A CHARACTERIZATION OF THE APPROXIMATELY CONTINUOUS DENJOY INTEGRAL

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1. Introduction. The approximately continuous integral which includes the Lebesgue integral has been considered by Burkill [**1**] and Ridder [**3**; **4**]. I [**2**] also defined the AD-integral of this kind which is more general than the AP-integral of Burkill [**1**] and the Denjoy integral in the wide sense. But this integral is equivalent to the β-integral of Ridder.

Our aim in this paper is to characterize the AD-integral in the following way: The AD-integral is the least general approximately continuous integral (Definition 1) which includes the Lebesgue integral and fulfils the Cauchy and Harnack conditions (Definition 2).

2. The AD-integral. A real-valued function $F(x)$ defined on the closed interval $[a, b]$ is said to be (ACG) on the interval if $[a, b]$ is the sum of a countable number of *closed* sets E_n such that $F(x)$ is absolutely continuous on each set E_n.

An extended real-valued function $f(x)$ is said to be AD-integrable on $[a, b]$ if there exists an approximately continuous function $F(x)$ such that $F(x)$ is (ACG) on $[a, b]$ and

$$\text{AD } F(x) = f(x) \quad \text{a.e.},$$

where AD $F(x)$ is the approximate derivative of $F(x)$.

The function $F(x)$ is called an indefinite integral of $f(x)$, and the definite integral on $[a, b]$, denoted by $(\text{AD})\int_a^b f(t)\, dt$, is defined as $F(b) - F(a)$ [**2**: III].

LEMMA 1. *Let E be a closed set contained in $[a, b]$. If f is a function which is absolutely continuous on E and is linear on each contiguous closed interval of E with respect to $[a, b]$, then f is absolutely continuous on $[a, b]$.*

Proof. Let $\{I_k = [a_k, b_k]\}$ be the sequence of contiguous closed intervals of E with respect to $[a, b]$. Since f is absolutely continuous on E, for a given $\epsilon > 0$ we can find $\delta > 0$ such that

$$\sum_k |f(\beta_k) - f(\alpha_k)| < \epsilon/2$$

for all finite non-overlapping sequences of intervals $\{(\alpha_k, \beta_k)\}$ with end points on E and $\sum_k (\beta_k - \alpha_k) < \delta$. The function f is linear on each I_k and is therefore

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absolutely continuous, so that we can determine such a positive number δ_k for each $\epsilon/2^{k+1}$ ($k = 1, 2, \ldots$). Let N be a natural number such that $\sum_{k=N+1}^{\infty} (b_k - a_k) < \delta$.

If we put $\delta_0 = \min(\delta, \delta_1, \ldots, \delta_N)$, then we see that for all finite sequences of non-overlapping intervals $\{(\gamma_k, \delta_k)\}$ contained in $[a, b]$ and $\sum_k (\delta_k - \gamma_k) < \delta_0$, we have:

$$\sum_k |f(\delta_k) - f(\gamma_k)| < \epsilon.$$

It follows from Lemma 1 that the AD-integral is equivalent to the β-integral [**3**, Definition 7].

We now establish some essential properties of the AD-integral. If $I = [\alpha, \beta]$, then I^o is the open interval (α, β).

THEOREM 1. *If $f(x)$ is* AD-*integrable on every interval* $[a, \beta]$, *where* $a < \beta < b$, *and*

$$\operatorname*{app\,lim}_{\beta \to b} (\mathrm{AD}) \int_a^\beta f(t)\,dt = l,$$

then $f(x)$ is AD-*integrable on* $[a, b]$ *and*

$$(\mathrm{AD}) \int_a^b f(t)\,dt = l.$$

Proof. Let $\{b_n\}$ ($n = 1, 2, \ldots$) be an increasing sequence converging to b and put $b_1 = a$. Since $f(x)$ is AD-integrable on each $I_n = [b_n, b_{n+1}]$, there exists a function $F_n(x)$ which is approximately continuous and (ACG) on I_n and AD $F(x) = f(x)$ a.e. It may be assumed that $F_n(b_n) = 0$ ($n = 1, 2, \ldots$).

Let $F(x)$ be the function defined on $[a, b]$ as follows:

$$\begin{aligned} F(x) &= F_1(x) && (x \in I_1), \\ &= F_n(x) + \sum_{k=1}^{n-1} F_k(b_{k+1}) && (x \in I_n,\ n \geqq 2). \end{aligned}$$

Then $F(x)$ is approximately continuous on $[a, b)$ and (ACG) on $[a, \beta]$ for $a < \beta < b$ and AD $F(x) = f(x)$ a.e., so that $F(x)$ is an indefinite integral of $f(x)$ on $[a, \beta]$. If we define $F(b) = l$, then by hypothesis, $F(x)$ is approximately continuous on $[a, b]$. It is clear that $F(x)$ is (ACG) on $[a, b]$. Hence $f(x)$ is AD-integrable on $[a, b]$ and

$$(\mathrm{AD}) \int_a^b f(t)\,dt = F(b) - F(a) = F(b) = l.$$

THEOREM 2. *Let E be a closed set in $I_0 = [a, b]$ and $\{I_k = [a_k, b_k]\}$ the sequence of contiguous closed intervals of E with respect to I_0, and let $f(x)$ be a function which is Lebesgue integrable* (L-*integrable*) *on E and* AD-*integrable on each I_k. Suppose that the following conditions are satisfied:*

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(i) $\sum_{k=1}^{\infty} |(\mathrm{AD})\int_{I_k} f(t)\,dt| < \infty$;

(ii) *if* $x \in E$ *is a limit point of* $\{I_k\}$, *then there exists a set* E_x *which has unit density at* x *and contains all the end points of* $\{I_k\}$ *in a sufficiently small neighbourhood of* x, *such that*

$$\lim_{k\to\infty} O(\mathrm{AD}, f, E_x \cap I_k) = 0,$$

where $O(\mathrm{AD}, f, E_x \cap I_k)$ *means the oscillation of the indefinite* AD-*integral of* f *on* $E_x \cap I_k$.

Then $f(x)$ *is* AD-*integrable on* I_0 *and we have:*

$$(\mathrm{AD}) \int_{I_0} f(t)\,dt = (\mathrm{L}) \int_E f(t)\,dt + \sum_{k=1}^{\infty} (\mathrm{AD}) \int_{I_k} f(t)\,dt.$$

Proof. Let $I(x)$ denote the interval $[a, x]$, where $a \leqq x \leqq b$, and let

$$F(x) = \sum_{k=1}^{\infty} (\mathrm{AD}) \int_{I_k \cap I(x)} f(t)\,dt.$$

We shall show that the function $F(x)$ is approximately continuous on $[a, b]$. If x is an interior point of some I_n, then we have:

$$F(x) = \sum_{I_k \subset [a, a_n]} (\mathrm{AD}) \int_{I_k} f(t)\,dt + (\mathrm{AD}) \int_{a_n}^{x} f(t)\,dt$$

for $a_n < x < b_n$, and thus $F(x)$ is approximately continuous at x. If x is an isolated point of E, it is the common end point of some consecutive intervals. Hence, $F(x)$ is approximately continuous at this point. Finally, we consider the case in which x is a limit point of $\{I_k\}$. By (i) and (ii), there exists a natural number K such that

$$\sum_{k>K} \left| (\mathrm{AD}) \int_{I_k} f(t)\,dt \right| < \epsilon, \tag{1}$$

and

$$O(\mathrm{AD}, f, E_x, \cap I_k) < \epsilon \qquad (k > K). \tag{2}$$

Except for the case in which x is the end point of some I_k $(k \leqq K)$, we can select $\delta > 0$ such that the interval $(x - \delta, x + \delta)$ does not contain the intervals I_k for $k \leqq K$ and such that the set E_x contains the end points of I_k in $(x - \delta, x + \delta)$. If $t \in (x, x + \delta) \cap E_x$ and $t \in E$, then we have:

$$|F(t) - F(x)| \leqq \sum_{I_k \subset [x, t]} \left| (\mathrm{AD}) \int_{I_k} f(t)\,dt \right| \leqq \sum_{k>K} \left| (\mathrm{AD}) \int_{I_k} f(t)\,dt \right| < \epsilon.$$

If $t \in (x, x + \delta) \cap E_x$ and $t \in I_n^{\circ}$, then we obtain

$$F(t) - F(x) = \sum_{I_k \subset [x, a_n]} (\mathrm{AD}) \int_{I_k} f(t)\,dt + (\mathrm{AD}) \int_{a_n}^{t} f(t)\,dt.$$

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Since the set E_x contains the point a_n, it follows from (2) that

$$\left|(\mathrm{AD})\int_{a_n}^{t} f(t)\,dt\right| < \epsilon,$$

and hence

$$|F(t) - F(x)| < 2\epsilon.$$

Similarly we obtain the above inequality for the case $t < x$. Therefore F is approximately continuous at x. Thus we have proved that $F(x)$ is approximately continuous on I_0.

Next we can prove (as in [**5**, p. 257]) that $F(x)$ is also (ACG) on I_0 and that

$$\begin{aligned}\mathrm{AD}\,F(x) &= 0 \text{ a.e.} \quad \text{for } x \in E,\\ &= f(x) \text{ a.e.} \quad \text{for } x \in I_0 - E.\end{aligned}$$

Let

$$H(x) = F(x) + (\mathrm{L})\int_{E\cap I(x)} f(t)\,dt.$$

Then we see that $H(x)$ is approximately continuous and (ACG) on I_0 and $\mathrm{AD}\,H(x) = f(x)$ a.e. Hence $f(x)$ is AD-integrable on I_0 and we obtain

$$\begin{aligned}(\mathrm{AD})\int_{I_0} f(t)\,dt &= H(b) - H(a)\\ &= (\mathrm{L})\int_E f(t)\,dt + \sum_{k=1}^{\infty} (\mathrm{AD})\int_{I_k} f(t)\,dt.\end{aligned}$$

THEOREM 3. *If $f(x)$ is* AD-*integrable on $I_0 = [a, b]$, then for any closed set $E \subset I_0$, there exists a portion $J^\circ \cap E$ which satisfies the following three conditions:*

(i) *$f(x)$ is* L-*integrable on $J \cap E$;*

(ii) *Let $\{I_k\}$ be the sequence of contiguous closed intervals of $J \cap E$ with respect to J. Then*

$$\sum_{k=1}^{\infty}\left|(\mathrm{AD})\int_{I_k} f(t)\,dt\right| < \infty;$$

(iii) *If x is a limit point of $\{I_k\}$, then there exists a set E_x which has unit density at x and contains all the end points of I_k in a sufficiently small neighbourhood of x, such that*

$$\lim_{k\to\infty} O(\mathrm{AD}, f, E_x \cap I_k) = 0.$$

Proof. Let $F(x) = (\mathrm{AD})\int_a^x f(t)\,dt$. Then $F(x)$ is (ACG) on I_0, so that I_0 is represented as the sum of a countable number of closed sets E_k on each of which $F(x)$ is absolutely continuous. Since $E = \bigcup_{k=1}^{\infty} (E \cap E_k)$, by Baire's category theorem, there exist an interval J with $J^\circ \cap E \neq \emptyset$ and a natural number n such that $J^\circ \cap E \subset E \cap E_n$. Hence F is absolutely continuous on $J \cap E$. Since F is also of bounded variation on $J \cap E$, we have:

$$\sum_{k=1}^{\infty}\left|(\mathrm{AD})\int_{I_k} f(t)\,dt\right| = \sum_{k=1}^{\infty} |F(b_k) - F(a_k)| < \infty,$$

where $I_k = [a_k, b_k]$. Hence we have proved (ii).

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To prove condition (i), we denote by $G(x)$ the function which coincides with $F(x)$ on $J \cap E$ and is linear on each I_k. Then the function $G(x)$ is absolutely continuous on J by Lemma 1 and hence $G'(x)$ is L-integrable. Since $G'(x) = F'(x) = f(x)$ at almost all points of $J \cap E$, $f(x)$ is L-integrable on $J \cap E$.

Next we shall show condition (iii). Suppose that there exists no such E_x. Since $F(x)$ is approximately continuous at x, there exists a measurable set A_x having unit density at x, on which $F(x)$ is continuous. The set A_x may contain all the end points of I_k in a sufficiently small neighbourhood of x, because F is absolutely continuous on $J \cap E$ and all the end points of I_k are in $J \cap E$. Given $\epsilon > 0$, we can find $\delta > 0$ such that $t \in A_x \cap (x, x + \delta)$ implies

$$|F(t) - F(x)| < \epsilon. \tag{3}$$

By assumption, $O(F, A_x \cap I_k)$ does not tend to 0 as $k \to \infty$, so that there exist a positive constant c and a natural number K such that

$$O(F, A_x \cap I_k) > c > 0 \qquad (k > K). \tag{4}$$

We may assume that the interval $(x, x + \delta)$ does not contain the intervals I_k for $k \leqq K$. We have from (4)

$$c/2 < \sup_{t \in A_x \cap I_k} |F(t) - F(a_k)|, \tag{5}$$

and hence there exists a point $t_0 \in A_x \cap I_k$ such that

$$c/2 - \epsilon < |F(t_0) - F(a_k)|. \tag{6}$$

It follows from (5), (6), and the relation $|F(a_k) - F(x)| < \epsilon$ that

$$|F(t_0) - F(x)| > c/2 - 2\epsilon.$$

Taking $c > 6\epsilon$, the above inequality contradicts (3), and the theorem is proved.

Remark. The property of closedness in (ACG) is used explicitly in the above proof but not in Theorems 1 and 2. However, it is essential in defining the AD-integral.

3. An approximately continuous integral. Throughout this section we let I and J be closed intervals.

Let $T = T(f, I)$ be a bilinear functional on a subset of $M \times N$, where $M = \{f\}$ is the space of functions defined on I_0, $N = \{I\}$ the collection of subintervals of I_0, and I_0 fixed. The set $\{f\colon (f, I) \in \text{domain of } T\}$ will be denoted by $K(T, I)$. We also use the notation $T_\alpha{}^\beta(f)$ in place of $T(f, I)$ when $I = [\alpha, \beta]$.

Definition 1. A functional T is termed an approximately continuous integral if the following conditions are fulfilled:

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(i) If $f \in K(T, I)$, then $f \in K(T, J)$ for all $J \subset I$;

(ii) If I_1 and I_2 are abutting intervals and if $f \in K(T, I_1) \cap K(T, I_2)$, then $f \in K(T, I_1 \cup I_2)$ and

$$T(f, I_1 \cup I_2) = T(f, I_1) + T(f, I_2);$$

(iii) The function $F(x) = T_\alpha^x(f)$ $(\alpha \leqq x \leqq \beta)$ is approximately continuous on $I = [\alpha, \beta]$.

The AD-integral is an approximately continuous integral.

If T is an approximately continuous integral, any function belonging to $K(T, I)$ is termed T-integrable on I, and the number $T(f, I)$ is called the definite T-integral of f on I.

Given two integrals T_1 and T_2, we shall say that the integral T_1 includes the integral T_2, written $T_2 \subset T_1$, if $f \in K(T_2, I_0)$ implies $f \in K(T_1, I_0)$ and $T_1(f, I) = T_2(f, I)$ for every $I \subset I_0$.

4. A characterization of the AD-integral.

Definition 2. The Cauchy (C) and Harnack (H) properties of an approximately continuous integral T are given by the following conditions:

(C) If f is T-integrable on every interval $[\gamma, \delta] \subset [\alpha, \beta]$ and

$$\operatorname*{app\,lim}_{\substack{\gamma \to \alpha+; \\ \delta \to \beta-}} T_\gamma^\delta(f)$$

is finite, then f is T-integrable on $[\alpha, \beta]$ and

$$T_\alpha^\beta(f) = \operatorname*{app\,lim}_{\substack{\gamma \to \alpha+; \\ \delta \to \beta-}} T_\gamma^\delta(f).$$

(H) Let Q be a closed set in I and let $\{I_k = [a_k, b_k]\}$ be the sequence of intervals contiguous to Q with respect to I. Let f be L-integrable on Q and T-integrable on each I_k. Suppose that the following conditions are satisfied:

(i) $\sum_{k=1}^\infty |T(f, I_k)| < \infty$;

(ii) if $x \in E$ is a limit point of $\{I_k\}$, then there exists a set E_x which has unit density at x such that

$$\lim_{k \to \infty} O(\mathrm{T}, f, E_x \cap I_k) = 0,$$

where the set E_x contains all the end points of I_k in a sufficiently small neighbourhood of x. Then f is T-integrable on I and we have:

$$T(f, I) = (\mathrm{L}) \int_Q f(t)\, dt + \sum_{k=1}^\infty T(f, I_k).$$

We have proved in Theorems 1 and 2 that the AD-integral has the properties (C) and (H).

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LEMMA 2. *Let* T *be an approximately continuous integral which has the properties* (C) *and* (H), *and* I *fixed. If for every* $x \in I^{\circ}$ *we can find an interval* J_x *containing* x *in its interior such that* $T \supset AD$ *on* J_x, *then* $T \supset AD$ *on* I.

Proof. First we show that $T \supset AD$ for any $J \subset I^{\circ}$. Since for every point $x \in J$, there corresponds an interval J_x, it follows from the Heine-Borel covering theorem that there exists a finite sequence of intervals $J_{x1}, \ldots, J_{x_n}$ such that $J \subset \bigcup_{k=1}^{n} J_{xk}$ and $T \supset AD$ on each J_{xk}. Hence, by Definition 1(ii), we obtain: $T \supset AD$ on J.

Next we show that $T \supset AD$ on I. Let J be any interval contained in I°. Then we have, for any $f \in K(AD, J)$,

$$\text{(AD)} \int_J f(t)\, dt = T(f, J).$$

It follows from (C)-property of T and AD-integrals that $f \in K(T, I)$ and $T(f, I) = AD(f, I)$. However, if I' is any subinterval of I, then similarly $f \in K(T, I')$ and $T(f, I') = AD(f, I')$. Hence the lemma is proved.

THEOREM 4. *Let* T_0 *be the* AD-*integral. If* T *is an approximately continuous integral which includes the* L-*integral and satisfies the conditions* (C) *and* (H), *then* $T \supset T_0$.

Proof. Let Q be the set of points in I_0 such that for $x \in Q$, $T \not\supset T_0$ on every interval containing x. Then Q is clearly closed. It follows from Theorem 3 that there exist a T_0-integrable function f and an interval J with $J^{\circ} \cap Q \neq \emptyset$ such that the following conditions are satisfied:

(i) f is L-integrable on $J \cap Q$;

(ii) if $\{I_k\}$ is the sequence of contiguous closed intervals of $J \cap Q$ with respect to J, then

$$\sum_{k=1}^{\infty} \left| (\mathrm{T}_0) \int_{I_k} f(t)\, dt \right| < \infty;$$

(iii) if $x \in Q$ is a limit point of $\{I_k\}$, then there exists a set E_x which has unit density at x such that

$$\lim_{k \to \infty} O(\mathrm{T}_0, f, E_x \cap I_k) = 0,$$

where the set E_x contains all the end points of I_k in a sufficiently small neighbourhood of x.

Hence, by (H)-property of the T_0-integral, f is T_0-integrable on J and

$$T_0(f, J) = \text{(L)} \int_{J \cap Q} f(t)\, dt + \sum_{k=1}^{\infty} T(f, I_k).$$

Since $I_k^{\circ} \cap Q = \emptyset$, it follows from Lemma 2 that

$$T_0(f, I_k) = T(f, I_k).$$

We have, by (H)-property of the T-integral, $f \in K(T, J)$ and

$$T(f, J) = (\mathrm{L}) \int_{J \cap Q} f(t)\, dt + \sum_{k=1}^{\infty} T(f, I_k).$$

Hence

$$T_0(f, J) = T(f, J).$$

The above identity also holds with J' replaced by any interval $J' \subset J$. But this contradicts the relation $J^\circ \cap Q \neq \emptyset$. Hence the theorem is proved.

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