^{(\infty)})$. We write $[z, g]$ for the point of $\mathrm{Sh}(\mathbb{C})$ given by $(z, g) \in \mathfrak{X} \times G(\mathbb{A}^{(\infty)})$. This pro-algebraic variety has a canonical model $\mathrm{Sh}(G, \mathfrak{X})$ defined over $\mathbb{Q}$, as we recall in this section.

3.1 Abelian varieties up to isogenies

Let F^2 be a column vector space, with its (finite) adelization $F^2_{\mathbb{A}^{(\infty)}}$. The group $G(\mathbb{A}^{(\infty)})$ acts on $F^2_{\mathbb{A}^{(\infty)}}$ by matrix multiplication. We consider the fibered category $\mathcal{A}_F^{\mathbb{Q}}$ over $\mathbb{Q}$-SCH defined by

(Object) abelian schemes with (rm1)–(rm4); (Morphism) $\mathrm{Hom}_F^{\mathbb{Q}}(A, A') = \mathrm{Hom}_O(A, A') \otimes_{\mathbb{Z}} \mathbb{Q}$.

For an object $A_{/S}$, we take a geometric point $s \in S$, consider the Tate module $T(A) = T_s(A) = \varprojlim_N A[N](k(s))$, and define $V(A) = V_s(A) = T(A) \otimes_{\widehat{\mathbb{Z}}} \mathbb{A}^{(\infty)}$. The module $V(A)$ is an $F_{\mathbb{A}^{(\infty)}}$-free module of rank two and has an $\widehat{O}$-stable lattice $T(A)$, where $\widehat{O} = O \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}} = \prod_{\ell:\mathrm{prime}} O_\ell$.

Picking a geometric point s in each connected component of S, a full level structure on A is an isomorphism $\eta : F^2_{\mathbb{A}^{(\infty)}} \cong V_s(A)$ of $F_{\mathbb{A}^{(\infty)}}$-modules. For a closed subgroup $K \subset G(\mathbb{A}^{(\infty)})$, a level K-structure is the (sheaf-theoretic) K-orbit $\overline{\eta} = \eta K$ of η for the right action $\eta \mapsto \eta \circ u$ ($u \in K$). Since $A[N]_{/S}$ is an étale finite group scheme, the algebraic fundamental group $\pi_1(S, s)$ with base point s acts on $A[N](k(s))$ for any integer N and hence on the full Tate module $V_s(A) = \varprojlim_N A[N](k(s)) \otimes \mathbb{Q}$. The level K-structure is defined over S if $\sigma \circ \overline{\eta} = \overline{\eta}$ for each $\sigma \in \pi_1(S, s)$. Polarizations $\lambda, \lambda' : A \to A^t$ are equivalent (written as $\lambda \sim \lambda'$) if $\lambda = a\lambda' = \lambda' \circ a$ for a totally positive $a \in F$. The equivalence class of a polarization λ defined over S is written as $\overline{\lambda}$.

For an open compact subgroup K, we consider the following functor from $\mathrm{SCH}_{/\mathbb{Q}}$ into SETS,

$$\mathcal{P}_K^{\mathbb{Q}}(S) = [(A, \overline{\lambda}, \overline{\eta})_{/S} \text{ with (rm1)–(rm4)}],$$

where $\overline{\eta}$ is a level K-structure as defined above, and $[\ ]=\{\ \}/\cong$ indicates the set of isomorphism classes in $\mathcal{A}_F^{\mathbb{Q}}$ of the objects defined over S in the brackets. For a compact subgroup K, $\mathcal{P}_K^{\mathbb{Q}}(S)$ is defined by the natural projective limit $\varprojlim_U \mathcal{P}_U^{\mathbb{Q}}(S)$ for U running over open compact subgroups containing K. An F-linear morphism $\phi \in \mathrm{Hom}_F^{\mathbb{Q}}(A, A')$ is an isomorphism between triples $(A, \overline{\lambda}, \overline{\eta})_{/S}$ and $(A', \overline{\lambda}', \overline{\eta}')_{/S}$ if it is compatible with all data; that is, $\phi \circ \overline{\eta} = \overline{\eta}'$ and $\phi^t \circ \overline{\lambda} = \overline{\lambda}' \circ \phi$.

Equip F^2 with an alternating form $\Lambda : F^2 \wedge_F F^2 \cong F$ given by $(x, y) = {}^t x J_1 y$ for $J_1 = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We define a $\mathbb{Q}$-alternating pairing $\langle \cdot, \cdot \rangle : F^2 \times F^2 \to \mathbb{Q}$ by $\mathrm{Tr}_{F/\mathbb{Q}} \circ \Lambda$. Suppose that the point $s \in S$ is a complex point $s \in S(\mathbb{C})$; so, we have the Betti homology group $H_1(A, \mathbb{Q}) := H_1(A(k(s)), \mathbb{Q})$. Then the polarization $\lambda : A \to A^t$ induces a nondegenerate alternating pairing $E_\lambda : \bigwedge^2 H_1(A, \mathbb{Q}) \to \mathbb{Q}$ (the Riemann form; see [Mum94, §§ 1 and 20]) with $E_\lambda(\alpha x, y) = E_\lambda(x, \alpha y)$ for all $\alpha \in F$. We write $e_\lambda : H_1(A, \mathbb{Q}) \wedge_F H_1(A, \mathbb{Q}) \cong F$ for a unique alternating form satisfying $\mathrm{Tr}_{F/\mathbb{Q}} \circ e_\lambda = E_\lambda$. The Hodge decomposition: $H^1(A, \mathbb{C}) = H^0(A(k(s)), \Omega^{\mathrm{an}}_{A/\mathbb{C}}) \oplus H^0(A(k(s)), \overline{\Omega}^{\mathrm{an}}_{A/\mathbb{C}})$ induces, by Poincaré duality, an embedding $h = h_A : \mathbb{C}^\times = \mathbb{S}(\mathbb{R}) \to \mathrm{Aut}_F(H_1(A, \mathbb{R})) \cong G(\mathbb{R})$ such that the following two conditions hold.

(i) The identity $h(z)\omega = z\omega$ for all $\omega \in \mathrm{Hom}_{\mathbb{C}}(H^0(A(k(s)), \Omega_{A/\mathbb{C}}), \mathbb{C})$ (and $h(z)\overline{\omega} = \overline{z\omega}$).

(ii) The pairing $E_\lambda(x, h(\sqrt{-1})y)$ is a positive definite Hermitian form on $H_1(A, \mathbb{R})$ ($\cong F^2_{\mathbb{R}} := F^2 \otimes_{\mathbb{Q}} \mathbb{R}$) under the complex structure given by h.

In this way, an abelian variety $A_{/\mathbb{C}}$ gives rise to $h_A \in \mathfrak{X}$. Starting from an $h_z \in \mathfrak{X} = (\mathbb{C} - \mathbb{R})^I$, via the multiplication by $h_z(z)$ on $F^2_{\mathbb{R}} = F^2 \otimes_{\mathbb{Q}} \mathbb{R}$, $F^2_{\mathbb{R}}$ can be considered as a complex vector space. By the theory of abelian variety over $\mathbb{C}$, for a lattice $L \subset F^2_{\mathbb{R}}$, the complex torus $F^2_{\mathbb{R}}/L$ gives rise to an abelian variety $A_{z/\mathbb{C}}$ with real multiplication by O such that $h_A = h_z$ and $A_z(\mathbb{C}) = F^2_{\mathbb{R}}/L$. Since $T(A) \cong \widehat{L} = L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}$ canonically, we have $\eta_0 : F^2_{\mathbb{A}^{(\infty)}} = V(A_z)$, and choosing a basis $\{v_1, v_2\}$ of $F^2_{\mathbb{A}^{(\infty)}}$ (over $F^{(\infty)}_{\mathbb{A}}$) is equivalent to a choice of $g \in G(\mathbb{A}^{(\infty)})$ bringing the standard basis to $\{v_1, v_2\}$. This gives rise to a full level structure $\eta = \eta_0 \circ g$ of A_z. In this way, (A_z, λ, η) with the polarization λ induced by Λ gives rise to a point $[z, g] \in \mathrm{Sh}(G, \mathfrak{X})(\mathbb{C})$, and we get an identification $\mathrm{Sh}(G, \mathfrak{X})(\mathbb{C}) \cong \mathcal{P}^{\mathbb{Q}}_{\mathbf{1}}(\mathbb{C})$ for the trivial subgroup $\mathbf{1} = \{1\} \subset G(\mathbb{A}^{(\infty)})$. This identification is actually valid not just for $S = \mathrm{Spec}(\mathbb{C})$ but actually for all $\mathbb{Q}$-schemes S. In other words, from [Shi66] and [Del71, 4.16–21] (see also [Hid04, § 4.2] and [Hid10, § 3]), we get the following theorem.

Theorem 3.1. *The canonical model* $\mathrm{Sh}(G, \mathfrak{X})_{/\mathbb{Q}}$ *represents the functor* $\mathcal{P}^{\mathbb{Q}}_{\mathbf{1}}$ *over* $\mathbb{Q}$ *for the trivial subgroup* $\mathbf{1}$ *made of the identity element of* $G(\mathbb{A}^{(\infty)})$.

Through the action of $G(\mathbb{A}^{(\infty)})$ on $F^2_{\mathbb{A}^{(\infty)}}$, $g \in G(\mathbb{A}^{(\infty)})$ acts on the level structure by $\eta \mapsto \eta \circ g$ and hence on the variety $\mathrm{Sh}(G, \mathfrak{X})$ from the right. If K is open and sufficiently small, the functor $\mathcal{P}^{\mathbb{Q}}_K$ is represented by the quotient $\mathrm{Sh}_K(G, \mathfrak{X}) := (\mathrm{Sh}(G, \mathfrak{X})/K)_{/\mathbb{Q}}$. In the complex uniformization, each point $[z, g]$ corresponds to the test triple $(A_z, \lambda_z, \eta_z \circ g)$, where $A_z(\mathbb{C}) = \mathbb{C}^I/(\mathfrak{d}^{-1} + Oz)$ and $\eta_z\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = bz - a$ identifying $T(A_z) = \widehat{\mathfrak{d}}^{-1} + \widehat{O}z$.

A key point of the proof of the representability (assuming that K is open-compact) is reducing it to the representability of a functor classifying abelian schemes up to isomorphisms not up to isogenies. Let $L \subset F^2$ be an O-lattice. We define the polarization ideal $\mathfrak{c}$ by $\mathfrak{c}^* = \Lambda(L \wedge L) \subset F$, where $\mathfrak{c}^*$ is the dual ideal $\{\xi \in F \mid \mathrm{Tr}(\xi\mathfrak{c}) \subset \mathbb{Z}\} = \mathfrak{c}^{-1}\mathfrak{d}^{-1}$. Let $\mathrm{Cl}^+(K) = F^\times_{\mathbb{A}^{(\infty)}}/\det(K)F^\times_+$, which is a finite group. We fix a complete representative set $\{c \in F^\times_{\mathbb{A}^{(\infty)}}\}$ for $\mathrm{Cl}^+(K)$ so that $c\widehat{O} \cap F = \mathfrak{c}$. We may choose L to be one of O-lattices $L_{\mathfrak{c}} = \mathfrak{c}^* \oplus O \subset F^2$ (indexed by $\mathfrak{c} \in \mathrm{Cl}^+(K)$) with $\Lambda(L_{\mathfrak{c}} \wedge L_{\mathfrak{c}}) = \mathfrak{c}^*$, and put $L = L_O$. Note that $L = L_{\mathfrak{c}} \cdot \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)$ in F^2. For each isogeny class of $(A,\overline{\lambda},\overline{\eta})_{/S}\in\mathcal{P}_K^{\mathbb{Q}}(S)$, we can functorially find a unique triple $(A',\lambda',\overline{\eta}')_{/S}$ and a polarization ideal $\mathfrak{c}$ such that $\eta'(\widehat{L}_{\mathfrak{c}})=T(A)$. See [Hid04, pp. 135–136] for the details of this process of finding a unique triple $(A',\lambda',\overline{\eta}')_{/S}$ in the isogeny class of $(A,\overline{\lambda},\overline{\eta})_{/S}$. If two such choices are isogenous, the isogeny between them has to be an isomorphism keeping the polarization. Thus we get an isomorphism of functors:

$$\mathcal{P}_K^{\mathbb{Q}}(S)\cong\mathcal{P}'_K(S):=\bigsqcup_{\mathfrak{c}\in\mathrm{Cl}^+(K)}\mathcal{P}'_{K,\mathfrak{c}}(S),\tag{3.2}$$

where $\mathfrak{c}$ runs over the ideal classes in $\mathrm{Cl}^+(K)=F_{\mathbb{A}}^{(\infty)\times}/F_+^{\times}\det(K)$, and

$$\mathcal{P}'_{K,\mathfrak{c}}(S)=\{(A',\lambda',\overline{\eta}')_{/S}\text{ with (rm1)–(rm4)}\mid\eta'(\widehat{L}_{\mathfrak{c}})=T(A')\text{ and }\mathfrak{c}(\lambda')=\mathfrak{c}\}/\cong.$$

Here $\cong$ means an isomorphism (not an isogeny) for a chosen polarization integral over the fixed lattice $L_{\mathfrak{c}}$ in the class of $\overline{\lambda}$ (in other words, λ induces a fixed alternating form on the space F^2 integral over $L_{\mathfrak{c}}$ up to units in $F\cap\det(K)$). If $K=G(\mathbb{Z}_p)\times K^{(p)}$ for an open compact subgroup $K^{(p)}$ of $G(\mathbb{A}^{(p\infty)})$, the above functor is well defined over $\mathcal{W}$-schemes S. This functor $\mathcal{P}'_K$ is proven (for example, by geometric invariant theory of Mumford) to be represented by a quasi-projective scheme, whose geometrically connected components $\mathfrak{M}(\mathfrak{c},K)$ representing $\mathcal{P}'_{K,\mathfrak{c}}$ are shown by Shimura's reciprocity law (Theorem 3.3) to be defined over a specific abelian extension k_K of $\mathbb{Q}$ dependent on K. If $K=G(\mathbb{Z}_p)\times K^{(p)}$ (with $K^{(p)}=\{x\in K\mid x_p=1\}$), $\mathfrak{M}(\mathfrak{c},K)$ is a geometrically connected scheme over $k_K\cap\mathcal{W}$. Hence $\pi_0(\mathrm{Sh}_K)\cong\mathrm{Cl}^+(K)$ canonically. See [Hid04, § 4.2] for details.

Now we recall the canonical p-integral model of the Shimura variety. We use the following variant (due to Kottwitz [Kot92]) of the functor $\mathcal{P}_K^{\mathbb{Q}}$. We fix a rational prime p unramified in $F/\mathbb{Q}$. This concerns an open-compact subgroup K *maximal at* p (i.e., $K=G(\mathbb{Z}_p)\times K^{(p)}$). We consider the following fibered category $\mathcal{A}_F^{(p)}$ over $\mathbb{Z}_{(p)}$-schemes:

(Object) abelian schemes with (rm1)–(rm4); (Morphism) $\mathrm{Hom}_{\mathcal{A}_F^{(p)}}(A,A')=\mathrm{Hom}_{\mathcal{A}_F}(A,A')\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$

for $\mathbb{Z}_{(p)}=\mathbb{Q}\cap\mathbb{Z}_p$. This means that to classify test objects, we now allow only isogenies with degree prime to p (i.e., 'prime-to-p isogenies'), and the degree of the polarization λ is supposed to be also prime to p. Polarizations are equivalent if $\lambda=a\lambda'=\lambda'\circ a$ for a totally positive $a\in F$ prime to p.

Fix an O-lattice $L\subset F^2$ with $\Lambda(L\wedge L)=\mathfrak{c}^*$, and assume self O_p-duality of $L_p=L\otimes_{\mathbb{Z}}\mathbb{Z}_p$ under the alternating pairing $\Lambda: F^2\wedge F^2\cong F$. Consider test objects $(A,\overline{\lambda},\overline{\eta}^{(p)})_{/S}$ with λ degree prime to p. Here $\eta^{(p)}:F^2_{\mathbb{A}^{(p\infty)}}\cong V^{(p)}(A)=T(A)\otimes_{\mathbb{Z}}\mathbb{A}^{(p\infty)}$ and $\lambda\in\overline{\lambda}$ are supposed to satisfy the following requirement: $V^{(p)}(A)\wedge V^{(p)}(A)\xrightarrow{e_\lambda}F_{\mathbb{A}}^{(p\infty)}$ is proportional to $\Lambda: F^2\wedge F^2\cong F$ up to scalars in $(F\otimes\mathbb{A}^{(p\infty)})^{\times}$. We write the $K^{(p)}$-orbit of $\eta^{(p)}$ as $\overline{\eta}^{(p)}$. Consider the following functor from $\mathbb{Z}_{(p)}$-schemes into SETS.

$$\mathcal{P}_K^{(p)}(S)=[(A,\overline{\lambda},\overline{\eta}^{(p)})_{/S}\text{ with (rm1)–(rm4)}].\tag{3.3}$$

We quote a result of Kottwitz [Kot92] from [Hid04, § 4.2.1] and [Hid10, § 3.1].

THEOREM 3.2. *The p-integral smooth canonical model $\mathrm{Sh}^{(p)}(G,\mathfrak{X})_{/\mathbb{Z}_{(p)}}$ over $\mathbb{Z}_{(p)}$ represents the functor $\mathcal{P}_{\mathbf{1}}^{(p)}$, and we have a canonical isomorphism:* $\mathrm{Sh}^{(p)}(G,\mathfrak{X})_{/\mathbb{Z}_{(p)}}\times_{\mathbb{Z}_{(p)}}\mathbb{Q}\cong\mathrm{Sh}(G,\mathfrak{X})/G(\mathbb{Z}_p)_{/\mathbb{Q}}$.

The continuous right action of $g \in G(\mathbb{A})$ on $\mathrm{Sh}(G,\mathfrak{X})_{/\mathbb{Q}}$ given by

$$(A,\overline{\lambda},\theta,\eta) \mapsto (A,\overline{\lambda},\theta,\eta \circ g^{(\infty)}) \tag{3.4}$$

is identical to the right multiplication by g on $\mathrm{Sh}(G,\mathfrak{X})(\mathbb{C})$ over $\mathbb{C}$. Since multiplication by $\xi \in F^\times$ gives a self isogeny on A, the center $Z(\mathbb{Q}) \subset G(\mathbb{Q})$ acts trivially. By definition, the action factors through $G(\mathbb{A}^{(\infty)}) = G(\mathbb{A})/G(\mathbb{R})$; so, the action factors through $G(\mathbb{A})/\overline{Z(\mathbb{Q})G(\mathbb{R})}$. Define

$$\mathcal{G} = \mathcal{G}(G,\mathfrak{X}) = \{g \in G(\mathbb{A}) \mid \det(g) \in \mathbb{A}^\times \overline{F^\times F^\times_{\infty+}}/\overline{F^\times F^\times_{\infty+}}\}$$

and

$$\overline{\mathcal{E}}(G,\mathfrak{X}) = \mathcal{G}(G,\mathfrak{X})/\overline{Z(\mathbb{Q})G(\mathbb{R})_+}.$$

Here $F^\times_{\infty+}$ is the subgroup of totally positive elements in $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. By (3.1) and (3.2), we have $\pi_0(\mathrm{Sh}(G,\mathfrak{X})(\mathbb{C})) \cong F^\times_{\mathbb{A}^{(\infty)}}/\overline{F^\times_+} \cong F^\times_{\mathbb{A}}/\overline{F^\times F^\times_{\infty+}} = \varprojlim_K \mathrm{Cl}^+(K)$. The action of $g \in G(\mathbb{A})$ permutes transitively connected components of $\mathrm{Sh}(G,\mathfrak{X})(\mathbb{C})$. See [Hid04, Theorem 4.14] for the following fact.

THEOREM 3.3 (Shimura). *The stabilizer in $G(\mathbb{A})/\overline{Z(\mathbb{Q})G(\mathbb{R})_+}$ of each geometrically irreducible component of $\mathrm{Sh}(G,\mathfrak{X})$ is given by $\overline{\mathcal{E}}(G,\mathfrak{X})$.*

When we regard $g \in \overline{\mathcal{E}}(G,\mathfrak{X})$ as an automorphism of $\mathcal{O}_{\mathrm{Sh}}$ or $\mathrm{Sh}(G,\mathfrak{X})_{/\mathbb{Q}}$, we write it as $\tau(g)$.

3.2 CM points

A point $x = [z,g] \in \mathrm{Sh}(G,\mathfrak{X})(\mathbb{C})$ is called a *CM point* if $z = (z_\sigma)_{\sigma \in I} \in \mathfrak{X} = (\mathbb{C} - \mathbb{R})^I \subset F \otimes_{\mathbb{Q}} \mathbb{C}$ generates a totally imaginary quadratic extension $M_x = F[z] \subset F \otimes_{\mathbb{Q}} \mathbb{C}$ of F (a CM field over F). Set $L = \mathfrak{d}^{-1} + Oz \subset M \subset F \otimes_{\mathbb{Q}} \mathbb{C}$. We write $\mathfrak{O} = \mathfrak{O}_x$ for the integer ring of M_x and $O_x = \{\alpha \in \mathfrak{O}_x \mid \alpha L \subset L\}$ (the order of L). Assume p is unramified in $M/\mathbb{Q}$, $L \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathfrak{O}_x \otimes_{\mathbb{Z}} \mathbb{Z}_p$ and $O_x \otimes_{\mathbb{Z}} \mathbb{Z}_p = \mathfrak{O}_x \otimes_{\mathbb{Z}} \mathbb{Z}_p$. Let $T_x = T_z$ be the torus $\mathrm{Res}_{\mathfrak{O}_{(p)}/\mathbb{Z}_{(p)}}\mathbb{G}_m$. The regular representation $\rho_z : T_x(\mathbb{Q}) = M_x^\times \to G(\mathbb{Q})$ given by $\binom{\alpha z}{\alpha} = \rho_z(\alpha)\binom{z}{1}$ gives rise to a representation $T_{x/\mathbb{Z}_{(p)}} \to G_{/\mathbb{Z}_{(p)}} = \mathrm{Res}_{O_{(p)}/\mathbb{Z}_{(p)}}\mathrm{GL}(2)$ because $(1,z)$ gives rise to a basis of $L \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. Since $\mathbb{A}^{(\infty)}$ is a $\mathbb{Z}_{(p)}$-algebra by the diagonal embedding, we may regard ρ_z as a representation $\widehat{\rho}_z : T_z \to G$ defined over $\mathbb{A}^{(\infty)}$. Now conjugating by g, we get $\widehat{\rho}_x : T_{x/\mathbb{A}^{(\infty)}} \to G_{/\mathbb{A}^{(\infty)}}$ defined over $\mathbb{A}^{(\infty)}$ given by $\widehat{\rho}_x(\alpha) = g^{-1}\widehat{\rho}_z(\alpha)g$. The abelian variety A_x has complex multiplication by $\mathfrak{O}_x$; that is, under the action of $T_x(\mathbb{Q})$ via $\widehat{\rho}_x$, $\widehat{L} \cdot g \cap F^2$ is identified with a fractional ideal of M_x prime to p. On the other hand, the level structure $\eta_x = \eta_z \circ g$ identifies $T(A_x)$ with $\widehat{L} \cdot g = \widehat{L}_{\mathfrak{c}}$ for a polarization ideal $\mathfrak{c}$ prime to p. Plainly $\widehat{\rho}_x(T_x(\mathbb{Q}))$ falls in $\mathcal{G}$ and hence fixes each geometrically connected component of Sh. Since $\alpha \in M^\times = T_x(\mathbb{Q})$ is a self isogeny $\alpha : A_x \to A_x$ with $\alpha \circ \eta_x = \eta_x \circ \widehat{\rho}_x(\alpha)$, $\tau(\widehat{\rho}_x(\alpha))$ fixes x.

We let $G(\mathbb{Q})$ act on the column vector space F^2 through the matrix multiplication. The action of T_x via ρ_z on F^2 makes F^2 a vector space over M_x of dimension one. Then the subspace V_x of $F^2 \otimes_{\mathbb{Q}} \mathbb{C}$ on which h_z acts by its restriction $\mu_x = h_z|_{\mathbb{G}_m \times 1}$ is preserved by multiplication by M_x, yielding an isomorphism class Σ_x of representations of M_x. Since the isomorphism class Σ_x is determined by its diagonal entries $\sigma_i : M_x \hookrightarrow \mathbb{C}$, we may identify Σ_x with a formal sum $\sum_i \sigma_i$. Since $\mu_x \times \overline{\mu}_x = h_z$, we find that $\{\sigma_i, c\sigma_i\}_{i=1,\dots,d}$ ($d = [F:\mathbb{Q}]$) is the total set I_x of complex embeddings of M_x into $\mathbb{C}$. The fiber $A = A_x$ at $x \in \mathrm{Sh}(\mathbb{C})$ of the universal abelian scheme over Sh has complex multiplication by M_x with CM type (M_x, Σ_x).

3.3 Reciprocity law for deformation spaces

We start with a fixed CM point $x=[z,g]$ and the associated CM abelian variety $(A_x,\overline{\lambda},\theta,\eta)$. We suppose that $\theta:O\hookrightarrow \mathrm{End}(A_x)$ extends to $\theta:\mathfrak{O}\hookrightarrow \mathrm{End}(A_x)$ for the integer ring $\mathfrak{O}$ of M. Write (M,Σ) for (M_x,Σ_x) and follow the convention in the introduction. Diagonalizing the action of M on $\mathrm{Lie}(A_x)_{/W}$, we may assume that $\sigma\in\Sigma$ embeds $\mathfrak{O}$ into W. Consider the reduction A_0 of $A_{x/W}$ modulo $\mathfrak{m}_W$. Suppose that A_0 is ordinary (i.e., we suppose the conditions (cm1)–(cm2) in the introduction). We pick a base of $M_{\mathbb{A}^{(\infty)}}$ over $F_{\mathbb{A}^{(\infty)}}$ and identify $M_{\mathbb{A}^{(\infty)}}$ with $F^2_{\mathbb{A}^{(\infty)}}$ so that the fixed lattice in the definition of $\mathcal{P}^{(p)}_K$ is a fractional ideal of M. If $x=[z,g]$, the choice of g is tantamount to the choice of the base of $M_{\mathbb{A}^{(\infty)}}$ over $F_{\mathbb{A}^{(\infty)}}$. Then the polarization λ induces an alternating pairing $(\alpha,\beta)=\mathrm{Tr}_{M/F}(\delta\alpha c(\beta)):L\times L\to F$ for the unique nontrivial automorphism c of M/F. Here $\delta\in M$ is as in (d1)–(d2) of the introduction. We have the polarization ideal $\mathfrak{c}_x$ given by $\mathfrak{c}_x^*=(L,L)$. We then have $A_x(\mathbb{C})=L\backslash(M\otimes_{\mathbb{Q}}\mathbb{R})$ for a fractional ideal $L\subset M$ with $L_p=\mathfrak{O}_p$ (identifying $M\otimes_{\mathbb{Q}}\mathbb{R}$ with $\mathbb{C}^\Sigma$ through $a\otimes t\mapsto(\sigma(a)t)_{\sigma\in\Sigma}$). This induces $\eta^{(p)}=\eta_z^{(p)}\circ g^{(p)}:M\otimes_{\mathbb{Q}}\mathbb{A}^{(p\infty)}\cong V^{(p)}(A_x)$. By reduction modulo $\mathfrak{m}_W$, $\eta^{(p)}$ induces a prime-to-p level structure $\eta_0^{(p)}$ on A_0. Let $(A,\iota_A,\lambda)_{/R}$ be any deformation of $(A_0,\iota_0,\lambda_0)_{/\mathbb{F}}$ over a local Artinian W-algebra R. Since $A[N]$ for N prime to p is étale over $\mathrm{Spec}(R)$, the level structure $\eta_0^{(p)}$ at the special fiber extends uniquely to a level structure $\eta_A^{(p)}$ on $A_{/R}$. Therefore, for the deformation functor

$$\widehat{\mathcal{P}}(R)=[(A,\iota_A,\theta,\lambda,\eta_A^{(p)})_{/R}\mid(A,\iota_A,\theta,\lambda,\eta_A^{(p)})\bmod\mathfrak{m}_R=(A_0,\iota_0,\theta_0,\lambda_0,\eta_0^{(p)})],$$

the forgetful morphism $(A,\iota_A,\theta,\lambda,\eta_A^{(p)})_{/R}\mapsto(A,\iota_A,\theta,\lambda)_{/R}$ of $\widehat{\mathcal{P}}$ into the original deformation functor $\widehat{\mathcal{P}}_{A_0,i_0,\lambda_0}$ induces an isomorphism of functors; so, $\widehat{\mathcal{P}}$ is represented by $\widehat{S}$ in Proposition 2.1.

We take the Kottwitz model $\mathrm{Sh}^{(p)}(G,\mathfrak{X})_{/W}$ representing (3.3) over W and consider $x=[z,g]$ as a point of $\mathrm{Sh}^{(p)}(G,\mathfrak{X})(W)$. We have the universal abelian scheme $\mathbf{A}\to\mathrm{Sh}^{(p)}$. Let

$$\mathrm{Sh}^{\mathrm{ord}}=\mathrm{Sh}^{\mathrm{ord}}(G,\mathfrak{X})=\mathrm{Sh}^{(p)}(G,\mathfrak{X})\left[\frac{1}{E}\right],$$

that is, we invert over $\mathrm{Sh}^{(p)}$ a lift E of a power of the Hasse invariant H. The formal completion $\mathrm{Sh}^{\mathrm{ord}}_\infty$ of $\mathrm{Sh}^{\mathrm{ord}}$ along $\mathrm{Sh}^{\mathrm{ord}}_1=\mathrm{Sh}^{\mathrm{ord}}\otimes_W\mathbb{F}$ is uniquely determined independently of the choice of E and gives the ordinary locus of $\mathbf{A}$. By (cm1)–(cm2) for (M_x,Σ_x), x is a point of $\mathrm{Sh}^{\mathrm{ord}}_\infty(W)$. We have the connected component $V_{/W}\subset\mathrm{Sh}^{(p)}_{/W}$ containing $x\in\mathrm{Sh}^{\mathrm{ord}}_\infty(W)$; so, $V=\varprojlim_K\mathfrak{M}(\mathfrak{c}_x,K)_{/W}$. Then $V_{/\mathbb{F}}=V\otimes_W\mathbb{F}$ is the connected component containing the point $\overline{x}$ carrying $(A_0,\iota_0,\theta_0,\lambda_0,\eta_0^{(p)})$. Let $\mathbf{p}=\prod_{v\in\Sigma_p}\mathfrak{p}_v$ for the prime $\mathfrak{p}_v$ associated to the valuation $v\in\Sigma_p$ (and $\overline{\mathbf{p}}=\prod_{v\in\Sigma_p}\mathfrak{p}_v^c$). Then we have $\overline{i}:\bigcup_j\overline{\mathbf{p}}^{-j}/\mathfrak{O}\cong A_x[\overline{\mathbf{p}}^\infty]$, which induces $\eta_p^{\mathrm{ord}}:O_p\cong\mathfrak{O}_{\overline{\mathbf{p}}}\cong\mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Q}_p/\mathbb{Z}_p,A_x[\overline{\mathbf{p}}^\infty])=TA_x[p^\infty]^{\mathrm{et}}$. We can therefore extend $\eta^{(p)}$ to

$$\eta^{\mathrm{ord}}:O_p\times(M_x\otimes_{\mathbb{Q}}\mathbb{A}^{(p\infty)})\cong TA_x[p^\infty]^{\mathrm{et}}\times V^{(p)}(A_x).$$

Let $\widehat{\mathcal{K}}$ be the field of fractions of W. Over the field $\widehat{\mathcal{K}}[\mu_{p^\infty}]$, we can further extend η_p^{ord} to

$$\eta_p:O_p\times O_p=\mathfrak{O}_p\cong TA_x[p^\infty]\quad\text{by identifying } i:\bigcup_j(\mathbf{p}^{-j}/\mathfrak{O})\otimes_{\mathbb{Z}}\mu_{p^j}\cong A_x[\mathbf{p}^\infty].$$

The map $i^*:=\overline{i}^{-1}$ and i are adjoint under the Weil pairing. This choice is tantamount to the choice of g_p which brings the base of L_p to the base given by the two idempotent $1_{\mathbf{p}}:=(1,0)$ of $\mathfrak{O}_{\mathbf{p}}$ and $1_{\mathbf{p}^c}:=(0,1)$ of $\mathfrak{O}_{\mathbf{p}^c}$ in $\mathfrak{O}_{\mathbf{p}}\times\mathfrak{O}_{\mathbf{p}^c}=O_p\times O_p$. We write $\eta=\eta_p\times\eta^{(p)}$ and $\eta^{\mathrm{ord}}=\eta_p^{\mathrm{ord}}\times\eta^{(p)}$.

https://doi.org/10.1112/S0010437X10005257 Published online by Cambridge University Press

Consider the formal completion V_∞^{ord} of $V[1/E]_{/W}$ along $V_{/\mathbb{F}}^{\mathrm{ord}} = V[1/E]_{/\mathbb{F}}$, and recall the Igusa tower $\mathrm{Ig}_{/V_\infty^{\mathrm{ord}}}$ given (in [Hid10, § 3.3]) by

$$\mathrm{Ig}_{/V_\infty^{\mathrm{ord}}} \cong \mathrm{Isom}_O(\mu_{p^\infty} \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}_{/V_\infty^{\mathrm{ord}}}, \mathbf{A}^{\mathrm{ord}}[p^\infty]^\circ_{/V_\infty^{\mathrm{ord}}}).$$

We can think of the deformation of $(A_0, \theta, \iota_0, \lambda_0, \eta_0^{\mathrm{ord}})_{/\overline{\mathbb{F}}_p}$ for $\eta_0^{\mathrm{ord}} = \eta^{\mathrm{ord}} \bmod p$. The p-part of the level p-structure η_0^{ord} provides the canonical identification of the deformation space $\widehat{S}$ with $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$. For any complete local W-algebra C and any deformation $A_{/C}$ of A_0, $A[p^\infty]^{\mathrm{et}}$ is étale over $\mathrm{Spec}(C)$; so, again the deformation is insensitive to the ordinary level structure. Thus we get

$$\text{a canonical immersion } \iota : \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} \hookrightarrow \mathrm{Ig} \text{ such that } \iota^* \underline{\mathbf{A}}^{\mathrm{ord}} = \underline{\mathcal{A}}^{\mathrm{ord}}. \tag{3.5}$$

Here $\underline{\mathbf{A}}^{\mathrm{ord}}$ and $\underline{\mathcal{A}}^{\mathrm{ord}}$ denote the universal test object over Ig and over $\widehat{S}$, respectively.

Write $\widehat{\mathbb{Z}_p[O]}$ for the formal completion of $\mathbb{Z}_p[O]$ at the origin $1 \in S(\mathbb{F})$ for $S = \mathbb{G}_m \otimes \mathfrak{d}^{-1}$ (here $\widehat{\mathbb{Z}_p[O]} \cong \mathbb{Z}_p[[(t^{\xi_1} - 1), \dots, (t^{\xi_d} - 1)]]$ for a base $\xi_1, \dots, \xi_d$ of O over $\mathbb{Z}$). Identify $\widehat{\mathbb{Z}_p[O]}$ with the ring made up of series: $\sum_{\xi \in O} a(\xi) t^\xi$ for $a(\xi) \in \mathbb{Z}_p$. Let $T = \mathrm{Res}_{O/\mathbb{Z}} \mathbb{G}_m$. Since we have $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} = \mathrm{Spf}(\widehat{\mathbb{Z}_p[O]})$, $s \in O_p^\times = T(\mathbb{Z}_p)$ acts on $\widehat{S}$ by the variable change $t \mapsto t^s$, which induces an automorphism of the formal group $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$.

The inclusion $O \hookrightarrow \mathfrak{O}$ induces an identification of p-adic rings $\mathfrak{O}_{\mathbf{p}}$ with O_p which we fix in this paper and use always in the following. Note that $\mathfrak{O}_p = \mathfrak{O}_{\mathbf{p}} \times \mathfrak{O}_{\mathbf{p}^c}$. This same inclusion: $O \hookrightarrow \mathfrak{O}$ induces an inclusion of $\mathbb{Z}_{(p)}$-tori $T \hookrightarrow T_x$. Let $\mathcal{T} := T_x/T$. By the identification above, the map $\mathfrak{O}_{(p)}^\times \to \mathfrak{O}_{\mathbf{p}}^\times$ given by $\alpha \mapsto \alpha^{1-c}$ induces an injective homomorphism

$$\mathcal{T}(\mathbb{Z}_{(p)}) \to O_p^\times = T(\mathbb{Z}_p). \tag{3.6}$$

Thereby, the action of $T(\mathbb{Z}_p)$ on $\widehat{S}$ and that of $\mathcal{T}(\mathbb{Z}_{(p)})$ are compatible. The torus $\mathcal{T}(\mathbb{Z}_{(p)})$ is isomorphic to the image (under $\widehat{\rho}_x$) of $T_x(\mathbb{Z}_{(p)})$ in $\overline{\mathcal{E}}(G, \mathfrak{X})$, and its action on $\widehat{S}$ factors through the action of the image of $T_x(\mathbb{Z}_{(p)})$ in $\overline{\mathcal{E}}(G, \mathfrak{X})$ on Ig via (3.5).

We regard $\mathrm{Sh}^{(p)}$ as a (pro-)scheme over $\mathcal{W}$. By the definition of $\widehat{\rho}_x$ given above, we have $\alpha \circ \eta_p = \eta_p \circ \widehat{\rho}_x(\alpha)$. If $\alpha \in T_x(\mathbb{Z}_{(p)})$, it acts on Ig as an automorphism via $\widehat{\rho}_x(\alpha)$. We quote the following fact from [Hid10, Proposition 3.4, Corollary 3.5].

Proposition 3.4. *Let $\mathcal{O}_{\mathrm{Ig}, \overline{x}/\mathbb{F}}$ be the stalk at the point $\overline{x} \in \mathrm{Ig}(\mathbb{F})$ (carrying $(A_x, \lambda_x, \theta_x, \eta_x^{(p)}) \times_W \mathbb{F}$) of the structure sheaf of $\mathrm{Ig}_{/\mathbb{F}}$. If $\alpha \in \mathfrak{O}_{(p)}^\times$ ($\cong T_x(\mathbb{Z}_{(p)})$), then $\tau(\widehat{\rho}_x(\alpha))$ fixes $\overline{x}$ and preserves $\mathcal{O}_{\mathrm{Ig}, \overline{x}/\mathbb{F}}$. The effect of $\widehat{\rho}_x(\alpha)$ on the canonical coordinate $t \in \widehat{S}$ is given by $t \mapsto t^{\alpha^{1-c}}$.*

Thus among the automorphisms $t \mapsto t^s$ of $\widehat{S}$ for $s \in O_p^\times$, those $s = \alpha^{1-c}$ with $\alpha \in \mathfrak{O}_{(p)}^\times$ preserve the p-integral $\mathcal{W}$-structure coming from the Shimura variety $\mathrm{Sh}^{(p)}$ (and the Igusa tower Ig).

3.4 Linear independence

To include modular forms in our scope, we need the datum of a nowhere vanishing differential. We look into the following functor $\mathcal{Q}_K$:

$$U \mapsto [(A, \overline{\lambda}, \theta, \overline{\eta}^{(p)}, \omega)_{/U} \mid \underline{A} \in \mathcal{P}'_K(U) \cong \mathcal{P}_K^{(p)}(U), \pi_* \Omega_{A/U} = (\mathcal{O}_U \otimes_{\mathbb{Z}} O)\omega], \tag{3.7}$$

where K is maximal at p and $\underline{A} = (A, \overline{\lambda}, \theta, \overline{\eta}^{(p)})$ is chosen in $\mathcal{P}'_K(U)$. Then $\mathcal{Q}_K$ is represented over $\mathcal{W}$ by a T-torsor $\mathcal{M}_K(G, X)$ over $\mathrm{Sh}^{(p)}(G, X)/K$. The torus $T = \mathrm{Res}_{O/\mathbb{Z}} \mathbb{G}_m$ acts on $\mathcal{Q}_K$

by $\omega \mapsto t\omega$ for $t \in T(\mathcal{O}_U) = (\mathcal{O}_U \otimes_{\mathbb{Z}} O)^\times$. Choose a nowhere vanishing differential ω_0 on $A_0 = A_x \otimes_W \mathbb{F}$, and consider the formal completion $\widehat{\mathcal{M}}_K$ of $\mathcal{M}_K$ along the closed point corresponding to $(A_0, \overline{\lambda}_0, \theta_0, \eta_0^{\mathrm{ord}}, \omega_0)$, which is a formal $\widehat{T}$-torsor over $\widehat{S} = \widehat{V}_x$. Here $\widehat{T}$ is the formal completion of T along the origin. Over W, once we choose a level p^∞-structure η_p^{ord}, it naturally induces an isomorphism of formal groups $\eta : \widehat{A}_x \cong \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} = \mathrm{Spf}(\widehat{W[q^\xi]}_{\xi \in O})$, which in turn gives a canonical differential ω_p on $A_{x/W}$ with $\omega_p|_{\widehat{A}_x} = \eta^* dq/q$. This $\eta_p^{\mathrm{ord}} \mapsto \omega_p$ splits the formal $\widehat{T}$-torsor $\widehat{\mathcal{M}}_K$ into a product $\widehat{T} \times_W \widehat{S}$ over $\widehat{S} \cong \widehat{\mathbb{G}}_m \otimes \mathfrak{d}^{-1}$. Thus the deformation functor

$$\widehat{\mathcal{Q}}(R) = [(A, \overline{\lambda}, \theta, \eta^{\mathrm{ord}}, \omega)_{/R} \mid (A, \overline{\lambda}, \theta, \eta^{\mathrm{ord}}, \omega) \times_R \mathbb{F} = (A_0, \overline{\lambda}_0, \theta_0, \eta_0^{\mathrm{ord}}, \omega_0)]$$

for Artinian local W-algebras R with residue field $\mathbb{F}$ is pro-represented by $\widehat{S} \times \widehat{T}$. In the above discussion, we may actually allow K of p-power level in (3.7) as long as K contains the monodromy group U_∞ of the infinity cusp in $G(\mathbb{Z}_p) \cap \mathcal{G}(G, \mathfrak{X})$, replacing $V \subset \mathrm{Sh}^{(p)}(G, \mathfrak{X})$ by the Igusa tower over V^{ord} and the level structure $\overline{\eta}^{(p)} = \eta^{(p)} K^{(p)}$ by $\eta^{\mathrm{ord}} K^{(p)}$. In this slightly more general case, the functor is again represented by a formal scheme $\widehat{S} \times \widehat{T}$, where $\widehat{S}$ is identified with infinitesimal neighborhood of x in the Igusa tower. Therefore in the following, we allow modular forms of finite p-power level of type $\Gamma_1(p^r)$.

We identify the character group $X^*(T)$ of T with the module of formal linear combinations $\kappa = \sum_\sigma \kappa_\sigma \sigma$ ($\kappa_\sigma \in \mathbb{Z}$) for field embeddings $\sigma : F \hookrightarrow \overline{\mathbb{Q}}$ so that $x^\kappa = \prod_\sigma \sigma(x)^{\kappa_\sigma}$ ($x \in T(\mathbb{Q})$). For each character κ of T and a p-adic W-algebra R, we write $G_\kappa(R)$ for the κ^{-1}-eigenspace of $\mathcal{O}_{\mathcal{M}/R}$. Thus $G_\kappa(R)$ is the union of R-integral modular forms of weight κ and of finite level (of $\Gamma_1(N)$-type for all positive integers N). Since p is unramified in O, T is smooth over $\mathbb{Z}_p$ and is diagonalizable over $\mathbb{Z}_p$. Therefore we have $\mathcal{O}_{\mathcal{M}/W} = \bigoplus_\kappa G_\kappa(W)$. By the above splitting, we may regard $G_\kappa(R) \subset \mathcal{O}_{\widehat{S}/R}$. In particular, $a \in T_x(\mathbb{Z}_p)$ acts on $f \in G_\kappa(\mathbb{F})$ through the identification $\mathcal{T}(\mathbb{Z}_p) = \mathrm{Aut}_O(\widehat{S}_{/\mathbb{F}})$, and we have $a(f) \in \mathcal{O}_{\widehat{S}/\mathbb{F}}$. We write $t - 1 = (t_j - 1)_j$ for the parameter at 1 of $\widehat{S}$. Each $\phi \in G_\kappa(R)$ has t-expansion given by

$$\phi(t) = \phi(\underline{\mathcal{A}}^{\mathrm{ord}}) \in R[[t-1]].$$

We quote the following result proven in [Hid10, Corollary 3.21].

THEOREM 3.5. *Fix a weight $\kappa > 0$. Let $a_0, \ldots, a_n \in T_x(\mathbb{Z}_p)$ and suppose that $a_i a_j^{-1} \notin T_x(\mathbb{Q})$ for all $i \neq j$. Let $I \subset \{0, 1, 2, \ldots, n\}$ be a subset of indices, and choose $0 \neq h_i \in G_\kappa(\mathbb{F})$ with $h_i(x) \neq 0$ for each $i \in I$. Let J be another finite index set. Then if $\{h_i, f_{ij} \in G_\kappa(\mathbb{F})\}_{j \in J}$ are linearly independent over $\mathbb{F}$ for each $i \in I$, then $\{a_i(f_{ij})\}_{i \in I, j \in J}$ in $\mathcal{O}_{\widehat{S}/\mathbb{F}}$ are linearly independent over $\mathbb{F}$.*

When κ is parallel, a canonical choice of h_i is H_κ for a Hasse invariant H_κ. The Hasse invariant H satisfies $H(t) = 1$. Since H is invertible on $\mathrm{Sh}^{\mathrm{ord}}$, for any given *parallel* weight $\kappa = \sum_\sigma k\sigma$ ($k \in \mathbb{Z}$), we have $H_\kappa \in G_\kappa(\mathbb{F})$ such that $H_\kappa(t) = 1$.

4. Eisenstein and Katz measure

We recall the Fourier expansion of Eisenstein series and Eisenstein measure from [Kat78a].

4.1 Geometric modular forms

Recall $\mathfrak{a}^* = \mathfrak{a}^{-1}\mathfrak{d}^{-1}$ for each ideal $\mathfrak{a} \subset F$. For a fixed fractional ideal $\mathfrak{c}$ prime to p of F, we consider the following triples $(A, \lambda, i)_{/S}$ formed by:

- an abelian scheme $\pi : A \to S$ with an algebra homomorphism: $O \hookrightarrow \mathrm{End}(A_{/S})$ making $\pi_*(\Omega_{A/S})$ a locally free $O \otimes_{\mathbb{Z}} \mathcal{O}_S$-module of rank one;
- an O-linear polarization $\lambda : A^t \cong A \otimes \mathfrak{c}$ (as explained in [Hid10, §§ 2.4 and 4.1], this condition is equivalent to λ having polarization ideal $\mathfrak{c}$);
- we have an O-linear closed immersion $i = i_{p^n} : \mu_{p^n} \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} \hookrightarrow A[p^n]$ of group schemes.

The Hilbert modular variety $\mathfrak{M}(\mathfrak{c}, p^n) := \mathfrak{M}(\mathfrak{c}, K)_{/\mathcal{W}}$ is the coarse moduli scheme of the functor $\mathcal{P}(S) = [(A, \lambda, i)_{/S}]$ from the category of $\mathcal{W}$-schemes S into the category SETS, where

$$K = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbb{A}^{(\infty)}) \mid g\widehat{L}_{\mathfrak{c}} = \widehat{L}_{\mathfrak{c}}, c \in p^n \widehat{O}, a_p \equiv 1 \bmod p^n O_p \right\},$$

and we call $(A, \lambda, i) \cong (A', \lambda', i')$ if we have an O-linear isomorphism $\phi : A_{/S} \to A'_{/S}$ such that $\lambda' = (\phi \otimes 1) \circ \lambda \circ \phi^t$ and $\phi \circ i = i'$. The quasi projective scheme $\mathfrak{M}(\mathfrak{c}, p^n)$ is a fine moduli if $n \gg 0$.

We could insist that $\pi_*(\Omega_{A/S})$ is free over $\mathcal{O}_S \otimes_{\mathbb{Z}} O$, and taking a generator ω with $\pi_*(\Omega_{A/S}) = (\mathcal{O}_S \otimes_{\mathbb{Z}} O)\omega$, we consider the following functor (A, λ, i, ω):

$$\mathcal{Q}(S) = [(A, \lambda, i, \omega)_{/S}]. \tag{4.1}$$

We let $a \in T(S) = H^0(S, (\mathcal{O}_S \otimes_{\mathbb{Z}} O)^\times)$ act on $\mathcal{Q}(S)$ by $(A, \lambda, i, \omega) \mapsto (A, \lambda, i, a\omega)$; so, $\mathcal{Q}$ is a T-torsor over $\mathcal{P}$; so, $\mathcal{Q}$ is representable by a scheme $\mathcal{M} = \mathcal{M}(\mathfrak{c}, p^n)_{/\mathcal{W}}$ affine over $\mathfrak{M} = \mathfrak{M}(\mathfrak{c}, p^n)$. By definition, $\mathcal{M}$ is a T-torsor over $\mathfrak{M}$. For each character $\kappa \in X^*(T) = \mathrm{Hom}_{\text{gp-sch}}(T, \mathbb{G}_m)$ and a given $\mathcal{W}$-algebra R, if $F \neq \mathbb{Q}$, the κ^{-1}-eigenspace of $H^0(\mathcal{M}_{/R}, \mathcal{O}_{\mathcal{M}/R})$ is the space of modular forms of weight κ integral over R, where $\mathcal{M}_{/R} = \mathcal{M} \times_{\mathbb{Z}} \mathrm{Spec}(R)$. We write $G_\kappa(\mathfrak{c}, p^n; R)$ for this space of R-integral modular forms, which is an R-module of finite type. An element $f \in G_\kappa(\mathfrak{c}, p^n; R)$ may be regarded as a morphism of functors: $\mathcal{Q} \to \mathbb{G}_a$; so, it is a rule assigning an element in an R-algebra C to each quadruple $(A, \lambda, i, \omega)_{/C}$ satisfying the following three conditions:

(G1) $f(A, \lambda, i, \omega) = f(A', \lambda', i', \omega') \in C$ if $(A, \lambda, i, \omega) \cong (A', \lambda', i', \omega')$ over C;

(G2) $f((A, \lambda, i, \omega) \otimes_{C,\rho} C') = \rho(f(A, \lambda, i, \omega))$ for each $\rho \in \mathrm{Hom}_{R\text{-alg}}(C, C')$;

(G3) $f(A, \lambda, i, a\omega) = \kappa(a)^{-1} f(A, \lambda, i, \omega)$ for $a \in T(C)$.

When $F = \mathbb{Q}$, we need to take the subsheaf of sections with logarithmic growth towards cusps.

We fix a fractional ideal $\mathfrak{c}$ prime to p and take two ideals $\mathfrak{a}$ and $\mathfrak{b}$ prime to p such that $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$. To this pair $(\mathfrak{a}, \mathfrak{b})$, as in [Kat78a, § 1.1], we can attach the Tate AVRM $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ defined over the completed group ring $\mathbb{Z}((\mathfrak{a}\mathfrak{b}))$ made of formal series $f(q) = \sum_{\xi \gg -\infty} a(\xi) q^\xi$ $(a(\xi) \in \mathbb{Z})$. Here ξ runs over all elements in $\mathfrak{a}\mathfrak{b}$, and there exists a positive constant C_0 (dependent on f) such that $a(\xi) = 0$ if $\sigma(\xi) + C_0 < 0$ for some $\sigma \in I$. We write $R[[(\mathfrak{a}\mathfrak{b})_{\geqslant 0}]]$ for the subring of $R((\mathfrak{a}\mathfrak{b}))$ made of formal series f (having coefficients in R) with $a(\xi) = 0$ for all ξ with $\sigma(\xi) < 0$ for at least one embedding $\sigma : F \hookrightarrow \mathbb{R}$. The scheme $\mathrm{Tate}(q)$ can be extended to a semi-abelian scheme over $\mathbb{Z}[[(\mathfrak{a}\mathfrak{b})_{\geqslant 0}]]$ with special fiber $\mathbb{G}_m \otimes \mathfrak{a}^*$ at the augmentation ideal $\mathfrak{A}$. As described in [Kat78a, § 1.1] (see also [Hid10, § 4.1]), $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ has a canonical $\mathfrak{c}$-polarization λ_{can}, a canonical level structure $i_{\mathrm{can}} : \mu_{p^\infty} \hookrightarrow \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)[p^\infty]$ and a canonical differential ω_{can}. Thus we can evaluate $f \in G_\kappa(\mathfrak{c}; R)$ at $(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, i_{\mathrm{can}}, \omega_{\mathrm{can}})$. The value $f(q) = f_{\mathfrak{a},\mathfrak{b}}(q)$, if $F \neq \mathbb{Q}$, actually falls in $R[[(\mathfrak{a}\mathfrak{b})_{\geqslant 0}]]$ by Koecher's principle and is called the q-expansion at the cusp $(\mathfrak{a}, \mathfrak{b})$. When $F = \mathbb{Q}$, we impose f to have values in $R[[(\mathfrak{a}\mathfrak{b})_{\geqslant 0}]]$ (the logarithmic growth condition).

We can think of a functor

$$\widehat{\mathcal{Q}}(R) = [(A, \lambda, i_p)_{/R}]$$

similar to $\mathcal{Q}$ in (3.7) defined over the category of p-adic W-algebras $R = \varprojlim_n R/p^nR$. The only difference here is that we consider an isomorphism of Barsotti–Tate groups $i_p : \mu_{p^\infty} \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} \cong A[p^\infty]^\circ$ (in place of a differential ω), which induces an isomorphism $\widehat{\mathbb{G}}_m \otimes \mathfrak{d}^{-1} \cong \widehat{A}$ formal schemes. It is a theorem (due to Deligne–Ribet and Katz) that this functor is representable by the formal completion $\widehat{\mathfrak{M}}(\mathfrak{c}, p^\infty)_{/W}$ of $\mathfrak{M}(\mathfrak{c}, p^\infty) = \varprojlim_n \mathfrak{M}(\mathfrak{c}, p^n)$ along its mod p fiber. Thus we can think of p-adic modular forms $f_{/R}$ which are functions of $(A, \lambda, i_p)_{/C}$ (for any p-adic R-algebra C) satisfying the following conditions:

(G_p1) $f(A, \lambda, i_p) = f(A', \lambda', i'_p) \in C$ if $(A, \lambda, i_p)_{/C} \cong (A', \lambda', i'_p)_{/C}$;

(G_p2) $f((A, \lambda, i_p) \otimes_{C,\rho} C') = \rho(f(A, \lambda, i_p))$ for each R-algebra homomorphism $\rho : C \to C'$;

(G_p3) $f_{\mathfrak{a},\mathfrak{b}}(q) \in R[[(\mathfrak{a}\mathfrak{b})_{\geqslant 0}]]$ for all $(\mathfrak{a}, \mathfrak{b})$ prime to p.

We write $V(\mathfrak{c}; R)$ for the space of p-adic modular forms satisfying (G_p1)–(G_p3). This $V(\mathfrak{c}; R)$ is a p-adically complete R-algebra. We have the q-expansion principle valid both for classical and p-adic modular forms f (see [DR80], [Hid04, Theorem 4.21] and [Hid09]):

(q-exp) *the q-expansion: $f \mapsto f_{\mathfrak{a},\mathfrak{b}}(q) \in R[[(\mathfrak{a}\mathfrak{b})_{\geqslant 0}]]$ determines f uniquely.*

Since $\mathbb{G}_m \otimes \mathfrak{d}^{-1} = \mathrm{Spec}(\mathbb{Z}[t^\xi]_{\xi \in O})$ has a canonical invariant differential dt/t, we have $\omega_p = i_{p,*}dt/t$ on A. This allows us to regard each $f \in G_\kappa(\mathfrak{c}; R)$ as a p-adic modular form by putting

$$f(A, \lambda, i_p) = f(A, \lambda, \omega_p).$$

We thus have a canonical embedding: $G_\kappa(\mathfrak{c}; R) \hookrightarrow V(\mathfrak{c}; R)$ preserving q-expansions.

Over $\mathbb{C}$, the category of quadruples (A, λ, i, ω) with $\mathfrak{c}$-polarization λ is equivalent to the category of triples $(\mathcal{L}, \lambda, i)$ made of the following data (see [Hid10, § 4.1]): $\mathcal{L}$ is an O-lattice in $O \otimes_{\mathbb{Z}} \mathbb{C} = \mathbb{C}^I$, an alternating form $\lambda : \mathcal{L} \wedge_O \mathcal{L} \cong \mathfrak{c}^*$ and $i : p^{-n}\mathfrak{d}^{-1}/\mathfrak{d}^{-1} \hookrightarrow F\mathcal{L}/\mathcal{L}$. The form λ is supposed to be positive in the sense that $\lambda(u, v)/\mathrm{Im}(uv^c)$ is totally positive in $O \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^I$. The differential ω can be recovered by $\iota : A(\mathbb{C}) = \mathbb{C}^I/\mathcal{L}$ so that $\omega = \iota^* du$ where $u = (u_\sigma)_{\sigma \in I}$ is the variable on $\mathbb{C}^I$. Conversely, if we start with a triple $(A, \lambda, \omega)_{/\mathbb{C}}$,

$$\mathcal{L}_A = \left\{ \int_\gamma \omega \in O \otimes_{\mathbb{Z}} \mathbb{C} \mid \gamma \in H_1(A(\mathbb{C}), \mathbb{Z}) \right\}$$

is a lattice in $\mathbb{C}^I$, and the polarization $\lambda : A^t \cong A \otimes \mathfrak{c}$ induces $\mathcal{L}_A \wedge \mathcal{L}_A \cong \mathfrak{c}^*$. Using this equivalence, we can relate our geometric definition of Hilbert modular forms with the classical analytic definition. Recall $\mathfrak{Z}$ which is the product $\mathfrak{Z} = \mathfrak{H}^I$ of I copies of the upper half complex plane $\mathfrak{H}$. For a cusp $(\mathfrak{a}, \mathfrak{b})$ and each $z \in \mathfrak{Z}$, we define $\mathcal{L}_z = 2\pi\sqrt{-1}(\mathfrak{b}z + \mathfrak{a}^*) \subset \mathbb{C}^I$,

$$\lambda_z(2\pi\sqrt{-1}(az + b), 2\pi\sqrt{-1}(cz + d)) = -(ad - bc) \in \mathfrak{c}^*$$

and $i_z : p^{-n}\mathfrak{d}^{-1}/\mathfrak{d}^{-1} = (p^n\mathfrak{a})^*/\mathfrak{a}^* \to F\mathcal{L}_z/\mathcal{L}_z$ by $i_z(a \bmod \mathfrak{d}^{-1}) = 2\pi\sqrt{-1}a \bmod \mathcal{L}_z$.

Consider the following congruence subgroup $\Gamma_{11}(p^n; \mathfrak{a}, \mathfrak{b})$ given by

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) \mid a, d \in O, b \in (\mathfrak{a}\mathfrak{b})^*, c \in p^n\mathfrak{a}\mathfrak{b}\mathfrak{d} \text{ and } d - 1 \in p^n \right\}.$$

Write $\Gamma_{11}(\mathfrak{c}; p^n) = \Gamma_{11}(p^n; O, \mathfrak{c}^{-1})$. We let $g = (g_\sigma) \in \mathrm{SL}_2(F \otimes_{\mathbb{Q}} \mathbb{R}) = \mathrm{SL}_2(\mathbb{R})^I$ act on $\mathfrak{Z}$ by linear fractional transformation of g_σ on each component z_σ. Then

$$(\mathcal{L}_z, \lambda_z, i_z) \cong (\mathcal{L}_w, \lambda_w, i_w) \Longleftrightarrow w = \gamma(z) \quad \text{for } \gamma \in \Gamma_{11}(p^n; \mathfrak{a}, \mathfrak{b}).$$

This implies

$$\mathfrak{M}(\mathfrak{c}, p^n)(\mathbb{C}) \cong \Gamma_{11}(\mathfrak{c}; p^n)\backslash\mathfrak{Z} \quad \text{canonically}.$$

The set of pairs $(\mathfrak{a}, \mathfrak{b})$ with $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$ is in bijection with the set of cusps of $\Gamma_{11}(\mathfrak{c}; 1)$. Two cusps are equivalent if they transform each other by an element in $\Gamma_{11}(\mathfrak{c}; p^n)$. A standard choice is $(O, \mathfrak{c}^{-1})$, which we call the infinity cusp of $\mathfrak{M}(\mathfrak{c}, p^n)$. For each ideal $\mathfrak{t}$, $(\mathfrak{t}, \mathfrak{t}^{-1}\mathfrak{c}^{-1})$ gives another cusp. The two cusps $(\mathfrak{t}, \mathfrak{t}^{-1}\mathfrak{c}^{-1})$ and $(\mathfrak{s}, \mathfrak{s}^{-1}\mathfrak{c}^{-1})$ are equivalent under $\Gamma_{11}(\mathfrak{c}; p^n)$ if $\mathfrak{t} = \alpha\mathfrak{s}$ for an element $\alpha \in F^\times$ with $\alpha \equiv 1 \bmod p^n O_p$ in $F_p^\times$.

Recall the identification $X^*(T)$ with $\mathbb{Z}[I]$ so that $\kappa(x) = \prod_\sigma \sigma(x)^{\kappa_\sigma}$. Regarding $f \in G_\kappa(\mathfrak{c}, p^n; \mathbb{C})$ as a holomorphic function of $z \in \mathfrak{Z}$ by $f(z) = f(\mathcal{L}_z, \lambda_z, i_z)$, it satisfies

$$f(\gamma(z)) = f(z) \prod_\sigma (c^\sigma z_\sigma + d^\sigma)^{\kappa_\sigma} \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{11}(\mathfrak{c}; p^n). \tag{4.2}$$

The holomorphy of f follows from (G2), and $f \in G_\kappa(\mathfrak{c}, p^n; \mathbb{C})$ has the Fourier expansion

$$f(z) = \sum_{\xi \in (\mathfrak{a}\mathfrak{b})_{\geqslant 0}} a(\xi) \mathbf{e}_F(\xi z)$$

at the cusp corresponding to $(\mathfrak{a}, \mathfrak{b})$. Here $\mathbf{e}_F(\xi z) = \exp(2\pi\sqrt{-1} \sum_\sigma \xi^\sigma z_\sigma)$. This Fourier expansion equals the q-expansion $f_{\mathfrak{a},\mathfrak{b}}(q)$ replacing $\mathbf{e}_F(\xi z)$ by q^ξ.

Shimura studied in his theory of arithmetic of Hecke L-values the effect on modular forms of the following differential operators on $\mathfrak{Z}$ indexed by $\kappa \in \mathbb{Z}[I]$:

$$\delta_\kappa^\sigma = \frac{1}{2\pi\sqrt{-1}} \left(\frac{\partial}{\partial z_\sigma} + \frac{\kappa_\sigma}{z_\sigma - \overline{z}_\sigma} \right) \quad \text{and} \quad \delta_\kappa^k = \prod_\sigma (\delta^\sigma_{\kappa_\sigma + 2k_\sigma - 2} \cdots \delta^\sigma_{\kappa_\sigma}), \tag{4.3}$$

where $k \in \mathbb{Z}[I]$ with $k_\sigma \geqslant 0$. To describe rationality property of δ_κ^k in [Shi00, III] and [Shi75], we recall the two embeddings $i_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $i_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ fixed in the introduction. Recall $\mathcal{W} = i_p^{-1}(W)$, which is a *discrete* valuation ring. Let $\underline{A} := (A, \lambda, \omega, i)_{/\mathcal{W}}$ be an ordinary quadruple of CM type (M, Σ) (having complex multiplication by the integer ring $\mathfrak{O} \subset M$). The complex uniformization: $\iota : A(\mathbb{C}) \cong \mathbb{C}^\Sigma / \Sigma(\mathfrak{A})$ induces a canonical base $\omega_\infty = \iota^* du$ of $\Omega_{A/\mathbb{C}}$ over $\mathfrak{O} \otimes_{\mathbb{Z}} \mathbb{R}$, where $u = (u_\sigma)_{\sigma \in \Sigma}$ is the standard variable on $\mathbb{C}^\Sigma$ and $\Sigma(\mathfrak{A}) = \{(\sigma(a))_{\sigma\in\Sigma} \in \mathbb{C}^\Sigma \mid a \in \mathfrak{A}\}$. We define the periods $\Omega_\infty \in \mathbb{C}^\Sigma = O \otimes_{\mathbb{Z}} \mathbb{C}$ by $\omega = \Omega_\infty \omega_\infty$. Here is the rationality result of Shimura:

$$\frac{(\delta_\kappa^k f)(A, \lambda, \omega_\infty, i)}{\Omega_\infty^{\kappa+2k}} = (\delta_\kappa^k f)(A, \lambda, \omega, i) \in \overline{\mathbb{Q}} \quad \text{for } f \in G_\kappa(\mathfrak{c}, p^n; \mathcal{W}). \tag{S}$$

Katz gave a purely algebro-geometric definition of the operator (see [Kat78a, ch. II]). Using this algebraization of δ_κ^k, he extended the operator to geometric modular forms and p-adic modular forms. We write his operator corresponding to Shimura's operator δ_*^k as $d^k : V(\mathfrak{c}, p^n; R) \to V(\mathfrak{c}, p^n; R)$. The level p-structure $i_p : \mu_{p^\infty} \otimes \mathfrak{d}^{-1} \hookrightarrow A[p^\infty]$ induces an isomorphism $\iota_p : \mathrm{Spf}(\widehat{W[q^\xi]}_{\xi \in O}) = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1} \cong \widehat{A}$ for the p-adic formal group $\widehat{A}_{/W}$ at the origin. Then $\omega = \Omega_p \omega_p$ ($\Omega_p \in O \otimes_{\mathbb{Z}} W = W^\Sigma$) for $\omega_p = \iota_{p,*} dq/q$. An important formula given in [Kat78a, (2.6.7)] is

$$\frac{(d^k f)(A, \lambda, \omega_p, i)}{\Omega_p^{\kappa+2k}} = (d^k f)(A, \lambda, \omega, i) = (\delta_\kappa^k f)(A, \lambda, \omega, i) \in \mathcal{W} = \overline{\mathbb{Q}} \cap W \quad \text{for } f \in G_\kappa(\mathfrak{c}, p^n; \mathcal{W}). \tag{K}$$

Let t be the canonical variable of the deformation space $\widehat{S}$ of $\underline{A}_0 = \underline{A} \times_{\mathcal{W}} \mathbb{F}$. Identifying $\widehat{S}$ with $\widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$ via i_p, t is the character $1 \in O = X^*(\mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}) = \mathrm{Hom}(\mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}, \mathbb{G}_m)$. Write $S = \mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$. We have $\widehat{S} = \mathrm{Spf}(\widehat{W[X(S)]})$ for the completion $\widehat{W[X(S)]}$ at the augmentation ideal of the monoid algebra $W[X(S)] = W[O]$ ($X(S) = \mathrm{Hom}_{\mathrm{alg-gp}}(S, \mathbb{G}_m)$), where $W[O]$ is the ring of formal finite sums $\sum_{\xi\in O} a(\xi)t^\xi$ $(a(\xi)\in W)$. We have the following interpretation of d^κ:

$$d^\kappa \sum_\xi a(\xi)t^\xi = \sum_\xi a(\xi)\xi^\kappa t^\xi. \tag{4.4}$$

See [Hid10, (4.5)] for a proof of (4.4). Similarly, by [Kat78a, (2.6.25)], the effect of d^κ on q-expansion is

$$d^\kappa \sum_\xi a(\xi)q^\xi = \sum_\xi a(\xi)\xi^\kappa q^\xi. \tag{4.5}$$

For each $f\in V(\mathfrak{c},p^n;R)$ (for a p-adic algebra R), we call the expansion

$$f(t):=f(\mathcal{A},\widehat{\lambda},\widehat{i})=\sum_{\xi\in O} a(\xi,f)t^\xi$$

as an element of $\widehat{R[O]}$ a *t-expansion* of f. Hereafter, we write this ring symbolically as $R[[t^\xi]]_{\xi\in O}$. Choosing a $\mathbb{Z}$-base $\{a_j\}$ of O, $T_j=t^{a_j}-1$ gives a complete set of local parameters at the point $x\in\widehat{\mathfrak{M}}(\mathfrak{c},p^n)_{/R}$ given by $\underline{A}$ and $\widehat{R[O]}\cong R[[T_1,\dots,T_d]]$. We have the following t-expansion principle.

(t-exp) *The t-expansion: $f\mapsto f(t)\in R[[t^\xi]]_{\xi\in O}$ determines f uniquely.*

The Taylor expansion of f with respect to the variables $T=(T_j)$ can be computed by applying differential operators $\partial_j=\partial/\partial T_j$ and evaluating the result at $x=\underline{A}$. Since ∂_j is a linear combination of the d^σ with coefficients in the field of fractions of R as long as R is of characteristic 0, we have, for $f,g\in V(\mathfrak{c},p^n;W)$,

$$d^\kappa f(\underline{A})=d^\kappa g(\underline{A})\quad\text{for all } \kappa\geqslant 0 \Longleftrightarrow f(t)=g(t). \tag{4.6}$$

4.2 The q-expansion of Eisenstein series

Let $\phi: O_p\times O_p\to\mathbb{C}$ be a locally constant function such that $\phi(\varepsilon^{-1}x,\varepsilon y)=N(\varepsilon)^k\phi(x,y)$ for all $\varepsilon\in O^\times$, where k is a positive integer. We suppose that all $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ are prime to p. We regard ϕ as a function on $X\times Y$ with $X=Y=O_p$. We put $X_\alpha=O/p^\alpha O$ and define the partial Fourier transform

$$P\phi:(F_p/\mathfrak{d}_p^{-1})\times Y=\left\{\bigcup_\alpha p^{-\alpha}\mathfrak{d}^{-1}/\mathfrak{d}^{-1}\right\}\times Y\to\mathbb{C}$$

of ϕ, taking α so that ϕ factors through $X_\alpha\times Y$, by

$$P\phi(x,y)=\begin{cases} p^{-\alpha[F:\mathbb{Q}]}\sum_{a\in X_\alpha}\phi(a,y)\mathbf{e}_F(ax) & \text{if } x\in p^{-\alpha}\mathfrak{d}^{-1}/\mathfrak{d}^{-1},\\ 0 & \text{if } x\notin p^{-\alpha}\mathfrak{d}^{-1}/\mathfrak{d}^{-1},\end{cases} \tag{4.7}$$

where $\mathbf{e}_F$ is the standard additive character of $F_\mathbb{A}$ restricted to the local component F_p at p.

We construct an Eisenstein series $E_k(z;\phi)$ for a positive integer k and ϕ as above as a function of triples $(\mathcal{L},\lambda,i)$ we have studied in the previous subsection. Actually k indicates the parallel weight $\kappa=\sum_\sigma k\sigma$; so, we sometimes write E_κ for E_k. Here $i:F_p/\mathfrak{d}_p^{-1}\hookrightarrow p^{-\infty}\mathcal{L}/\mathcal{L}$ is the level p^∞-structure. We define an O_p-submodule $\mathrm{PV}(\mathcal{L})\subset\mathcal{L}\otimes_O F_p$ specified by the following conditions:

(pv) $\mathrm{PV}(\mathcal{L})\supset\mathcal{L}\otimes_O O_p$ and $\mathrm{PV}(\mathcal{L})/(\mathcal{L}\otimes_O O_p)=\mathrm{Im}(i)$.

Consider $i^{-1}:\mathrm{PV}(\mathcal{L})\twoheadrightarrow\mathrm{PV}(\mathcal{L})/(\mathcal{L}\otimes_O O_p)\cong F_p/\mathfrak{d}_p^{-1}$. By Pontryagin duality under $\mathrm{Tr}\circ\lambda$, the dual map i' of i gives rise to $i':\mathrm{PV}(\mathcal{L})\twoheadrightarrow O_p$. Then we may regard $P\phi$ as a function

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on $p^{-\infty}\mathcal{L} \cap \mathrm{PV}(\mathcal{L}) = (\bigcup_\alpha p^{-\alpha}\mathcal{L}) \cap \mathrm{PV}(\mathcal{L})$ by

$$P\phi(w) = \begin{cases} P\phi(i^{-1}(w), i'(w)) & \text{if } (w \bmod \mathcal{L}) \in \mathrm{Im}(i), \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

For each $w = (w_\sigma) \in F \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}^I$, the norm map $N(w) = \prod_{\sigma \in I} w_\sigma$ is well defined. Writing $\underline{\mathcal{L}} = (\mathcal{L}, \lambda, i)$ for simplicity, we define the value $E_k(\underline{\mathcal{L}}; \phi, \mathfrak{c})$ by

$$E_k(\underline{\mathcal{L}}; \phi, \mathfrak{c}) = \frac{\{(-1)^k \Gamma(k+s)\}^{[F:\mathbb{Q}]}}{\sqrt{|D_F|}} \sum_{w \in p^{-\infty}\mathcal{L}/O^\times}{}' \frac{P\phi(w)}{N(w)^k |N(w)|^{2s}}\bigg|_{s=0}. \tag{4.9}$$

Here '$\sum'$' indicates that we are excluding $w = 0$ from the summation. This type of series for ϕ with $\phi(a, 0) = 0$ for all a is convergent when the real part of s is sufficiently large and is continued to a meromorphic function well-defined at $s = 0$. If $\phi(a, 0) = 0$ for all a, the function $E_k(\mathfrak{c}, \phi)$ gives an element in $G_\kappa(\mathfrak{c}, p^\infty; \mathbb{C})$ ($\kappa = k \sum_{\sigma: F \hookrightarrow \overline{\mathbb{Q}}} \sigma$) without constant term in its q-expansion, and as computed in [Kat78a, Chapter III] the expansion is given by

$$N(\mathfrak{a})^{-1} E_k(\phi, \mathfrak{c})_{\mathfrak{a},\mathfrak{b}}(q) = \sum_{0 \ll \xi \in \mathfrak{ab}} \sum_{\substack{(a,b) \in (\mathfrak{a} \times \mathfrak{b})/O^\times \\ ab = \xi}} \phi(a, b) \frac{N(a)^k}{|N(a)|} q^\xi. \tag{4.10}$$

4.3 Eisenstein measure

We recall the Eisenstein measure in [Kat78a] with values in $V(\mathfrak{c}; W)$. Recall $\mathbf{p} = \prod_{\mathfrak{P} \in \Sigma_p} \mathfrak{P}$. We split $\mathfrak{O}_p = \mathfrak{O}_{\mathbf{p}} \times \mathfrak{O}_{\mathbf{p}^c} \cong O_p \times O_p$ and write the variable on $\mathfrak{O}_p$ as $(x; y)$ for $x, y \in O_p$. Here $\mathfrak{O}_{\mathbf{p}} = \prod_{\mathfrak{P} \in \Sigma_p} \mathfrak{O}_{\mathfrak{P}}$. We take the closure $\overline{O^\times}$ in $\mathfrak{O}_p$. We let $\varepsilon \in \overline{O^\times}$ act on $\mathfrak{O}_p$ by multiplication: $\varepsilon(x; y) = (\varepsilon x; \varepsilon y)$. Then we define $T = \mathfrak{O}_p/\overline{O^\times}$ and $T^\times = \mathfrak{O}_p^\times/\overline{O^\times}$.

For each continuous function $\phi(x; y)$ on $T^\times$, we consider

$$\phi^\circ(x; y) := \phi(x^{-1}; y). \tag{4.11}$$

The map $\phi \mapsto \phi^\circ$ is a bounded linear operator acting on the space $\mathcal{C}(T^\times; W)$ of all continuous functions on $T^\times$ with values in W into the space of continuous functions on $O_p^\times \times O_p^\times$ with values in W invariant under the following modified action of $\overline{O^\times}$. Indeed the function ϕ° satisfies the following property:

$$\phi^\circ(\varepsilon x; \varepsilon^{-1} y) = \phi^\circ(x; y) \quad \text{for all } \varepsilon \in \overline{O^\times}.$$

This is the property required to define Eisenstein series (for even weight k) in the previous subsection. Then there exists a unique measure $\mathbf{E}_\mathfrak{c} : \mathcal{C}(T^\times; W) \to V(\mathfrak{c}; W)$ with the following two properties.

(E1) If ϕ has values in $\overline{\mathbb{Q}}$ equipped with the discrete topology, then for each positive integer $k > 0$,

$$\mathbf{E}_\mathfrak{c}(N^{-k}\phi) = E_k(\phi^\circ; \mathfrak{c}),$$

where $N : T \to \mathbb{Z}_p^\times$ is given by $N(x; y) = N_{F/\mathbb{Q}}(x)$ for the norm map $N_{F/\mathbb{Q}} : O_p \to \mathbb{Z}_p$.

(E2) The q-expansion of $\mathbf{E}_\mathfrak{c}(\phi)$ at the cusp $(\mathfrak{a}, \mathfrak{b})$ is given by

$$N(\mathfrak{a}) \sum_{0 \ll \xi \in \mathfrak{ab}} q^\xi \sum_{(a,b) \in (\mathfrak{a} \times \mathfrak{b})/O^\times, ab = \xi} \phi^\circ(a; b) |N(a)|^{-1},$$

where $|N(a)|$ is the (complex) absolute value of the norm $N(a)$ of $a \in \mathfrak{a}$, and $\varepsilon \in O^\times$ acts on $(a, b) \in (\mathfrak{a} \times \mathfrak{b})/\overline{O^\times} \subset T$ by $(a, b) \mapsto (\varepsilon a, \varepsilon^{-1} b)$.

The existence and the uniqueness of the measure $\mathbf{E}_\mathfrak{c}$ is a consequence of the q-expansion principle.

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4.4 Katz measure

We evaluate p-adic modular forms f at any test object $(A, \lambda, i_p)_{/W}$ defined over W. This gives rise to a linear form $Ev: V(\mathfrak{c}; W) \to W$ given by $Ev(f) = f(A, \lambda, i_p)$. The evaluation $Ev \circ \mathbf{E}_{\mathfrak{c}}$ is a bounded measure on $\mathcal{C}(T; W)$ with values in W. Now we choose a specific test object. Let $x = [z, g]$ be an ordinary CM point of the Shimura variety. We take the abelian scheme (A, λ, i_p) sitting over $x \in \mathfrak{M}(\mathfrak{c}, p^\infty)$. Thus A is of CM type $(M = M_x, \Sigma)$. For a lattice $\mathfrak{A} \subset M$, let $\Sigma(\mathfrak{A}) = \{(a^\sigma)_{\sigma\in\Sigma} \in \mathbb{C}^\Sigma \mid a \in \mathfrak{A}\}$ which is a lattice in $\mathbb{C}^\Sigma$. The complex manifold $A(\mathbb{C})$ is given by $\mathbb{C}^\Sigma/\Sigma(\mathfrak{A})$ and we can find a model A defined over $\mathcal{W}$. The model with p^∞ level structure i_p is unique (up to isomorphisms; see [Hid10, § 4.3]). We write this model $A(\mathfrak{A})_{/\mathcal{W}}$.

We recall briefly the construction of the Katz measure interpolating the L-values of arithmetic Hecke characters of conductor dividing p^∞. We suppose the following four conditions.

(i) The ideal $\mathfrak{A}$ is a fractional ideal of M prime to p.

(ii) Choose $\delta \in M$ as in (d1)–(d2). Then the alternating form $\langle u, v\rangle = (u^c v - uv^c)/2\delta$ induces $\mathfrak{c}(\mathfrak{A}\mathfrak{A}^c)^{-1}$-polarization $\lambda = \lambda(\mathfrak{A})$ on $A(\mathfrak{A})$.

(iii) Identify $M_{\mathbf{p}} = F_p$ via $F \subset M$, and compose $i_p : \mu_{p^\infty} \otimes \mathfrak{d}^{-1} \cong A(\mathfrak{O})[\mathbf{p}^\infty]$ with the isogeny $A(\mathfrak{O}) \to A(\mathfrak{A})$ inducing the identity: $M_{\mathbf{p}}/\mathfrak{A}_{\mathbf{p}} = M_{\mathbf{p}}/\mathfrak{O}_{\mathbf{p}}$ (as $\mathfrak{A}$ is prime to p). In this way, we get $i'_p : \mu_{p^\infty} \otimes \mathfrak{d}^{-1} \cong A(\mathfrak{A})[\mathbf{p}^\infty]$. We put $i(\mathfrak{A})(x) = i'_p(2\delta x)$.

(iv) Fix a differential $\omega = \omega(\mathfrak{O})$ on $A(\mathfrak{O})_{/\mathcal{W}}$ so that $H^0(A(\mathfrak{O}), \Omega_{A(\mathfrak{O})/\mathcal{W}}) = (\mathcal{W} \otimes_{\mathbb{Z}} O)\omega$. Since $\mathfrak{A}_p = \mathfrak{O}_p$, $A(\mathfrak{O} \cap \mathfrak{A})$ is an étale covering of both $A(\mathfrak{A})$ and $A(\mathfrak{O})$; so, $\omega(\mathfrak{O})$ induces a differential $\omega(\mathfrak{A})$ first by pull-back to $A(\mathfrak{O} \cap \mathfrak{A})$ and then by pull-back inverse from $A(\mathfrak{O} \cap \mathfrak{A})$ to $A(\mathfrak{A})$.

As long as the projection $\pi : A(\mathfrak{O} \cap \mathfrak{A}) \twoheadrightarrow A(\mathfrak{A})$ is étale, the pull-back inverse $(\pi^*)^{-1}$: $\Omega_{A(\mathfrak{O}\cap\mathfrak{A})/\mathcal{W}} \to \Omega_{A(\mathfrak{A})/\mathcal{W}}$ is a surjective isomorphism. We thus have

$$H^0(A(\mathfrak{A}), \Omega_{A(\mathfrak{O})/\mathcal{W}}) = (\mathcal{W} \otimes_{\mathbb{Z}} O)\omega(\mathfrak{A}).$$

Let $Z = \mathrm{Cl}_M(p^\infty) = \varprojlim_n \mathrm{Cl}_M(p^n)$ for the ray-class group $\mathrm{Cl}_M(p^n)$ of M modulo p^n, and write $\mathrm{Cl}_M(1)$ as Cl_M. Identifying $\mathfrak{O}_{\mathbf{p}}$ (respectively $\mathfrak{O}_{\mathbf{p}^c}$) with the first (respectively last) component O_p of $\mathfrak{O}_p$, we bring $T^\times$ into Z. Then we have the exact sequence:

$$T^\times \xrightarrow{\iota} Z \to \mathrm{Cl}_M \to 1 \quad \text{with finite kernel } \mathrm{Ker}(\iota).$$

We write $[\mathfrak{A}]$ for the image of the class of an ideal $\mathfrak{A}$ prime to p in Z. For $\alpha \in \mathfrak{O}$, we have $[(\alpha)] = \alpha^{-1}$, where the right-hand side is the image of the inclusion $\mathfrak{O}_p^\times \to Z$. Choosing a complete representative set $\{\mathfrak{A}\}$ for Cl_M, we have a decomposition $Z = \bigsqcup_{\mathfrak{A}} \mathrm{Im}(\iota)[\mathfrak{A}]^{-1}$. For each function $\phi \in \mathcal{C}(Z; W)$, we define $\phi_{\mathfrak{A}} \in \mathcal{C}(T; W)$ in the following way: $\phi_{\mathfrak{A}}(t) = \phi(t[\mathfrak{A}]^{-1})$ for $t \in T^\times$ and extend it by 0 outside $T^\times$. Then define

$$\int_Z \phi\, d\varphi = \sum_{\mathfrak{A}} \int_T \phi_{\mathfrak{A}}\, d\mathbf{E}_{\mathfrak{c}_{\mathfrak{A}}}(A(\mathfrak{A}), \lambda(\mathfrak{A}), i(\mathfrak{A})), \tag{4.12}$$

where $\mathfrak{c}_{\mathfrak{A}} = \mathfrak{c}(\mathfrak{A}\mathfrak{A}^c)^{-1}$. We write $E_{\mathfrak{A}}(\phi)$ for $\mathbf{E}_{\mathfrak{c}_{\mathfrak{A}}}(\phi)$ for functions $\phi \in \mathcal{C}(T^\times; W)$.

5. Proof of Theorem II

Recall the quadratic CM extension M/F and the CM types (Σ, Σ_p) introduced in § 1. As explained in the introduction, we may (and will) assume that M/F is everywhere unramified at finite places and that the branch character ψ_0 has prime-to-p conductor 1 and order prime to p; so, ψ_0 has values in $W(\mathbb{F})$. Take, as a base ring, a sufficiently large discrete valuation ring $W \subset \mathbb{C}_p$ over $W(\mathbb{F})$. We write ψ for an arithmetic Hecke character with $\widehat{\psi}|_\Delta = \psi_0$ having prime-to-p conductor 1. We now prove Theorem II, and the proof concludes in § 5.5.

5.1 Splitting the Katz measure

We recall the splitting of the Katz measure introduced in [Hid10, § 5.1] to compute the q-expansion of corresponding Eisenstein series. We assume that $p > 2$. Let the triple $(A(\mathfrak{A}), \lambda(\mathfrak{A}), i(\mathfrak{A}))$ be the abelian variety of CM type (M, Σ) with the polarization ideal $\mathfrak{c}_\mathfrak{A} = \mathfrak{c}(\mathfrak{A}\mathfrak{A}^c)^{-1}$ as in § 4.4. We consider the measure $E_\mathfrak{A} : \phi \mapsto \int_T \phi\, d\mathbf{E}_{\mathfrak{c}_\mathfrak{A}}(A(\mathfrak{A}), \lambda(\mathfrak{A}), i(\mathfrak{A}))$ (on the image of $T^\times$ in $\mathrm{Cl}_M(p^\infty)$). For $\alpha \in M$ prime to p, as seen in [Hid10, (5.1)], we have

$$\int_{T^\times} \phi(\alpha t)\, dE_\mathfrak{A}(t) = \int_{T^\times} \phi(t)\, dE_{\alpha\mathfrak{A}}(t), \tag{5.1}$$

where $\alpha(x; y) = (\alpha x, \alpha a; \alpha^c y, \alpha^c b)$ for $t = (x; y)$.

For each function ϕ on $\mathrm{Im}(\iota)[\mathfrak{A}]^{-1}$, we define

$$\phi_\mathfrak{A}(x) = \phi(x[\mathfrak{A}]^{-1}). \tag{5.2}$$

Now we decompose, for an open subgroup H of $T^\times$ containing $\mathrm{Ker}(\iota)$,

$$\iota(T^\times)[\mathfrak{A}]^{-1} = \bigsqcup_{\mathfrak{B}} \iota(H)[\mathfrak{B}]^{-1} \Longleftrightarrow T^\times = \bigsqcup_{\mathfrak{B}} H[\mathfrak{B}^{-1}\mathfrak{A}].$$

Thus, we have

$$\begin{aligned}\int_{T^\times} \phi_\mathfrak{A}(t)\, dE_\mathfrak{A}(t) &= \sum_{\mathfrak{B}} \int_{T^\times} \chi_{H[\mathfrak{B}^{-1}\mathfrak{A}]}\phi_\mathfrak{A}(t)\, dE_\mathfrak{A}(t) \\ &= \sum_{\mathfrak{B}} \int_{T^\times} \chi_H(t[\mathfrak{B}\mathfrak{A}^{-1}])\phi_\mathfrak{B}(t[\mathfrak{B}\mathfrak{A}^{-1}])\, dE_\mathfrak{A}(t) = \sum_{\mathfrak{B}} \int_T \chi_H(t)\phi_\mathfrak{B}(t)\, dE_\mathfrak{B}(t),\end{aligned}$$

where χ_H is the characteristic function of H. Note here that we have $\mathfrak{B} = \alpha\mathfrak{A}$ for $\alpha \in M^\times$.

Recall the decomposition $Z = \Gamma \times \Delta$ in [Hid10, § 5.1] such that the following two conditions hold.

(i) The group $Z = \Gamma \times \Delta$ with torsion-free Γ and a finite group Δ.

(ii) The groups Γ and Δ are stable under c.

For each $z \in Z$, we define $\pi_-(z) = [z]^- := z^{1-c}$. Recall the torus $T_x = \mathrm{Res}_{M/\mathbb{Q}}\mathbb{G}_m \subset G$ fixing the closed point $x \in \mathrm{Sh}^{\mathrm{ord}}$ over $(A(\mathfrak{O}), \lambda(\mathfrak{O}), i(\mathfrak{O})) \in V$ and its quotient $\mathcal{T}$ as in (3.6) (with the injection: $\mathcal{T}(\mathbb{Z}_{(p)}) \hookrightarrow T(\mathbb{Z}_p) = O_p^\times$ sending $\alpha \in T_x(\mathbb{Z}_{(p)})$ to $\alpha^{1-c} \in O_p^\times$). We have an exact sequence:

$$1 \to \mathfrak{O}_p^\times/\overline{\mathfrak{O}^\times} \to Z \to \mathrm{Cl}_M \to 1,$$

where Cl_M is the class group of M. Since $O^\times$ is a subgroup of $\mathfrak{O}^\times$ of finite index and p is unramified in $M/\mathbb{Q}$, $\pi_-(\mathfrak{O}^\times)$ is a finite group of order prime to p. By this fact, we see that

$$\Gamma^- \cap \pi_-(\mathfrak{O}_p^\times/\overline{\mathfrak{O}^\times}) \hookrightarrow \mathfrak{O}_p^\times[-1] \cong O_p^\times,$$

where $\mathfrak{O}_p^\times[-1] = \{a \in \mathfrak{O}_p^\times \mid c(a) = a^{-1}\}$. In particular, identifying $\mathfrak{O}_\mathbf{p}$ with O_p, for a principal ideal (α) prime to p, $[(\alpha)]^- = \alpha_\mathbf{p}^{c-1} \in T(\mathbb{Z}_p) = \mathfrak{O}_\mathbf{p}^\times = O_p^\times$, where $\mathbf{p} = \prod_{\mathfrak{p}\in\Sigma_p} \mathfrak{p}$. Therefore we have, regarding $\mathcal{T}(\mathbb{Z}_{(p)}) \subset T(\mathbb{Z}_p) = O_p^\times$ by (3.6),

$$[\mathfrak{A}]^- \in \mathcal{T}(\mathbb{Z}_{(p)}) \Longleftrightarrow [\mathfrak{A}] \in \{[(\alpha)] \mid \alpha \in \mathfrak{O}_{(p)}^\times\} \cdot (\Gamma^+ \times \Delta^+) \subset Z, \tag{5.3}$$

where $\Delta^+ = H^0(\mathrm{Gal}(M/F), \Delta)$, and $\mathfrak{O}_{(p)} \subset M$ is the localization (not the completion) of $\mathfrak{O}$ at p.

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The translation $\phi(z) \mapsto \phi(z\zeta)$ by $\zeta \in Z$ gives an action of Z on the space of continuous functions $\mathcal{C}(Z; W)$ on Z with values in W. For each character ψ of Z, we write $\mathcal{C}(Z; \mathcal{O})[\psi]$ for the ψ-eigenspace. Then the restriction of continuous functions on Z to Γ gives rise to an isomorphism $\mathrm{Res}: \mathcal{C}(Z; W)[\psi] \cong \mathcal{C}(\Gamma; W)$. We write Inf_ψ for Res^{-1}. For a given measure φ on Z, the ψ-projection $\varphi_\psi \in W[[\Gamma]]$ is defined by

$$\int_\Gamma \phi \, d\varphi_\psi = \int_Z \mathrm{Inf}_\psi \phi \, d\varphi.$$

By definition, we have $\int_\Gamma \phi \, d\varphi_{\psi\epsilon} = \int_\Gamma \phi\epsilon \, d\varphi_\psi$ if ϵ factors through Γ.

More generally, for any finite order character ϵ of $H \cong \mathbb{Z}_p^r$ with values in $W^\times$ and a measure φ on H, we define $\varphi_\epsilon = \epsilon * \varphi$ by $\int_\Gamma \phi \, d\varphi_\epsilon = \int_\Gamma \phi\epsilon \, d\varphi$. It is plain that $\mu(\varphi) = \mu(\varphi_\epsilon)$. However, if we have a nontrivial projection $\pi : H \to H'$ for $H' \cong \mathbb{Z}_p^s$ $(0 < s < r)$ and if ϵ does not factor through π, we often have $\mu(\pi_*\varphi) \neq \mu(\pi_*(\epsilon * \varphi))$ as the convolution product $\varphi \mapsto \epsilon * \varphi$ does not commute with the push-forward along π. Since $0 \leqslant \mu(\varphi) \leqslant \mu(\pi_*(\epsilon * \varphi))$ for any ϵ, we get the following lemma.

LEMMA 5.1. *We have $\mu(\varphi) = 0$ if and only if $\liminf_\epsilon \mu(\pi_*(\epsilon * \varphi)) = 0$, where ϵ runs over all finite order characters of H.*

5.2 Good representatives

We want to choose good representatives D so that $D \cong Z/\Gamma'$ for the intersection Γ' of Γ and the image of $\mathfrak{O}_p^\times$. Let $\mathcal{I}(p)$ be the group of fractional ideals of M prime to p, and define $\mathcal{I}(p)^+ = \{\mathfrak{A} \in \mathcal{I}(p) \mid \mathfrak{A}^{1-c} = \alpha^{1-c}\mathfrak{O}$ for $\alpha \in M^\times\}$. Since $\mathfrak{A}$ is prime to p, α^{1-c} is prime to p. Thus if a prime factor $\mathfrak{P}$ of p divides the principal ideal (α), its conjugate $\mathfrak{P}^c$ divides (α) with the equal multiplicity. Thus $\alpha = \beta\gamma$ for $\gamma \in F^\times$ with β prime to p, as M/F is everywhere unramified. In other words, $(\beta^{1-c}) = (\alpha^{1-c}) = \mathfrak{A}^{1-c}$, and hence we can write

$$\mathcal{I}(p)^+ = \{\mathfrak{A} \in \mathcal{I}(p) \mid \mathfrak{A}^{1-c} = \alpha_{\mathfrak{A}}^{1-c}\mathfrak{O} \text{ for } \alpha_{\mathfrak{A}} \in M^\times \text{ prime to } p\}. \tag{5.4}$$

The quotient of $\mathcal{I}(p)^+$ by principal ideals prime to p is a subgroup of the class group Cl_M of M. If $\mathfrak{A} \in \mathcal{I}(p)^+$, we have $(\mathfrak{A}\alpha_{\mathfrak{A}}^{-1})^c = \mathfrak{A}\alpha_{\mathfrak{A}}^{-1}$. Since M/F is everywhere unramified, this group is the image $\overline{\mathrm{Cl}}_F$ of Cl_F in Cl_M. We see easily that

$$\overline{\mathrm{Cl}}_F = \frac{\mathcal{I}(p)^+}{\text{principal ideals}} \subset H^0(\mathrm{Gal}(M/F), \mathrm{Cl}_M).$$

We take a complete representative set D^- (respectively D^+) for $\mathrm{Cl}_M/\overline{\mathrm{Cl}}_F$ (respectively $\overline{\mathrm{Cl}}_F$ in F-ideals). Hereafter, as convention, we use lower case Gothic letters for fractional F-ideals and upper case for a fractional M-ideal (which may come from an F-ideal). Thus we write $\mathfrak{a} \in D^+$ since we have chosen a representative in D^+ from F-ideals. We write Γ' for the intersection of Γ with the image of $\mathfrak{O}_p^\times$ in the group $Z = \mathrm{Cl}_M(p^\infty)$. Write $\mathcal{D}$ for a complete representative set in the localization (not the completion) $\mathfrak{O}_{(p)}^\times$ for $\pi(\mathfrak{O}_p^\times)/\Gamma'$ with the projection $\pi : \mathfrak{O}_p^\times \to Z$. Then we have

$$\int_Z \phi \, d\varphi = \sum_{\mathfrak{a} \in D^+} \sum_{\alpha \in \mathcal{D}} \sum_{\mathfrak{B} \in D^-} \int_{\Gamma'} \phi_{\mathfrak{a}\alpha\mathfrak{B}}(z) \, dE_{\mathfrak{a}\alpha\mathfrak{B}}.$$

5.3 Operation by ideals on p-adic modular forms

The Katz measure is a finite sum of evaluation of the Eisenstein measure at different CM points. We introduce two algebraic operations by ideals on modular forms in order to make the evaluating CM point unique.

Central action. Since we have, for an O-ideal $\mathfrak{a}$ prime to the level p^n of $(A(\mathfrak{B}), \lambda(\mathfrak{B}), i(\mathfrak{B}))$

$$A(\mathfrak{B}\mathfrak{a})(\mathbb{C}) = \mathbb{C}^\Sigma/\Sigma(\mathfrak{a}\mathfrak{B}) = \mathbb{C}^\Sigma/\Sigma(\mathfrak{B}) \otimes_O \mathfrak{a} = A(\mathfrak{B})(\mathbb{C}) \otimes_O \mathfrak{a},$$

we conclude $A(\mathfrak{a}\mathfrak{B}) = A(\mathfrak{B}) \otimes_O \mathfrak{a}$. There is another construction of $A(\mathfrak{A}) \otimes \mathfrak{a}$: tensoring $A(\mathfrak{B})$ with the exact sequence, $\mathfrak{a} \hookrightarrow O \twoheadrightarrow O/\mathfrak{a}$, we get another one,

$$0 \to \mathrm{Tor}_1(O/\mathfrak{a}, A(\mathfrak{B})) \xrightarrow{i} A(\mathfrak{B}) \otimes_O \mathfrak{a} \to A(\mathfrak{B}) \to 0.$$

Since O is a Dedekind domain, we have $\mathrm{Tor}_1(O/\mathfrak{a}, A(\mathfrak{B})) \cong A(\mathfrak{B})[\mathfrak{a}]$ canonically. Thus i brings $A(\mathfrak{B})[\mathfrak{a}]$ onto $(A(\mathfrak{B}) \otimes_O \mathfrak{a})[\mathfrak{a}]$. Since $\lambda(\mathfrak{B})$ is a $\mathfrak{c}_\mathfrak{B}$-polarization for $\mathfrak{c}_\mathfrak{B} = \mathfrak{c}(\mathfrak{B}\mathfrak{B}^c)^{-1}$, we have $A(\mathfrak{B})^t \xrightarrow{\lambda(\mathfrak{B})} A(\mathfrak{B}) \otimes \mathfrak{c}_\mathfrak{B}$. This induces

$$\lambda(\mathfrak{B}) \otimes \mathfrak{a} : (A(\mathfrak{B}) \otimes \mathfrak{a})^t \cong A(\mathfrak{B})^t/A(\mathfrak{B})^t[\mathfrak{a}] \cong (A(\mathfrak{B}) \otimes_O \mathfrak{c}_\mathfrak{B}) \otimes_O \mathfrak{a}^{-1} = (A(\mathfrak{B}) \otimes_O \mathfrak{a}) \otimes \mathfrak{c}_{\mathfrak{a}\mathfrak{B}}.$$

We check $\lambda(\mathfrak{B}) \otimes \mathfrak{a} = \lambda(\mathfrak{a}\mathfrak{B})$. Since $\mathfrak{a}$ is prime to p, the quotient process does not alter the level structure; so, $i(\mathfrak{B})$ induces $i(\mathfrak{a}\mathfrak{B}) = i(\mathfrak{B}) \otimes \mathfrak{a}$.

The above process of making $(A(\mathfrak{a}\mathfrak{B}), \lambda(\mathfrak{a}\mathfrak{B}), i(\mathfrak{a}\mathfrak{B}))$ can be performed for general triples (A, λ, i) (even without complex multiplication) and yields a functorial map from test object (A, λ, i) with polarization ideal $\mathfrak{c}$ to test objects $(A \otimes_O \mathfrak{a}, \lambda \otimes \mathfrak{a}, i \otimes \mathfrak{a})$ with polarization ideal $\mathfrak{c}\mathfrak{a}^{-2}$. For a p-adic modular form $f \in V(\mathfrak{c}\mathfrak{a}^{-2}, \mathfrak{N}; R)$, we define $f|\langle\mathfrak{a}\rangle \in V(\mathfrak{c}, \mathfrak{N}; R)$ by

$$f|\langle\mathfrak{a}\rangle(A, \lambda, i) = f(A \otimes_O \mathfrak{a}, \lambda \otimes \mathfrak{a}, i \otimes \mathfrak{a}) \tag{5.5}$$

for a fractional ideal $\mathfrak{a}$ of F prime to $\mathfrak{N}$ (see [Hid04, 4.1.9]). This shows

$$\int_{\Gamma'} \phi_{\mathfrak{a}\mathfrak{B}}\, dE_{\mathfrak{a}\mathfrak{B}} = (\mathbf{E}(\chi_{\Gamma'}\phi_{\mathfrak{a}\mathfrak{B}})|\langle\mathfrak{a}\rangle)(A(\mathfrak{B}), \lambda(\mathfrak{B}), i(\mathfrak{B})) \quad \text{for } \mathbf{E} = \mathbf{E}_{\mathfrak{c}_{\mathfrak{a}\mathfrak{B}}}, \text{ if } \mathfrak{a} \subset F. \tag{5.6}$$

Level raising action. We can construct another operator $[\mathfrak{q}] : V(\mathfrak{c}\mathfrak{q}, \mathfrak{N}; R) \to V(\mathfrak{c}, \mathfrak{N}\mathfrak{q}; R)$ in the following way. Here we assume that $\mathfrak{q}$ is an integral ideal prime to $\mathfrak{c}p$. For each test object $(A, \lambda, \omega, i)_{/C}$ (over a p-adic R-algebra C) of level $\mathfrak{N}\mathfrak{q}p^\infty$ with polarization ideal $\mathfrak{c}$, we define a new test object $(A', \lambda', \omega', i')$. First define $A' = A/i(\mathfrak{q}^*/\mathfrak{d}^{-1})$. The quotient exists over C, since $i(\mathfrak{q}^*/\mathfrak{d}^{-1})$ is an étale subgroup of A (because C is a p-adic ring). The level structure $i : (F_p/\mathfrak{d}_p^{-1}) \times ((\mathfrak{N}\mathfrak{q})^*/\mathfrak{d}^{-1}) \to A$ composed with the quotient map $\pi : A \to A'$ induces, modulo $\mathfrak{q}^*/\mathfrak{d}^{-1}$, the level structure $i' : F_p/\mathfrak{d}_p^{-1} \times \mathfrak{N}^*/\mathfrak{d}^{-1} \to A'$ defined over C. The $\mathfrak{c}\mathfrak{q}$-polarization $\lambda' : A'^t \cong A' \otimes \mathfrak{c}\mathfrak{q}$ is defined as follows: tensoring the sequence $0 \to \mathfrak{q} \to O \to O/\mathfrak{q} \to 0$ with $A^t = A \otimes \mathfrak{c}$, we have another exact sequence

$$0 \to A \otimes \mathfrak{c}\mathfrak{q}[\mathfrak{q}] \to A \otimes \mathfrak{c}\mathfrak{q} \to A \otimes \mathfrak{c} \to 0.$$

Taking dual of the quotient map $\pi : A \to A'$, we have one more exact sequence

$$0 \to \mathrm{Hom}(i(\mathfrak{q}^*/\mathfrak{d}^{-1}), \mathbb{G}_m) \to A'^t \xrightarrow{\pi^t} A^t \to 0,$$

which gives rise to the following exact sequence:

$$0 \to \mathrm{Hom}(i(\mathfrak{q}^*/\mathfrak{d}^{-1}), \mathbb{G}_m) \to A'^t[\mathfrak{q}] \xrightarrow{\lambda\circ\pi^t} i(\mathfrak{q}^*/\mathfrak{d}^{-1}) \otimes \mathfrak{c} \to 0.$$

Since $\mathfrak{q}$ is prime to $\mathfrak{c}$, the kernel of the composite: $(\pi \otimes \mathrm{id}) \circ \lambda \circ \pi^t : A'^t \to A' \otimes \mathfrak{c}$ is the entire $\mathfrak{q}$-torsion subgroup $A'^t[\mathfrak{q}]$. Since $A'^t/A'^t[\mathfrak{q}] = A'^t \otimes \mathfrak{q}^{-1}$, we have constructed an isomorphism:

$$(\pi \otimes \mathrm{id}) \circ \lambda \circ \pi^t : A'^t \otimes \mathfrak{q}^{-1} \cong A' \otimes \mathfrak{c}.$$

Tensoring $\mathfrak{q}$ with this isomorphism, we get the desired $\lambda' : A'^t \cong A' \otimes \mathfrak{c}\mathfrak{q}$. Since $\mathfrak{q}$ is prime to p, on a p-adic algebra C, $\mathrm{Lie}(A) \cong \mathrm{Lie}(A')$, which implies that $\omega' = \pi^*\omega$ is a well-defined generator of $\Omega_{A'/C}$. The association $(A,\lambda,\omega,i)_{/C}\mapsto(A',\lambda',\omega',i')_{/C}$ is functorial (i.e., a morphism between the functors $\mathcal{Q}$ in (4.1) with respect to $(\mathfrak{c},\mathfrak{N}\mathfrak{q}p^\infty)$ and $(\mathfrak{c}\mathfrak{q},\mathfrak{N}p^\infty)$). We have

$$[\mathfrak{q}]:V(\mathfrak{c}\mathfrak{q},\mathfrak{N};R)\to V(\mathfrak{c},\mathfrak{N}\mathfrak{q};R)\quad\text{and}\quad[\mathfrak{q}]:G_\kappa(\mathfrak{c}\mathfrak{q},\mathfrak{N};R)\to G_\kappa(\mathfrak{c},\mathfrak{N}\mathfrak{q};R)$$

by $f|[\mathfrak{q}](A,\lambda,\omega,i)=f(A',\lambda',\omega',i')$.

We compute $[\mathfrak{q}](A(\mathfrak{A}),\lambda(\mathfrak{A}),i(\mathfrak{A}))_{/\mathcal{W}}$ for a fractional ideal $\mathfrak{A}\subset M$, supposing that all prime factors of $\mathfrak{q}$ are split in M/F. Choose an integral ideal $\mathfrak{Q}$ in M such that the inclusion $O\hookrightarrow\mathfrak{O}$ induces $O/\mathfrak{q}\cong\mathfrak{O}/\mathfrak{Q}$. Then $\mathfrak{Q}+\mathfrak{Q}^c=\mathfrak{O}$. Consider $(A(\mathfrak{A}),\lambda(\mathfrak{A}),i(\mathfrak{A}))$ with the level $\mathfrak{q}p^\infty$-structure $i(\mathfrak{A})$ sending $x\in\mathfrak{q}^*/\mathfrak{d}^{-1}$ to $2\delta x\in\mathfrak{Q}^{-1}\mathfrak{A}/\mathfrak{A}$. Then $A(\mathfrak{A})[\mathfrak{Q}]=i(\mathfrak{A})(\mathfrak{q}^*/\mathfrak{d}^{-1})$ and hence $A(\mathfrak{A})/i(\mathfrak{A})(\mathfrak{q}^*/\mathfrak{d}^{-1})=A(\mathfrak{A}\mathfrak{Q}^{-1})$ and $i(\mathfrak{A})'=i(\mathfrak{A}\mathfrak{Q}^{-1})$, which are the level p^∞-structure. Since $\mathfrak{q}$ is prime to $p\mathfrak{c}$, using the fact that $\mathfrak{Q}\mathfrak{Q}^c=\mathfrak{q}$, we can verify that

$$[\mathfrak{q}](A(\mathfrak{A}),\lambda(\mathfrak{A}),\omega(\mathfrak{A}),i(\mathfrak{A}))_{/\mathcal{W}}\cong(A(\mathfrak{A}\mathfrak{Q}^{-1}),\lambda(\mathfrak{A}\mathfrak{Q}^{-1}),\omega(\mathfrak{A}\mathfrak{Q}^{-1}),i(\mathfrak{A}\mathfrak{Q}^{-1}))_{/\mathcal{W}},\tag{5.7}$$

where $i(\mathfrak{A})$ is the level $\mathfrak{q}p^\infty$-structure as above and $i(\mathfrak{A}\mathfrak{Q}^{-1})$ is the induced level p^∞-structure. We can always choose $\mathfrak{Q}\in D^+$ so that $\mathfrak{Q}^c+\mathfrak{Q}=\mathfrak{O}$ and $\mathfrak{O}/\mathfrak{Q}\cong O/\mathfrak{q}$ for $\mathfrak{q}=\mathfrak{Q}\cap F$. This shows

$$\int_{\Gamma'}\phi_{\mathfrak{Q}^{-1}\mathfrak{B}}\,dE_{\mathfrak{Q}^{-1}\mathfrak{B}}=(\mathbf{E}(\chi_{\Gamma'}\phi_{\mathfrak{Q}^{-1}\mathfrak{B}})|[\mathfrak{q}])(A(\mathfrak{B}),\lambda(\mathfrak{B}),i(\mathfrak{B}))\quad\text{for }\mathfrak{Q}\text{ and }\mathfrak{q}\text{ as above},\tag{5.8}$$

where $\mathbf{E}=\mathbf{E}_{\mathfrak{c}_{\mathfrak{A}\mathfrak{B}}}$. As for the effect of $\alpha\in M^\times$, we may assume either $\alpha\in O$ or $\alpha\in\mathfrak{O}-O$. When $\alpha\in\mathfrak{O}-O$, we assume that $(\alpha)+(\alpha^c)=\mathfrak{O}$. Then we have, for the characteristic function $\chi_{\Gamma'}$ of Γ',

$$\int_{\Gamma'}\phi_{\alpha\mathfrak{B}}\,dE_{\alpha\mathfrak{B}}=(\mathbf{E}(\chi_{\Gamma'}\phi_{\alpha\mathfrak{B}})|\langle\alpha\rangle)(A(\mathfrak{B}),\lambda(\mathfrak{B}),i(\mathfrak{B}))\quad\text{if }\alpha\in O\cap F^\times,\tag{5.9}$$

$$\int_{\Gamma'}\phi_{\alpha^{-1}\mathfrak{B}}\,dE_{\alpha^{-1}\mathfrak{B}}=(\mathbf{E}(\chi_{\Gamma'}\phi_{\alpha^{-1}\mathfrak{B}})|[\alpha\alpha^c])(A(\mathfrak{B}),\lambda(\mathfrak{B}),i(\mathfrak{B}))\quad\text{if }\alpha\notin O,\tag{5.10}$$

where $\mathbf{E}=\mathbf{E}_\mathfrak{c}$ for $\mathfrak{c}=\mathfrak{c}_{\alpha\mathfrak{B}}$ for (5.9) and $\mathfrak{c}=\mathfrak{c}_{\alpha^{-1}\mathfrak{B}}$ for (5.10).

5.4 Operation by ideles on p-adic modular forms

We interpret the ideal action as pull-back of the action of $G(\mathbb{A}^{(\infty)})$ on the Shimura variety. Pick an element $g\in G(\mathbb{A}^{(\infty)})$ with totally positive $\det(g)\in F$. Then g induces an automorphism of the Shimura variety (see (3.4)), and hence the functorial action of g on test objects. We write

$$g(A,\lambda,i)=(A,\lambda_g,i_g)$$

for the image of a test object (A,λ,i) under the action of g. Here, writing $T(A)=\varprojlim_N A[N]$ for the Tate module, the level structure is an isomorphism $i:F^2_{\mathbb{A}^{(\infty)}}\cong T(A)\otimes_\mathbb{Q}\mathbb{A}^{(\infty)}$, where $F^2_{\mathbb{A}^{(\infty)}}$ is made up of row vectors on which $G(\mathbb{A}^{(\infty)})$ acts from the right. Then we have $i_g=i\circ g$ and $\lambda_g=\det(g)\lambda$. When $g=\gamma\in G(\mathbb{Q})_+$, we have an isogeny $\widetilde{\gamma}:(A',\lambda',\widetilde{\gamma}\circ i')\to(A,\lambda_\gamma,i_\gamma=i\circ\gamma)$ for a suitable A' (see below (L2)). Thus we can interpret the action as an action of an isogeny in this case. This follows from the following three facts for $\gamma\in G(\mathbb{Q})_+$ and test objects over $\mathbb{C}$.

(L1) Writing $\mathcal{L}_z^{\mathfrak{b},\mathfrak{a}^*}=(\mathfrak{b},\mathfrak{a}^*)^t(z,1)=\mathfrak{b}z+\mathfrak{a}^*$ and $i_z((b)\bmod\mathfrak{b}\oplus\mathfrak{a}^*)=bz+a\bmod\mathcal{L}_z^{\mathfrak{b},\mathfrak{a}^*}$, we have $\mathcal{L}_{\gamma(z)}^{(\mathfrak{b},\mathfrak{a}^*)\gamma^{-1}}\cong\mathcal{L}_z^{\mathfrak{b},\mathfrak{a}^*}$ by $w\mapsto w(cz+d)^{-1}$, where $\gamma=\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)$.

(L2) The identity $i_{\gamma(z)}=(cz+d)i_z\circ\gamma$; so, $A'=\mathbb{C}^I/\mathcal{L}_{\gamma(z)}^{(\mathfrak{b},\mathfrak{a}^*)\gamma^{-1}}$ and $\widetilde{\gamma}(w+\mathcal{L}_{\gamma(z)}^{(\mathfrak{b},\mathfrak{a}^*)\gamma^{-1}})=(cz+d)^{-1}w+\mathcal{L}_z^{\mathfrak{b},\mathfrak{a}^*}$.

(L3) We have the identity of the Tate module via i_z:

$$T(\mathbb{C}^I/\mathcal{L}_z^{\mathfrak{b},\mathfrak{a}^*})\cong\widehat{\mathfrak{b}}\oplus\widehat{\mathfrak{a}}^*\quad\text{and}\quad T(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q))=\widehat{\mathfrak{b}}\oplus\widehat{\mathfrak{a}}^*\ (\widehat{\mathfrak{x}}=\mathfrak{x}\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}).$$

If we have an isogeny $\alpha:A\to A$, we have $\alpha(A,\lambda,i)=(A,\lambda',i')$ given by $\lambda'=\alpha\alpha^*\lambda$ and $i'(x)=\alpha i(x)$. Here $\alpha^*=\lambda\circ\alpha^t\circ\lambda^{-1}$, which is $\alpha\alpha^*\mathfrak{c}$-polarization. In other words, defining $\rho(\alpha)\in G(\mathbb{Q})$ by $\alpha i=i\circ\rho(\alpha)$, we find that $\rho(\alpha)^{-1}(\alpha(A,\lambda,i))=(A,\lambda,i)$. Since the Shimura variety classifies the triples up to isogeny, $\alpha(A,\lambda,i)$ and (A,λ,i) are equal as a point of $\mathrm{Sh}(G,\mathfrak{X})$, and $\mathrm{Im}(\rho)$ gives rise to the stabilizer of the point of $\mathrm{Sh}(G,\mathfrak{X})$ represented by (A,λ,i) (see Proposition 3.4 and [Hid10, Corollary 3.5]). When we consider the level structure i modulo a subgroup $K\subset G(\mathbb{A}^{(\infty)})$, we write (A,λ,i_K). Then $g(A,\lambda,i_K)=(A,\lambda_g,(i\circ g)_{g^{-1}Kg})$ is well defined for $g\in G(\mathbb{A}^{(\infty)})$ with $\det(g)\in F_+^\times$.

We now consider the Tate AVRM: $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$. For each positive integer N, we have a canonical exact sequence [Kat78a, (1.1.15)], $1\to\mu_N\otimes\mathfrak{a}^*\to\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)[N]\to\mathfrak{b}/N\mathfrak{b}\to 0$. We therefore have a canonical level structure i_{can} modulo an (integral) upper unipotent subgroup $U=U(\widehat{\mathbb{Z}})\subset G(\mathbb{A}^{(\infty)})$, which is represented by the following exact sequence tensored by $\mathbb{A}^{(\infty)}$ (over $\widehat{\mathbb{Z}}$):

$$0\to\widehat{\mathfrak{a}}^*(1)\to T(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q))\to\widehat{\mathfrak{b}}\to 0,$$

where $\widehat{\mathfrak{b}}=\widehat{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathfrak{b}$ and $\widehat{\mathfrak{a}}^*(1)=\widehat{\mathbb{Z}}(1)\otimes_{\mathbb{Z}}\mathfrak{a}^*$. For the row vector space $\widehat{\mathfrak{b}}\oplus\widehat{\mathfrak{a}}^*$, let

$$K=K_{\mathfrak{a},\mathfrak{b}}=\{g\in G(\mathbb{A}^{(\infty)})\mid(\widehat{\mathfrak{b}}\oplus\widehat{\mathfrak{a}}^*)g=\widehat{\mathfrak{b}}\oplus\widehat{\mathfrak{a}}^*\}.$$

Thus $\Gamma_{11}(O;\mathfrak{a},\mathfrak{b})=\mathrm{SL}_2(F)\cap K_{\mathfrak{a},\mathfrak{b}}$. Define

$$K(\mathfrak{N})=K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{N})=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in K_{\mathfrak{a},\mathfrak{b}}\;\middle|\;c\in\widehat{\mathfrak{N}\mathfrak{a}\mathfrak{b}\mathfrak{d}},\,a\equiv d\equiv 1\bmod\mathfrak{N}\widehat{O}\right\}.$$

Then we have $\Gamma_{11}(\mathfrak{N};\mathfrak{a},\mathfrak{b})=\mathrm{SL}_2(F)\cap K(\mathfrak{N})$. For each given $g\in G(\mathbb{A}^{(\infty)})$ with totally positive $\det(g)\in F$ (so, $g\in\overline{\mathcal{E}}(G,\mathfrak{X})$), we can find finite ideles $a(g),b(g)\in\mathbb{A}^{(\infty)}$ such that $g=u(g)\left(\begin{smallmatrix}b(g)&*\\0&a(g)\end{smallmatrix}\right)$ with $u(g)\in K\cap\mathrm{SL}_2(F_{\mathbb{A}})$ and $\left(\begin{smallmatrix}1&*\\0&1\end{smallmatrix}\right)\in K$. Let (A,λ,i) be as in (L1)–(L3), and put $i_{K(\mathfrak{N})}=(i\bmod K(\mathfrak{N}))$. Having $(A,\lambda,i_{K(\mathfrak{N})})$ is equivalent to having $T(A)=i(\widehat{\mathfrak{b}}\oplus\widehat{\mathfrak{a}}^*)$ and $i_{K(\mathfrak{N})}:(\mathfrak{N}\mathfrak{a})^*/\mathfrak{a}^*\hookrightarrow A[\mathfrak{N}]$. The ideles $a(g)$ and $b(g)$ are determined uniquely modulo multiple of units in $\widehat{O}$. We assume here that $a(g)_{\mathfrak{N}}=b(g)_{\mathfrak{N}}=1$.

Write simply $\mathfrak{a}'=a(g)^{-1}\mathfrak{a}$ and $\mathfrak{b}'=b(g)\mathfrak{b}$ and $K^g=g^{-1}Kg$. We have a canonical identification $\widehat{\mathfrak{b}'}\oplus\widehat{\mathfrak{a}'}^*(1)=T(\mathrm{Tate}_{\mathfrak{a}',\mathfrak{b}'}(q))$ and

$$i^{\mathfrak{a}',\mathfrak{b}'}_{\mathrm{can},K^g}:\mu_{\mathfrak{N}}\otimes(\mathfrak{N}\mathfrak{a}')^*/\mathfrak{a}'*\hookrightarrow\mathbb{G}_m\otimes(\mathfrak{a}')^*\twoheadrightarrow\mathrm{Tate}_{\mathfrak{a}',\mathfrak{b}'}(q).$$

Since $\mathfrak{b}'\oplus\mathfrak{a}'^*$ and $\mathfrak{b}\oplus\mathfrak{a}^*$ are commensurable, the two Tate AVRMs $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ and $\mathrm{Tate}_{\mathfrak{a}',\mathfrak{b}'}(q)$ are isogenous. Since $\widehat{(a(g)\mathfrak{a}^*)}=a(g)^{-1}\widehat{\mathfrak{a}}^*$ and $b(g)\widehat{\mathfrak{b}}=\widehat{b(g)\mathfrak{b}}$, up to isogenies, we have from (L3)

$$\begin{aligned}&g(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q),\lambda^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}},i^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can},K(\mathfrak{N})})\\&\quad=(\mathrm{Tate}_{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}}(q),\det(g)\lambda^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}}=\lambda^{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}}_{\mathrm{can}},i^{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}}_{\mathrm{can},K(\mathfrak{N})^g}\circ u(g)).\end{aligned}\tag{5.11}$$

If $g\in F^\times$, then $a(g)=b(g)=g$, and we have an isogeny

$$g:(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q),\lambda^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}},i^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}})\to(\mathrm{Tate}_{g^{-1}\mathfrak{a},g\mathfrak{b}}(q),\lambda^{g^{-1}\mathfrak{a},g\mathfrak{b}}_{\mathrm{can}},g\circ i^{g^{-1}\mathfrak{a},g\mathfrak{b}}_{\mathrm{can}})$$

https://doi.org/10.1112/S0010437X10005257 Published online by Cambridge University Press

induced by $q \mapsto q^g$ (or equivalently, by $\mathbb{G}_m \otimes \mathfrak{a}^* \to \mathbb{G}_m \otimes (g^{-1}\mathfrak{a}^*)$ given by $x \otimes a \mapsto x \otimes ga$). Therefore the central rational element acts on the Tate AVRM trivially.

For the p-adic Eisenstein series $E(\phi) = E_0(\phi; \mathfrak{c})$ (weight 0) of a function $\phi(x; y)$ ($(x; y) \in \mathfrak{a}_p \times \mathfrak{b}_p$), we find from the above computation (assuming $a(g)_p = b(g)_p = 1$):

$$E(\phi)(g(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}}, i^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}})) = E(\phi|u(g))(\mathrm{Tate}_{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}}(q), \lambda^{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}}_{\mathrm{can}}, i^{a(g)^{-1}\mathfrak{a},b(g)\mathfrak{b}}_{\mathrm{can}}),$$

where $\phi|u(g)(x; y) = P^{-1}(P\phi((x; y)u(g)))$ (letting the 2×2-matrix $u(g)$ act from the right on the row vector $(x; y)$) for the partial Fourier transform $\phi \mapsto P\phi$ as in § 4.2.

We compute the q-expansion of $E(\phi)|\langle\mathfrak{u}\rangle$ for an integral ideal $\mathfrak{u}$ of F. This is the special case of (5.11) when g is a scalar matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ with $a_{\mathfrak{N}} = 1$ (and $a\widehat{O} = \widehat{\mathfrak{u}}$). By construction, we have a homomorphism $\underline{q} : \mathfrak{b} \hookrightarrow \mathbb{G}_m \otimes_{\mathbb{Z}} \mathfrak{a}^*$. Since the $\mathfrak{u}$-torsion points of $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)$ is given by $\underline{q}(\mathfrak{b}\mathfrak{u}^{-1}/\mathfrak{b}) \oplus (\mu_{\mathfrak{N}} \otimes \mathfrak{a}^*)[\mathfrak{u}]$. Thus $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) \otimes \mathfrak{u}^{-1} = \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)/\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q)[\mathfrak{u}] = \mathrm{Tate}_{\mathfrak{a}\mathfrak{u},\mathfrak{b}\mathfrak{u}^{-1}}(q)$. From this, it is easy to see (cf. [Hid04, (4.53)])

$$(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) \otimes \mathfrak{u}, \lambda^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}} \otimes \mathfrak{u}, i^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}} \otimes \mathfrak{u}) = (\mathrm{Tate}_{\mathfrak{a}\mathfrak{u}^{-1},\mathfrak{b}\mathfrak{u}}(q), \lambda^{\mathfrak{a}\mathfrak{u}^{-1},\mathfrak{b}\mathfrak{u}}_{\mathrm{can}}, i^{\mathfrak{a}\mathfrak{A}^{-1},\mathfrak{b}\mathfrak{A}}_{\mathrm{can}}). \tag{5.12}$$

We compute $[\mathfrak{q}](\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda_{\mathrm{can}}, \omega_{\mathrm{can}}, i_{\mathrm{can}})$ for an ideal $\mathfrak{q} \subset O$. Recall $\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) = \mathbb{G}_m \otimes \mathfrak{a}^*/\underline{q}(\mathfrak{b})$. Tensoring $\mathbb{G}_m \otimes \mathfrak{a}^*$ with $0 \to O \to \mathfrak{q}^{-1} \to \mathfrak{q}^{-1}/O \to 0$, we have another exact sequence:

$$0 \to (\mathbb{G}_m \otimes \mathfrak{a}^*)[\mathfrak{q}] \to \mathbb{G}_m \otimes \mathfrak{a}^* \to \mathbb{G}_m \otimes (\mathfrak{a}\mathfrak{q})^* \to 0.$$

Taking the quotient by $\underline{q}(\mathfrak{b})$, we get the following exact sequence:

$$0 \to (\mathbb{G}_m \otimes \mathfrak{a}^*)[\mathfrak{q}] \xrightarrow{i_{\mathrm{can}}} \mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q) \to \mathrm{Tate}_{\mathfrak{a}\mathfrak{q},\mathfrak{b}}(q) \to 0.$$

Then going back to the construction of the Tate quadruples in [Kat78a, 1.1] (and [HT93, 1.7]), we verify

$$[\mathfrak{q}](\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}(q), \lambda^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}}, \omega^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}}, i_{\mathrm{can},K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{N}\mathfrak{q})}) = (\mathrm{Tate}_{\mathfrak{a}\mathfrak{q},\mathfrak{b}}(q), \lambda^{\mathfrak{a}\mathfrak{q},\mathfrak{b}}_{\mathrm{can}}, \omega^{\mathfrak{a}\mathfrak{q},\mathfrak{b}}_{\mathrm{can}}, i_{\mathrm{can},K_{\mathfrak{a}\mathfrak{q},\mathfrak{b}}(\mathfrak{N})}). \tag{5.13}$$

The above action $[\mathfrak{q}]$ corresponds to the action of $g = \left(\begin{smallmatrix} 1 & 0 \\ 0 & q^{-1} \end{smallmatrix}\right)$ for a finite idele q with $q\widehat{O} = \widehat{\mathfrak{q}}$ and $q_{\mathfrak{N}} = 1$. This follows from (5.11) combined with the fact that $K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{N}\mathfrak{q})^g = K_{\mathfrak{a}\mathfrak{q},\mathfrak{b}}(\mathfrak{N})$.

Now we further suppose that $K_{\mathfrak{a},\mathfrak{b}}g = K_{\mathfrak{a},\mathfrak{b}}\gamma$ for $\gamma \in G_+(\mathbb{Q})$ and $g_{\mathfrak{N}} = 1$. Then $u = g\gamma^{-1} \in K_{\mathfrak{a},\mathfrak{b}}$, and hence $u_{\mathfrak{N}} = \gamma_{\mathfrak{N}}^{-1}$. This shows

$$g(\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}, \lambda^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}}, \omega^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}}, i_{\mathrm{can},K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{N})}) = (\mathrm{Tate}_{\mathfrak{a},\mathfrak{b}}, \lambda^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}}, \omega^{\mathfrak{a},\mathfrak{b}}_{\mathrm{can}}, i_{\mathrm{can},K_{\mathfrak{a},\mathfrak{b}}(\mathfrak{N})} \circ \gamma_{\mathfrak{N}}^{-1}). \tag{5.14}$$

5.5 Linear independence of Eisenstein series

We look at $\mathbf{E}_{\mathfrak{c}}(N^{-k}\phi)$ for a positive k. Take an arithmetic Hecke character (of conductor p) $\eta = \eta_k$ with $\eta((\alpha)) = \alpha^{-k}$ if $\alpha \equiv 1 \bmod p$. Recall $T = (O_p^\times) \times (O_p^\times)/\overline{O^\times}$. Then the variable on T is written as $(x; y)$ with $x, y \in O_p$. Write $\pi_x : T \to O_p^\times/\overline{O^\times}$ for the projection given by $\pi_x(x, y) = x \bmod \overline{O^\times}$. Let $\epsilon_0 : O_p^\times/\overline{O^\times} \to \mu_{p^r}(\mathbb{C}_p)$ be a character and $\epsilon : Z \twoheadrightarrow \Gamma \to \mu_{p^\infty}(\mathbb{C}_p)$ be a character extending $\epsilon_0 \circ \pi_x$. Enlarge W if necessary so that ϵ has values in $W^\times$. Define a character $\widehat{\psi}_{k,\epsilon} : Z \to W^\times$ by $\widehat{\psi}_{k,\epsilon}(\zeta\gamma) = \widehat{\eta}_k(\gamma)\epsilon(\gamma)\psi_0(\zeta)$. Then $\widehat{\psi}_{k,\epsilon}$ is the p-adic avatar of an arithmetic Hecke character $\psi_{k,\epsilon}$ of conductor p. We would like to show that the t-expansion of this Eisenstein series at $x = (A(\mathfrak{A}), \lambda(\mathfrak{A}), i(\mathfrak{A}))$ gives the Iwasawa power series of φ^-_{ψ} for $\psi = \psi_{k,\epsilon}$ restricted to $[\mathfrak{A}]\Gamma$.

If confusion is not likely, we just write ψ for $\psi_{k,\epsilon}$. Recall that $D \cong Z/\Gamma'$ for the intersection Γ' (in Z) of Γ with the image of $\mathfrak{O}_p^\times$. Let $\chi_{\Gamma'}$ be the characteristic function of $\Gamma' \subset Z$. We put $\phi = \mathrm{Inf}_{\psi_0} \widehat{\psi}\chi_{\Gamma'}$ for a character $\psi_0 : \Delta \to W^\times$. We can apply Theorem 3.5 to $\mathbf{E}_{\mathfrak{c}}(\phi)$ to compute the μ-invariant. Recall that we have written $E(\phi^\circ)$ for $E_0(\phi^\circ;\mathfrak{c}) = \mathbf{E}_{\mathfrak{c}}(\phi)$ for a suitable choice of $\mathfrak{c}$ in the context. Let $\chi^\circ_{\Gamma'}(x;y) = \chi_{\Gamma'}(x^{-1};y) = \chi(x;y)$, where χ is the characteristic function of $\{(x;y) \mid \pi(x;y) \in \Gamma'\}$ for $\pi : \mathfrak{O}_p^\times \to Z$.

Recall $\mathcal{D}$ which is a complete representative set in the localization (not the completion) $\mathfrak{O}_{(p)}^\times$ for $\pi(\mathfrak{O}_p^\times)/\Gamma'$ with the projection $\pi : \mathfrak{O}_p^\times \to Z$. We split further $\mathcal{D} = \bigsqcup_{\alpha\in\mathcal{D}^-} \alpha\mathcal{D}^+$ where $\mathcal{D}^+$ is the subset of $\mathcal{D}$ represented by elements of $F^\times$:

$$\mathcal{D}^+ = \{\alpha \in \mathcal{D} \mid \alpha\Gamma' = \beta\Gamma' \text{ with } \beta \in F^\times \cap \mathfrak{O}_p^\times\}.$$

We choose $\alpha \in \mathcal{D}^-$ so that $(\alpha) = \mathfrak{Q}$ is a prime ideal split in M over F. Then $K_{\mathfrak{a},\mathfrak{b}}\rho(\alpha) = K_{\mathfrak{a},\mathfrak{b}}g_{\mathfrak{Q}}$ for $g_{\mathfrak{Q}} = \left(\begin{smallmatrix}1&0\\0&q\end{smallmatrix}\right)$ for a finite idele $q \in \widehat{O}$ with $q\widehat{\mathfrak{O}} = \widehat{\mathfrak{Q}}\widehat{\mathfrak{Q}}^c$ and $q_p = 1$. Define $\mathcal{S}$ (respectively $\mathcal{R}$) by a subset $\{\beta_p \mid \beta \in \mathcal{D}^+\}$ (respectively $\{\alpha_p \mid \alpha \in \mathcal{D}^-\}$) in the completion $\mathfrak{O}_p^\times$. Note that $\widehat{\psi}(s) = \psi((\beta))^{-1}$ because $s = \beta_p$. Similarly $\widehat{\psi}(r) = \psi(\alpha)^{-1}$ ($r = \alpha_p \in \mathfrak{O}_p$).

By (5.6), (5.9) and (5.14), we see that

$$\sum_{\beta\in\mathcal{D}^+} \psi(\beta)^{-1}E(N^{-k}\epsilon^{-1}\chi^\circ_{\Gamma'})|\langle(\beta)\rangle|\beta = E(\Phi^\circ_+),$$

where '$|\beta$' is the action of the scalar element $\beta \in Z(\mathbb{Q})$, and by (5.14)

$$\Phi^\circ_+(x;y) = \sum_{s\in\mathcal{S}} \widehat{\psi}(sx)\chi^\circ(s^{-1}x^{-1};sy), \tag{5.15}$$

since $s = \beta_p \in \mathcal{S} \subset \mathfrak{O}_p^\times$ (for $\beta \in \mathcal{D}^+$). We further sum up over $\mathcal{D}^-$:

$$\sum_{\alpha\in\mathcal{D}^-} \psi(\alpha)E(\Phi^\circ_+)|[\alpha\alpha^c]|\rho(\alpha)^{-1} = E(\Phi^\circ), \tag{5.16}$$

where we have chosen (α) to be an integral ideal with $\mathfrak{O}/(\alpha) \cong O/((\alpha)\cap F)$ and $(\alpha)\cap F = (\alpha\alpha^c)$, and Φ° is given by

$$\Phi^\circ(x;y) = \sum_{r\in\mathcal{R}} \widehat{\psi}(r)^{-1}\Phi^\circ_+(r(x;y)). \tag{5.17}$$

Since we have $E(\Phi^\circ_+(r(x;y))) = E(\Phi^\circ_+)|\rho(r)$ for $\rho(r) \in K_{\mathfrak{a},\mathfrak{b}}$, we have by (5.14) that

$$\sum_{\alpha\in\mathcal{D}^-} \psi(\alpha)\Bigg(\sum_{\beta\in\mathcal{D}^+} \psi(\beta)^{-1}E(N^{-k}\epsilon^{-1}\chi^\circ_{\Gamma'})|\langle\beta\rangle|\beta\Bigg)|[\alpha\alpha^c]|\rho(\alpha)^{-1} = \sum_{r\in\mathcal{R}} \widehat{\psi}(r)^{-1}E(\Phi^\circ_+)|\rho(r)^{-1}. \tag{5.18}$$

We have computed $E(\Phi^\circ)$ as a linear combination of transforms of the weight k Eisenstein series $E(N^{-k}\epsilon^{-1}\chi^\circ_{\Gamma'})$. By definition of φ_ψ, Φ as above is the restriction of $\mathrm{Inf}_\psi\,\widehat{\psi}\chi_{\Gamma'}$ to $Z_0 = \mathfrak{O}_p^\times/\overline{\mathfrak{O}^\times} \subset Z$.

Recall that $\mathfrak{a} \in D^+$ is a fractional F-ideal; so, the operator $\langle\mathfrak{a}\rangle$ makes sense. Similarly, we have chosen $\mathfrak{B} \in D^-$ among prime ideals of M split over F; so, the operator $[\mathfrak{B}\mathfrak{B}^c]$ regarding $\mathfrak{B}\mathfrak{B}^c$ as a prime ideal of F also makes sense.

Theorem 5.2. *Suppose $p > 2$, $\mathfrak{C} = 1$ and that M/F is everywhere unramified. Let t be the canonical variable of the Serre–Tate deformation space $\widehat{S} = \widehat{\mathbb{G}}_m \otimes_{\mathbb{Z}} \mathfrak{d}^{-1}$ of $(A(\mathfrak{O}),\lambda(\mathfrak{O}),i(\mathfrak{O}))_{/W}$ so that the parameters $(t^{\xi_1}-1,\ldots,t^{\xi_d}-1)$ (for a base $\{\xi_i\}_i$ of O over $\mathbb{Z}$) give the coordinate around the origin $1 \in \widehat{S}$. Write Φ for the restriction of $\mathrm{Inf}_\psi\,\widehat{\psi}\chi_{\Gamma'}$ to $Z_0 \subset Z$. Put for each $\mathfrak{B} \in D^-$*

$$E_{\mathfrak{B}}(t) = \sum_{\mathfrak{a}\in D^+} \psi(\mathfrak{a})^{-1}E(\Phi^\circ)|\langle\mathfrak{a}\rangle(t) \in \mathcal{O}_{\widehat{S}},$$

where we have chosen $\mathfrak{a} \subset F$ prime to p. Then the t-expansion of

$$\mathcal{E} = \sum_{\mathfrak{B} \in D^-} \psi(\mathfrak{B}) E_{\mathfrak{B}} |[\mathfrak{B}\mathfrak{B}^c](t^{[\mathfrak{B}]^-})$$

at $(A(\mathfrak{O}), \lambda(\mathfrak{O}), i(\mathfrak{O}))$ gives (up to an automorphism of $W[[Z]]$) the t-expansion of the anticyclotomic measure φ_ψ^-. Moreover, the μ-invariant $\mu(\varphi_\psi^-)$ is given by

$$\mu(\psi) = \mathrm{Inf}_{\mathfrak{n}} v \biggl(\prod_{\mathfrak{q} | \mathfrak{n}} \frac{1 - (\psi(\mathfrak{q}) N(\mathfrak{q}))^{e(\mathfrak{q})+1}}{1 - \psi(\mathfrak{q}) N(\mathfrak{q})} \biggr), \tag{5.19}$$

where $\mathfrak{n}$ runs over all integral ideals of the form $\mathfrak{c}(\mathfrak{A}\mathfrak{A}^c)$ for ideals $\mathfrak{A}$ of M prime to p.

For any anticyclotomic character ϵ and every fractional F-ideal $\mathfrak{q}$, we have $\epsilon(\mathfrak{q}) = 1$ and $\psi\epsilon(\mathfrak{q}) = \psi(\mathfrak{q})$; thus, $\mu(\psi) = \mu(\psi\epsilon)$ as long as ϵ is anticyclotomic. This explains well $\mu(\varphi_\psi^-) = \mu(\varphi_\psi^- * \epsilon)$ for ϵ factoring through Γ^-; so, we need to twist φ by a nontrivial cyclotomic character to lower $\mu(\varphi_\psi^-)$.

The Eisenstein series $E_{\mathfrak{B}}$ defined in the theorem really depends on $\mathfrak{B} \in D^-$ since the polarization ideal of $E(\Phi^\circ)$ in the sum depends on $\mathfrak{B}$.

Proof. We first show that the t-expansion of $\mathcal{E}$ gives (up to an automorphism of $W[[Z]]$) the t-expansion of the Katz measure. We said 'up to an automorphism of $W[[Z]]$', because of the following reason: in the definition of the level structure $i(\mathfrak{A})$, $x \in F_p/\mathfrak{d}^{-1}$ is sent to $2\delta x \in M_{\mathbf{p}}/\mathfrak{A}$. This has the following harmless effect: the t-expansion of $E(\Phi_+^\circ((x;y)\mathrm{diag}[2\delta, (2\delta)^{-1}]))$ actually coincides with the t-expansion of the measure. The variable change $(x;y) \mapsto (x;y)\mathrm{diag}[2\delta, 2\delta^{-1}]$ corresponds to the automorphism: $z \mapsto 2\delta z$ of the topological space Z (since 2δ is chosen to be prime to $p\mathfrak{C}$), which gives rise to an automorphism of $W[[Z]]$. Hence we forget about the effect of this unit 2δ.

We are going to compute the κ-derivatives $d^\kappa \mathcal{E}$ of the Eisenstein series at $\underline{A}(\mathfrak{a}^{-1})$ for applying the t-expansion principle. Let $\psi^{(\kappa)} = \psi_{k,\epsilon}^{(\kappa)}$ be a unique Hecke character of Z such that $\psi_{k,\epsilon}^{(\kappa)}(\beta) = \beta^{(1-c)\kappa} \psi_{k,\epsilon}(\beta)$ for all $\beta \equiv 1 \bmod p$, $\widehat{\psi}_{k,\epsilon}^{(\kappa)}|_\Delta = \psi$ and $\psi_{k,\epsilon}^{(\kappa)}(\mathfrak{a}) = \alpha_{\mathfrak{a}}^{(1-c)\kappa} \psi_{k,\epsilon}(\mathfrak{a}) = \psi_{k,\epsilon}(\mathfrak{a})$, as $\mathfrak{a}$ is an F-ideal and $\alpha_{\mathfrak{a}} = 1$ as in (5.4) for all $\mathfrak{a} \in D^+$. We write $\langle z \rangle$ for the projection of $z \in W(\mathbb{F})^\times$ to the p-profinite part of $W(\mathbb{F})^\times$. Then we have

$$\langle (x,y)^{\kappa(c-1)} \phi \rangle^\circ = \langle (x^{-1}y)^\kappa \phi \rangle^\circ = \langle (xy)^\kappa \rangle \phi^\circ.$$

Recall the following definition:

$$\phi_{\mathfrak{A}}(x) = \phi(x[\mathfrak{A}]^{-1}).$$

Also recall (5.9):

$$\int_{\Gamma'} \phi_{\alpha\mathfrak{B}} \, dE_{\alpha\mathfrak{B}} = (\mathbf{E}(\chi_{\Gamma'} \phi_{\alpha\mathfrak{B}}) | \langle \alpha \rangle)(A(\mathfrak{B}), \lambda(\mathfrak{B}), i(\mathfrak{B})) \quad \text{if } \alpha \in O \cap F^\times,$$
$$\int_{\Gamma'} \phi_{\alpha^{-1}\mathfrak{B}} \, dE_{\alpha^{-1}\mathfrak{B}} = (\mathbf{E}(\chi_{\Gamma'} \phi_{\alpha^{-1}\mathfrak{B}}) | [\alpha\alpha^c])(A(\mathfrak{B}), \lambda(\mathfrak{B}), i(\mathfrak{B})) \quad \text{if } \alpha \notin O.$$

Because of these formulas, we may replace each term

$$\psi(\alpha)\psi(\beta^{-1}) E(N^{-k} \chi_{\Gamma'}^\circ) | \langle \beta \rangle | \beta | [\alpha\alpha^c] | \rho(\alpha)^{-1}$$

of (5.18) by

$$\begin{aligned}&\psi(\alpha)\psi(\beta^{-1})d^{\kappa}(E(N^{-k}\epsilon^{-1}\chi^{\circ}_{\Gamma'})|\langle\beta\rangle|\beta|[\alpha\alpha^c]|\rho(\alpha)^{-1})(\underline{A}(\mathfrak{a}^{-1}))\\ &\overset{(*)}{=}\psi(\alpha)\psi(\beta^{-1})(\alpha\beta^{-1})^{\kappa(c-1)}E((xy)^{\kappa}N^{-k}\epsilon^{-1}(\chi^{\circ}_{\Gamma'}|[\alpha\beta^{-1}]))(\underline{A}((\alpha^{-1}\beta)\mathfrak{a}^{-1}))\\ &=\psi(\alpha\beta^{-1})(\alpha\beta^{-1})^{\kappa(c-1)}E(((x^{-1}y)^{\kappa}x^k\epsilon(\chi_{\Gamma'}\circ(\alpha\beta^{-1})))^{\circ})(\underline{A}((\alpha^{-1}\beta)\mathfrak{a}^{-1}))\\ &\overset{(**)}{=}\psi^{(\kappa)}(\alpha\beta^{-1})E((\langle(x^{-1}y)^{\kappa}x^k\rangle\epsilon\chi_{\Gamma'})^{\circ}|[\alpha\beta^{-1}])(\underline{A}((\alpha^{-1}\beta)\mathfrak{a}^{-1}))\\ &=\psi^{(\kappa)}(\alpha\beta^{-1})E((\langle(x^{-1}y)^{\kappa}x^k\rangle\epsilon\chi_{\Gamma'})^{\circ})(\underline{A}(\mathfrak{a}^{-1})),\end{aligned} \tag{5.20}$$

where $\phi|[\alpha](x;y)=\phi(\alpha^{-1}x,\alpha^{-1}a;\alpha y,\alpha b)$ and $\phi\circ\alpha(t)=\phi(\alpha t)$ for $t\in T$. The above equality indicated by $(*)$ (respectively $(**)$) follows from (5.1) and the formulas: $d^{\kappa}(q^{xy})=(xy)^{\kappa}q^{xy}$ in (4.5) and $d^{\kappa}(t^a)=a^{\kappa}t^a$ in (4.4) (respectively the fact that $\chi_{\Gamma'}\circ(\alpha\beta^{-1})$ is the characteristic function of $(\alpha^{-1}\beta)\Gamma'$).

Let $\mathcal{F}=(\mathrm{Inf}_{\psi}((x^{-1}y)^{\kappa}\chi_{\Gamma'})|_{Z_0})^{\circ}=\langle(xy)^{\kappa}\rangle\Phi^{\circ}$. By the computation given in [HT93, (4.9)] and by (5.6), the partial L-value for the character $\psi^{(\kappa)}$ and for the ideal class of $\mathfrak{a}^{-1}$ is given by

$$\psi^{(\kappa)}(\mathfrak{a})^{-1}E(\mathcal{F})(\underline{A}(\mathfrak{a}^{-1}))=\psi^{(\kappa)}(\mathfrak{a})^{-1}d^{\kappa}(E(\Phi^{\circ})|\langle\mathfrak{a}\rangle)(\underline{A}(\mathfrak{O}))\quad\text{for } E(\mathcal{F})=E_{\mathfrak{c}_{\mathfrak{a}^{-1}}}(\mathcal{F})$$

and

$$d^{\kappa}E(\underline{A}(\mathfrak{O}))=\sum_{\mathfrak{a}\in D^+}\psi^{(\kappa)}(\mathfrak{a})^{-1}E(\mathcal{F})(\underline{A}(\mathfrak{a}^{-1})) \tag{5.21}$$

for all $\kappa\geqslant 0$. Hence we have $E(\phi)|\langle\mathfrak{a}\rangle(\underline{A}(\mathfrak{O}))=E(\phi)(\underline{A}(\mathfrak{a}^{-1}))$.

Now we apply the operator $[\mathfrak{B}\mathfrak{B}^c]$ and make variable change: $t\mapsto t^{[\mathfrak{B}]^-}$ in (5.21). We may assume that $[\mathfrak{B}]^-\in\Gamma'$. By the computation given in [HT93, (4.9)], the partial L-value for the character ψ_{κ} and for the ideal class of $\mathfrak{B}^{-1}$ is given by, for $E(\mathcal{F})=E_{\mathfrak{c}_{\mathfrak{B}^{-1}}}(\mathcal{F})$,

$$\psi^{(\kappa)}(\mathfrak{B})E(\mathcal{F})(\underline{A}(\mathfrak{B}^{-1}))=\psi^{(\kappa)}(\mathfrak{B})d^{\kappa}(E(\Phi^{\circ})|[\mathfrak{B}\mathfrak{B}^c](t^{[\mathfrak{B}]^-}))(\underline{A}(\mathfrak{O}))$$

and

$$d^{\kappa}E(\underline{A}(\mathfrak{O}))=\sum_{\mathfrak{B}\in D^-}\psi^{(\kappa)}(\mathfrak{B})E(\mathcal{F})(\underline{A}(\mathfrak{B}^{-1})) \tag{5.22}$$

for all $\kappa\geqslant 0$. Hence we have

$$E(\phi)|[\mathfrak{B}\mathfrak{B}^c](\underline{A}(\mathfrak{O}))=E(\phi)(\underline{A}(\mathfrak{B}^{-1})).$$

We obtain, by the effect of the differential operator d^{κ},

$$d^{\kappa}(E_{\mathfrak{B}}|[\mathfrak{B}\mathfrak{B}^c](t^{[\mathfrak{B}]^-}))(\underline{A}(\mathfrak{O}))=\sum_{\mathfrak{a}\in D^+}\psi^{(\kappa)}(\mathfrak{a}\mathfrak{B})E(\mathcal{F})(\underline{A}((\mathfrak{a}\mathfrak{B})^{-1})). \tag{5.23}$$

This combined with the evaluation formula (1.1) (and [HT93, (4.9)]) shows that the function in the theorem, after applying d^{κ} and evaluating at $\underline{A}(\mathfrak{O})$, has the property satisfied by the measure φ^-_{ψ}; so, the first assertion follows from (4.6).

As explained below Theorem 3.5, we have a unique element $H_{\kappa}\in G_{\kappa}(\mathbb{F})$ whose t-expansion is the constant 1 (identical to the t-expansion of the Hasse invariant). To apply Theorem 3.5, we take $h_i=H_{\kappa}$. Abusing terminology, we call H_{κ} the Hasse invariant. We will show $p^{\mu(\psi)}|E_{\mathfrak{B}}|[\mathfrak{B}\mathfrak{B}^c]$ in the q-expansion ring. Then we want to apply Theorem 3.5 taking $\{a_i\}_i=\{[\mathfrak{B}]^-\}_{\mathfrak{B}\in D^-}$ and $\{f_{ij}\}=\{E_{\mathfrak{B}}|[\mathfrak{B}\mathfrak{B}^c]/p^{\mu(\psi)} \bmod \mathfrak{m}_W\}$ for each $i=\mathfrak{B}\in D^-$. In other words, for each index i with $a_i=[\mathfrak{B}]^-$, $\{f_{ij}\}$ is given by the single element $\overline{E}_{\mathfrak{B}}:=(E_{\mathfrak{B}}|[\mathfrak{B}\mathfrak{B}^c]/p^{\mu(\psi)} \bmod \mathfrak{m}_W)$.

To verify the assumption (of Theorem 3.5) of linear independence (over $\mathbb{F}$) of $\{H_{\kappa},f_{ij}\}_j$ for each i, we need to show that for each $\mathfrak{B}\in D^-$, $\overline{E}_{\mathfrak{B}}/p^{\mu(\psi)}$ is linearly independent from the Hasse invariant $H_\kappa(t)=1$, using the t-expansion principle and the q-expansion principle. Once this is done, by Theorem 3.5, $\{\overline{E}_{\mathfrak{B}}(t^{[\mathfrak{B}]^-})/p^{\mu(\psi)}\}_{\mathfrak{B}\in D^-}$ is linearly independent over $\mathbb{F}$, and hence we conclude the nonvanishing of $\mathcal{E}/p^{\mu(\psi)}$ (i.e., $\mu(\varphi^-_\psi)=\mu(\psi)$) by Theorem 3.5 (which requires the unramifiedness of p in $F/\mathbb{Q}$), since elements in $\{[\mathfrak{B}]^-\}_{\mathfrak{B}\in D^-}$ are distinct modulo $T_x(\mathbb{Q})$. We show the linear independence of $E_{\mathfrak{B}}/p^{\mu(\psi)}$ from H_κ by showing that $v(a(\xi, E_{\mathfrak{B}}))\geqslant \mu(\psi)$ for any $\xi\in F$ with equality for some ξ.

Write $\pi: T^\times\to\Delta$ for the projection $Z\to\Delta$ composed with $\iota: T^\times\to Z$. Let Ψ be the function on $T^\times$ given by $\Psi(x;y)=\psi\circ\pi(x^{-1};y)$. By our assumption, $\Phi^\circ(x;y)=\Psi$. The q-expansion coefficient of $\xi\in\mathfrak{ab}$ of $E(\Psi)$ at the cusp $(\mathfrak{a},\mathfrak{b})$ is given by

$$\sum_{(a,b)\in(\mathfrak{a}\times\mathfrak{b})/O^\times, ab=\xi}\Psi(a,b)|N(a)|^{-1}.$$

We fix $\mathfrak{B}\in D^-$ and use the symbol $\mathfrak{a}$ to indicate the F-ideals running over D^+. The ideal $\mathfrak{B}$ determines $\mathfrak{c}_{\mathfrak{B}^{-1}}$ which is the polarization ideal of $\lambda(\mathfrak{B}^{-1})$ on $A(\mathfrak{B}^{-1})$. If we write $\mathfrak{c}$ for the polarization ideal of $\lambda(\mathfrak{O})$, we know $\mathfrak{c}_{\mathfrak{B}^{-1}}=\mathfrak{c}(\mathfrak{B}\mathfrak{B}^c)$. We choose $\mathfrak{c}^{-1}_{\mathfrak{B}^{-1}}$ to be a prime $\mathfrak{l}$ prime to p (this is possible by changing it in its strict ideal class and choosing $\delta\in M$ suitably).

We now take a totally positive $0\ll\xi\in O$ so that $(\xi)=\mathfrak{ln}$ ($\mathfrak{l}=\mathfrak{c}^{-1}_{\mathfrak{B}^{-1}}$: a prime by our choice) for an integral ideal $\mathfrak{n}$ prime to p. We pick a pair $(a,b)\in F^2$ with $ab=\xi$ for $a\in\mathfrak{a}^{-1}$ and $b\in\mathfrak{la}$. Then $(a)=\mathfrak{a}^{-1}\mathfrak{x}$ for an integral ideal $\mathfrak{x}$ and $(b)=\mathfrak{aly}$. Since $(ab)=\mathfrak{ln}$, we find that $\mathfrak{xy}=\mathfrak{n}$. Thus for each factor $\mathfrak{x}$ of $\mathfrak{n}$, we could have a pair $(a_{\mathfrak{x}},b_{\mathfrak{x}})$ with $a_{\mathfrak{x}}b_{\mathfrak{x}}=\xi$ such that

$$((a_{\mathfrak{x}})=\mathfrak{a}_{\mathfrak{x}}^{-1}\mathfrak{x}, (b_{\mathfrak{x}})=(\xi a_{\mathfrak{x}}^{-1})=\mathfrak{a}_{\mathfrak{x}}\mathfrak{ln}\mathfrak{x}^{-1})$$

for $\mathfrak{a}_{\mathfrak{x}}\in D^+$ representing the ideal class of the ideal $\mathfrak{x}$. We then write down the q-expansion coefficient of q^ξ at the cusp $(O,\mathfrak{l})$ of $E_{\mathfrak{B}}$ as in the theorem:

$$\begin{aligned}\psi_{\mathbf{p}^c}(\xi)a(\xi,E_{\mathfrak{B}})&=\sum_{\mathfrak{x}|\mathfrak{n}}\frac{\psi_{\mathbf{p}^c}(\xi)}{\psi(\mathfrak{a}_{\mathfrak{x}})}a(\xi,E(\Phi^\circ)|\langle\mathfrak{a}_{\mathfrak{x}}\rangle)\overset{\text{(E2)}}{=}\sum_{\mathfrak{x}|\mathfrak{n}}(\psi N(\mathfrak{a}_{\mathfrak{x}}))^{-1}\frac{1}{\psi(a_{\mathfrak{x}})|N(a_{\mathfrak{x}})|}\\&=\sum_{\mathfrak{x}|\mathfrak{n}}\frac{1}{\psi N(\mathfrak{x})}=\prod_{\mathfrak{q}|\mathfrak{n}}\left(\sum_{j=0}^{e(\mathfrak{q})}(\psi N(\mathfrak{q}))^{-j}\right)=\frac{1}{\psi N(\mathfrak{n})}\prod_{\mathfrak{q}|\mathfrak{n}}\frac{1-(\psi N(\mathfrak{q}))^{e(\mathfrak{q})+1}}{1-\psi N(\mathfrak{q})},\end{aligned}$$

where $\mathfrak{n}=\prod_{\mathfrak{q}|\mathfrak{n}}\mathfrak{q}^{e(\mathfrak{q})}$ is the prime factorization of $\mathfrak{n}$.

Recall, for the valuation v of W (normalized so that $v(p)=1$)

$$\mu(\psi)=\mathrm{Inf}_{\mathfrak{n}}v\left(\prod_{\mathfrak{q}|\mathfrak{n}}\frac{1-(\psi(\mathfrak{q})N(\mathfrak{q}))^{e(\mathfrak{q})+1}}{1-\psi(\mathfrak{q})N(\mathfrak{q})}\right),\tag{5.24}$$

where $\mathfrak{n}$ runs over all integral ideals of the form $\mathfrak{c}(\mathfrak{A}\mathfrak{A}^c)$ for ideals $\mathfrak{A}$ of M. Here $\mathfrak{c}$ is the polarization ideal of $A(\mathfrak{O})$. Then, by moving around $\mathfrak{B}$ in D^-, the μ-invariant $\mu(\varphi^-_\psi)$ of the ψ-branch of the anticyclotomic Katz measure φ^- is equal to $\mu(\psi)$. This finished the proof of Theorem 5.2. $\square$

COROLLARY 5.3. *For a fixed integer k, we have $\liminf_\epsilon \mu(\psi_{k,\epsilon})=0$, and hence $\mu(\varphi_{\psi_0})=0$.*

Proof. Choose a prime $\mathfrak{q}$. We take ϵ so that $\epsilon(\mathfrak{q})$ is a primitive p^rth root of unity. Since $\mathfrak{O}^\times_{\mathbf{p}}/\overline{\mathfrak{O}^\times}$ covers $O^\times_p/\overline{O^\times}$ which has p-profinite infinite cyclic quotient $\mathbb{Z}^\times_p$ via norm map in which $N(\mathfrak{q})$ has infinite order, we can make the order p^r of the root of unity $\epsilon(\mathfrak{q})$ whatever large. For any given number $a \in \mathbb{C}_p$, $\liminf_{\zeta \in \mu_{p^\infty}(\mathbb{C}_p)} v(a - \zeta) = 0$, and hence

$$\liminf_{\epsilon} v\left(1 + \frac{1}{\psi_{k,\epsilon}(\mathfrak{q})N(\mathfrak{q})}\right) = \liminf_{\epsilon} v\left(\epsilon(\mathfrak{q}) + \frac{1}{\psi_k(\mathfrak{q})N(\mathfrak{q})}\right) = 0.$$

This implies $\mu(\varphi_{\psi_0}) = 0$ by Lemma 5.1. □

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