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TWO RATIONALES BEHIND THE 'BUY-AND-HOLD OR SELL-AT-ONCE' STRATEGY

S. C. P. YAM,* *The Hong Kong Polytechnic University*
S. P. YUNG ** AND
W. ZHOU,** *** *The University of Hong Kong*

Abstract

The trading strategy of 'buy-and-hold for superior stock and sell-at-once for inferior stock', as suggested by conventional wisdom, has long been prevalent in Wall Street. In this paper, two rationales are provided to support this trading strategy from a purely mathematical standpoint. Adopting the standard binomial tree model (or CRR model for short, as first introduced in Cox, Ross and Rubinstein (1979)) to model the stock price dynamics, we look for the optimal stock selling rule(s) so as to maximize (i) the chance that an investor can sell a stock precisely at its ultimate highest price over a fixed investment horizon $[0, T]$; and (ii) the expected ratio of the selling price of a stock to its ultimate highest price over $[0, T]$. We show that both problems have exactly the same optimal solution which can literally be interpreted as 'buy-and-hold or sell-at-once' depending on the value of p (the going-up probability of the stock price at each step): when $p > \frac{1}{2}$, selling the stock at the last time step N is the optimal selling strategy; when $p = \frac{1}{2}$, a selling time is optimal if the stock is sold either at the last time step or at the time step when the stock price reaches its running maximum price; and when $p < \frac{1}{2}$, time 0, i.e. selling the stock at once, is the unique optimal selling time.

Keywords: p-random walk; binomial tree model; optimal stopping; martingales

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1. Introduction

Suppose that an investor holds a stock and thinks of selling it before a finite time T. If he is seeking the highest possible return, what is the right time for him to sell? Conventional wisdom would suggest that if the stock is good enough then the investor should buy-and-hold; on the other hand, if the stock is inferior, selling it immediately could be wisest. In this work, from a purely mathematical standpoint, we shall provide two rationales behind the 'buy-and-hold or sell-at-once' strategy; moreover, a simple index for classifying a stock as 'superior' or 'inferior' will also be given.

From here on, the standard N-step binomial tree model (CRR model) is used to model the stock price dynamics so that both the theoretical mean and variance of the logarithm of the stock price process match with those of the 'actual' stock price. The binomial tree model has been extensively used as a model of stock price processes in the context of option pricing; see, for example, Rubinstein (1994). In addition to binomial tree processes, discrete-time processes

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* Postal address: Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong.
** Postal address: Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong.
*** Email address: h0483715@hkusua.hku.hk

exhibiting momentum have also been proposed in the existing literature as a model of stock price processes; for example, Allaart (2004) used a correlated random walk to model a stock price process and looked for the optimal selling time so as to maximize the expected discounted return. Under the same model, Allaart and Monticino (2008) considered a multiple buy/sell trading strategy to maximize the expected value of total return. The intention of the present paper is not to argue which processes we should use to model the stock price process. Rather, under the simple model we use, we aim to provide two insights behind the 'buy-and-hold or sell-at-once' rule. Write $\triangle t = T/N$, and suppose that the random changes of logarithm of the stock price in all steps are independent and identically distributed with mean $\widehat{\mu} \triangle t$ and variance $\widehat{\sigma}^2 \triangle t$. Following the notation in Cox *et al.* (1979), the stock price process $(V_n^p)_{0\le n\le N}$ with V_n^p denoting the stock price at time $n \triangle t$ is assumed to satisfy the following recursive relation:

$$V_0^p = 1, \qquad V_{n+1}^p = \begin{cases} V_n^p u & \text{with (going-up) probability } p, \\ V_n^p d & \text{with (going-down) probability } q = 1 - p, \end{cases}$$

where p is the going-up probability and $ud = 1$ with $u > 1$. Under this model of stock price, we want to find the optimal selling rule(s) so as to maximize (i) the chance that the investor can sell a stock precisely at the highest price of the stock over $[0, T]$; and (ii) the expected ratio of the selling price of a stock to its highest price over $[0, T]$. We formulate these two problems as follows:

$$V_1^* = \sup_{0\le\tau\le N} \mathrm{P}\bigg(\frac{V_\tau^p}{M_N^p} = 1\bigg) \tag{1.1}$$

and

$$V_2^* = \sup_{0\le\tau\le N} \mathrm{E}\bigg[\frac{V_\tau^p}{M_N^p}\bigg], \tag{1.2}$$

where $M_n^p = \max_{0\le i\le n} V_i^p$ is the running maximum of the stock price, and the supremum is taken over all possible stopping times $0 \le \tau \le N$ adapted to the natural filtration $(\mathcal{F}_n)_{0\le n\le N}$ of $(V_n^p)_{0\le n\le N}$.

Both problems (1.1) and (1.2) belong to the context of optimal stopping theory since one aims to optimize the expectation of a functional of a stopped process over a class of stopping times. For the general theory of optimal stopping, we refer the reader to Chow *et al.* (1971) and Shiryaev (1978). Explicit solutions to an optimal stopping problem are sometimes available for a problem with an infinite-time horizon; for example, Gerber and Shiu (1994) obtained a closed-form solution to the perpetual American option problem using a Lévy process to model the stock price. However, this is not usually the case for optimal stopping problems with a finite-time horizon, and closed-form solutions can rarely be found in general. In this regard, various numerical methods are often required to approximate the optimal stopping boundary; for example, Lai and Lim (2002) discussed a few efficient numerical approximations of the exercising boundary of American lookback options.

Problem (1.1) can be regarded as a Markovian variant of the classical secretary problem (see Dynkin (1963) and Lindley (1961) for solutions to the classical secretary problem). To the best of the authors' knowledge, this question has not been treated in the existing literature except for the special case $p = \frac{1}{2}$, which was solved in Hlynka and Sheahan (1988). The continuous version of problem (1.2) has been considered by various authors. Under the assumption that the stock price follows a geometric Brownian motion with constant drift α and volatility σ, Du Toit and Peskir (2008) solved the problem for general α and σ, Shiryaev *et al.* (2008) solved the same problem for $\alpha \le 0$ and $\alpha \ge \sigma^2/2$, and Yam *et al.* (2008) also provided an independent proof for the case in which $\alpha \geq \sigma^2/2$. In the present work, despite providing original proofs for problem (1.1) for general p and problem (1.2) for $p \geq \frac{1}{2}$, we shall show that solutions to the two problems actually coincide and that the common solution can literally be interpreted as the 'buy-and-hold for superior stocks or sell-at-once for inferior stocks' strategy. Mathematically, we shall establish the following claims for both problems (1.1) and (1.2): (i) if $p > \frac{1}{2}$ then $\tau^* = N$ is an optimal selling time, i.e. the investor should buy and then hold the stock until the terminal time; (ii) if $p = \frac{1}{2}$, a necessary and sufficient condition for selling the stock optimally is to sell it either at the terminal time N or at any time when the stock price is at its running maximum; and (iii) if $p < \frac{1}{2}$, $\tau^* = 0$ is the unique optimal selling time, i.e. the investor should sell the stock at once. Consequently, we call a stock superior if $p > \frac{1}{2}$, neutral if $p = \frac{1}{2}$, and inferior if $p < \frac{1}{2}$. In a nutshell, any investor adopting the 'buy-and-hold or sell-at-once' strategy can solve two problems at one time with a single action: the investor can attain the biggest chance to sell his stock precisely at the ultimate highest price and also minimize the average relative error of the stock selling price to its ultimate highest price.

The rest of this paper is organized as follows. In Section 2 we shall provide a review of some preliminary knowledge of random walks and prove two lemmas that are key for obtaining the solutions to both problems (1.1) and (1.2). The optimality of the 'buy-and-hold or sell-at-once' rule for problems (1.1) and (1.2) will be discussed in Sections 3 and 4, respectively. We conclude the paper and discuss some future research directions in Section 5.

2. Preliminaries of p-random walks

For any $p \in [0, 1]$, we define

$$B_n^p := \frac{\ln(V_n^p / V_0)}{\ln u}.$$

In view of the assumption that $(V_n^p)_{0 \leq n \leq N}$ follows the standard CRR model, B_n^p is a p-random walk such that

$$B_0^p = 0, \qquad B_{n+1}^p = \begin{cases} B_n^p + 1 & \text{with probability } p, \\ B_n^p - 1 & \text{with probability } q = 1 - p. \end{cases}$$

Also, define $\delta := \ln u$ and $S_n^p := \max_{0 \leq i \leq n} B_i^p$, the running maximum of $B_\cdot^p$. Then we shall have $M_N^p = \exp(\delta S_N^p)$. In the rest of this paper, the process $(S_n^p)_{0 \leq n \leq N}$ will be called the running maximum of $B_\cdot^p$, while S_N^p will be referred as the ultimate maximum of $B_\cdot^p$ over the investment horizon $[0, T]$. Both problems (1.1) and (1.2) can now be formulated as

$$V_1^* = \sup_{0 \leq \tau \leq N} \mathrm{P}(B_\tau^p = S_N^p) \tag{2.1}$$

and

$$V_2^* = \sup_{0 \leq \tau \leq N} \mathrm{E}[\exp(\delta(B_\tau^p - S_N^p))]. \tag{2.2}$$

The next lemma converts problems (2.1) and (2.2) into the standard formulation of optimal stopping problems.

Lemma 2.1. *The following two identities hold:*

$$\mathrm{P}(B_\tau^p = S_N^p) = \mathrm{E}[g(\tau, X_\tau^p)],$$
$$\mathrm{E}[\exp(\delta(B_\tau^p - S_N^p))] = \mathrm{E}[G(\tau, X_\tau^p)],$$

where

$$X_n^p := S_n^p - B_n^p,$$

and

$$\begin{aligned} g(n,i) &= \mathrm{P}(S_{N-n}^p = 0)\mathbf{1}_{\{i=0\}}, \\ G(n,i) &= \mathrm{E}[\exp(-\delta(i \vee S_{N-n}^p))] \\ &= \mathrm{E}[\exp(-\delta S_{N-n}^p)\mathbf{1}_{\{S_{N-n}^p \geq i\}} + \mathrm{e}^{-\delta i}\mathbf{1}_{\{S_{N-n}^p < i\}}]. \end{aligned} \tag{2.3}$$

Proof. Using the fact that, for any stopping time $\tau \leq N$, $\widetilde{B}_n^p := B_{\tau+n}^p - B_\tau^p$ is also a p-random walk independent of the σ-algebra $\mathcal{F}_\tau$, we have

$$\begin{aligned} \mathrm{P}(B_\tau^p = S_N^p \mid \mathcal{F}_\tau) &= \mathrm{P}\Big(\Big(S_\tau^p \vee \max_{\tau \leq n \leq N} B_n^p\Big) - B_\tau^p = 0 \;\Big|\; \mathcal{F}_\tau\Big) \\ &= \mathrm{P}\Big(X_\tau^p \vee \max_{0 \leq n \leq N-\tau} \widetilde{B}_n^p = 0 \;\Big|\; \mathcal{F}_\tau\Big) \\ &= \mathrm{P}(S_{N-n}^p = 0)\mathbf{1}_{\{i=0\}}|_{i=X_\tau^p,\, n=\tau} \\ &= g(\tau, X_\tau^p). \end{aligned}$$

Similar to the above arguments, we also have

$$\begin{aligned} \mathrm{E}[\exp(\delta(B_\tau^p - S_N^p)) \mid \mathcal{F}_\tau] &= \mathrm{E}\Big[\exp\Big(\delta\Big(B_\tau^p - \Big(S_\tau^p \vee \max_{\tau \leq n \leq N} B_n^p\Big)\Big)\Big) \;\Big|\; \mathcal{F}_\tau\Big] \\ &= \mathrm{E}\Big[\exp\Big(-\delta\Big(X_\tau^p \vee \max_{\tau \leq n \leq N}(B_n^p - B_\tau^p)\Big)\Big) \;\Big|\; \mathcal{F}_\tau\Big] \\ &= \mathrm{E}[\exp(-\delta(i \vee \widetilde{S}_{N-n}^p))]|_{i=X_\tau^p,\, n=\tau} \\ &= G(\tau, X_\tau^p), \end{aligned}$$

where $\widetilde{S}_n^p = \max_{0 \leq i \leq n} \widetilde{B}_i^p$. Hence, using the tower property of conditional expectations, we have

$$\begin{gathered} \mathrm{P}(B_\tau^p = S_N^p) = \mathrm{E}[\mathrm{P}(B_\tau^p = S_N^p \mid \mathcal{F}_\tau) = \mathrm{E}[g(\tau, X_\tau^p)], \\ \mathrm{E}[\exp(\delta(B_\tau^p - S_N^p))] = \mathrm{E}[\exp(\delta(B_\tau^p - S_N^p)) \mid \mathcal{F}_\tau] = \mathrm{E}[G(\tau, X_\tau^p)]. \end{gathered}$$

In view of Lemma 2.1, both problems (2.1) and (2.2) can now be converted to

$$\begin{aligned} V_1^* &= \sup_{0 \leq \tau \leq N} \mathrm{E}[g(\tau, X_\tau^p)], \\ V_2^* &= \sup_{0 \leq \tau \leq N} \mathrm{E}[G(\tau, X_\tau^p)]. \end{aligned}$$

We shall need the joint probability density function of $(B_n^p,\ S_n^p)$, which can be computed via the reflection principle of a random walk (see Feller (1968, p. 369, Problem 20) or Kijima (2002, Chapter 6.4)):

$$\mathrm{P}(B_n^p = l,\ S_n^p = k) = \bigg(C\Big(n, \frac{n+2k-l}{2}\Big) - C\Big(n, \frac{n+2(k+1)-l}{2}\Big)\bigg)(pq)^{n/2}\Big(\frac{p}{q}\Big)^{l/2}, \tag{2.4}$$

where $C(n,k) = n!/k!\,(n-k)!$. The next lemma states that the probability density function of $(S_n^p)_{0 \leq n \leq N}$ satisfies a certain recursive relation.

Lemma 2.2. *For any $p \in [0, 1]$, the following statements holds.*

1. *For $n > 0$, $(S_n^q - B_n^q, S_n^q)$ and $(S_n^p, S_n^p - B_n^p)$ are equal in law. In particular, for any $i > 0$,*
$$\mathrm{P}(S_n^p = i) = \mathrm{P}(X_n^q = i).$$
2. *For $n > 0$ and $i > 0$,*
$$\mathrm{P}(S_n^p = i) = p\,\mathrm{P}(S_{n-1}^p = i - 1) + q\,\mathrm{P}(S_{n-1}^p = i + 1)$$
and
$$\mathrm{P}(S_n^p = 0) = q[\mathrm{P}(S_{n-1}^p = 0) + \mathrm{P}(S_{n-1}^p = 1)].$$
3. *For $n > 0$ and $i > 0$,*
$$\mathrm{P}(S_n^p \le i) = p\,\mathrm{P}(S_{n-1}^p \le i - 1) + q\,\mathrm{P}(S_{n-1}^p \le i + 1).$$

Proof. 1. Let us first note that $(S_n^q - B_n^q, S_n^q) = (\max_{0 \le k \le n}(B_k^q - B_n^q), \max_{0 \le k \le n} B_k^q)$, but $(B_k^q - B_n^q, B_k^q)_{0 \le k \le n}$ is equal in law to $(B_{n-k}^p, B_{n-k}^p - B_n^p)_{0 \le k \le n}$; therefore, $(S_n^q - B_n^q, S_n^q)$ is equal in law to
$$\Big(\max_{0 \le k \le n} B_{n-k}^p, \max_{0 \le k \le n}(B_{n-k}^p - B_n^p)\Big) = (S_n^p, S_n^p - B_n^p).$$

2. For $n > 0$, by considering the two possible changes of B_{n-1}^q after one step we have

(a) for $i > 0$,
$$\{X_n^q = i\} = \{X_{n-1}^q = i + 1,\ B_n^q = B_{n-1}^q + 1\} \cup \{X_{n-1}^q = i - 1,\ B_n^q = B_{n-1}^q - 1\};$$

(b) $\{X_n^q = 0\} = \{S_{n-1}^q - B_{n-1}^q = 0,\ B_n^q = B_{n-1}^q + 1\} \cup \{S_{n-1}^q - B_{n-1}^q = 1,\ B_n^q = B_{n-1}^q + 1\}$.

Therefore,
$$\mathrm{P}(X_n^q = i) = q\,\mathrm{P}(X_{n-1}^q = i + 1) + p\,\mathrm{P}(X_{n-1}^q = i - 1), \qquad i > 0,$$
$$\mathrm{P}(X_n^q = 0) = q\,\mathrm{P}(X_{n-1}^q = 1) + q\,\mathrm{P}(X_{n-1}^q = 0).$$

Then the second claim follows by applying the first claim to the above two equalities.

3. Finally, the last assertion of the lemma can be easily deduced either from the first two statements or by a simple one-step analysis as above.

3. Problem (1.1): maximizing the chance

In this section we shall tackle problem (1.1), which was shown in Lemma 2.1 to be equivalent to
$$V_1^* = \sup_{0 \le \tau \le N} \mathrm{E}[g(\tau, X_\tau^p)], \tag{3.1}$$
with the payoff function g as given in Lemma 2.1.

For each stopping time $\tau \le N$, we define a new stopping time:
$$\rho_\tau := \inf_n \{\tau \le n : B_n^p = S_\tau^p\} \wedge N. \tag{3.2}$$

To put this definition into words, ρ_τ is the first time after or equal to τ that the p-random walk reaches its running maximum, and in the case when the p-random walk does not hit its running

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maximum before N, we simply take ρ_τ to be N. This new class of stopping times plays an essential role in finding the optimal solution to problem (3.1). Equivalently, ρ_τ can also be defined as $\rho_\tau := \inf_n\{\tau \le n \colon X_n^p = 0\} \wedge N$. Since $g(n,i)$ is supported on $\{(n,i) \colon i = 0\}$, we always have $g(\rho_\tau, X_{\rho_\tau}^p) \ge g(\tau, X_\tau^p)$ for any stopping time $\tau \le N$; hence, ρ_τ always 'dominates' τ in the sense that

$$\mathrm{E}[g(\rho_\tau, X_{\rho_\tau}^p)] \ge \mathrm{E}[g(\tau, X_\tau^p)].$$

This observation suggests that the optimal stopping time must be of the form ρ_τ for some τ. Therefore, we are motivated to study ρ_τ for different stopping times τ, and this can be achieved via the following two propositions. Let $\mathrm{P}_{n,i}$ be the probability measure of the Markov process (n, X_n) under which the process starts at (n,i), and let $\mathrm{E}_{n,i}$ be the expectation taken with respect to $\mathrm{P}_{n,i}$. Also, let $\tau_0 \ge 0$ be the first time the process $(X_n^p)_{0\le n\le N}$ hits 0.

Proposition 3.1. *Define* $f(n,i) := \mathrm{P}(S_{N-n}^p = i)$. *For any stopping time* $\tau \le N$*,*

$$\mathrm{E}[g(\rho_\tau, X_{\rho_\tau}^p) \mid \mathcal{F}_\tau] = \mathrm{E}_{\tau, X_\tau^p}[g(\tau_0 \wedge (N-\tau), X_{\tau_0\wedge(N-\tau)}^p)] = f(\tau, X_\tau^p),$$

and, hence,

$$\mathrm{E}[g(\rho_\tau, X_{\rho_\tau}^p)] = \mathrm{E}[f(\tau, X_\tau^p)].$$

Proof. Since ρ_τ is a hitting time, by its definition, $\rho_\tau = \tau + \tau_0 \circ \theta_\tau$, where θ is the shift operator defined by $\theta_n(\omega)(k) = \omega(n+k)$. Using the strong Markov property of the process $((n, X_n^p))_{0\le n\le N}$, we have

$$\mathrm{E}[g(\rho_\tau, X_{\rho_\tau}^p) \mid \mathcal{F}_\tau] = \mathrm{E}_{\tau, X_\tau^p}[g(\tau_0 \wedge (N-\tau), X_{\tau_0\wedge(N-\tau)}^p)].$$

To prove the second equality, we define $T_i = \inf\{k > 0 \colon B_k^p = i\}$. Then we can see that

$$\mathrm{P}_{n,i}(\tau_0 = k) = \mathrm{P}(T_i = k \mid B_n^p = 0).$$

Also, observe that

$$\mathrm{P}(S_{N-n-k}^p = 0) = \mathrm{P}\Big(\max_{n+k\le l\le N} B_l^p = i \;\Big|\; B_{n+k}^p = i\Big),$$

owing to the space homogeneity of $(B_n^p)_{0\le n\le N}$. With the above two observations and (2.3), we have, for each $i > 0$,

$$\begin{aligned}
&\mathrm{E}_{n,i}[g(\tau_0 \wedge (N-n), X_{\tau_0\wedge(N-n)}^p)] \\
&\quad = \sum_{k=1}^{N-n} \mathrm{P}(S_{N-n-k}^p = 0)\, \mathrm{P}_{n,i}(\tau_0 = k) \\
&\quad = \sum_{k=1}^{N-n} \mathrm{P}\Big(\max_{n+k\le l\le N} B_l^p = i \;\Big|\; B_{n+k}^p = i\Big) \mathrm{P}(T_i = k \mid B_n^p = 0) \\
&\quad = \mathrm{P}(S_{N-n}^p = i) \\
&\quad = f(n,i).
\end{aligned}$$

Finally, for $i = 0$, in accordance with the definition of g, we have

$$f(n,0) = g(n,0) = \mathrm{E}_{n,0}[g(\tau_0 \wedge (N-n), X_{\tau_0\wedge(N-n)}^p)].$$

Combining the above results, our claim then follows immediately.

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Proposition 3.2. *The process* $(f(n, X_n^p))_{0\le n\le N}$ *is* (i) *a submartingale if* $p > \frac{1}{2}$*;* (ii) *a martingale if* $p = \frac{1}{2}$*; and* (iii) *a supermartingale if* $p < \frac{1}{2}$.

Proof. Note that, if $X_n^p = i > 0$, we have

$$X_{n+1}^p = \begin{cases} X_n^p - 1 & \text{with probability } p, \\ X_n^p + 1 & \text{with probability } q = 1 - p. \end{cases}$$

The second assertion in Lemma 2.2 suggests that

$$\begin{aligned} \mathrm{E}[f(n+1, X_{n+1}^p) \mid \mathcal{F}_n] &= pf(n+1, X_n^p - 1) + qf(n+1, X_n^p + 1) \\ &= f(n, X_n^p), \end{aligned}$$

provided that $X_n^p > 0$. If $X_n^p = 0$ for some $n \ge 0$ then we have

$$X_{n+1}^p = \begin{cases} 0 & \text{with probability } p, \\ 1 & \text{with probability } q = 1 - p. \end{cases}$$

Therefore,

$$\begin{aligned} \mathrm{E}[f(n+1, X_{n+1}^p) \mid \mathcal{F}_n] &= pf(n+1, 0) + qf(n+1, 1) \\ &= p\,\mathrm{P}(S_{N-n-1}^p = 0) + q\,\mathrm{P}(S_{N-n-1}^p = 1) \\ &= \mathrm{P}(S_{N-n}^p = 0) + (p-q)\,\mathrm{P}(S_{N-n-1}^P = 0) \\ &\begin{cases} > f(n, 0) & \text{if } p > \frac{1}{2}, \\ = f(n, 0) & \text{if } p = \frac{1}{2}, \\ < f(n, 0) & \text{if } p < \frac{1}{2}, \end{cases} \end{aligned} \tag{3.3}$$

from which the lemma follows.

We now conclude this section with our first main theorem.

Theorem 3.1. *Suppose that the dynamics of a stock price are modeled by the CRR model.*

1. *If* $p > \frac{1}{2}$ *then* $\tau^* = N$ *is an optimal selling time for problem (1.1), and*

$$V_1^* = \mathrm{P}(B_N^p = S_N^p).$$

2. *If* $p = \frac{1}{2}$ *then any* τ *satisfying* $\tau = \rho_\tau$ *almost surely is an optimal selling time for problem (1.1), and*

$$V_1^* = \mathrm{P}(B_\tau^{1/2} = S_N^{1/2})$$

   for any stopping time τ *satisfying* $\tau = \rho_\tau$*. In particular,*

$$V_1^* = \mathrm{P}(B_0^{1/2} = S_N^{1/2}) = \mathrm{P}(B_N^{1/2} = S_N^{1/2}).$$

3. *If* $p < \frac{1}{2}$ *then* $\tau^* = 0$ *is the unique optimal selling time for problem (1.1), and*

$$V_1^* = \mathrm{P}(B_0^p = S_N^p).$$

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Proof. 1. When $p > \frac{1}{2}$, $f(n, X_n^p)$ is a submartingale. Using the optional stopping theorem together with the fact that $\rho_N = N$, we have

$$\begin{aligned} P(B_\tau^p = S_N^p) &\le P(B_{\rho_\tau}^p = S_N^p) \\ &= E[f(\tau, X_\tau^p)] \\ &\le E[f(N, X_N^p)] \\ &= P(B_{\rho_N}^p = S_N^p) \\ &= P(B_N^p = S_N^p), \end{aligned}$$

which implies that $V_1^* = P(B_N^p = S_N^p)$ and that $\tau^* = N$ is an optimal selling time.

2. When $p = \frac{1}{2}$, $f(n, X_n^{1/2})$ is a martingale; hence, the above chain of (in)equalities remains valid except that the inequality

$$E[f(\tau, X_\tau^{1/2})] \le E[f(N, X_N^{1/2})]$$

can now be replaced by an equality. As a consequence, the optimal value $V_1^* = P(B_N^{1/2} = S_N^{1/2})$ can be achieved by any stopping time τ satisfying $\tau = \rho_\tau$; in particular, both 0 and N are optimal stopping times.

3. When $p < \frac{1}{2}$, $f(n, X_n^p)$ is a supermartingale. Applying the optional stopping theorem again, together with the fact that $\rho_0 = 0$, we have

$$P(B_\tau^p = S_N^p) \le E[f(0, X_0^p)] = P(B_0^p = S_N^p),$$

which implies that $V_1^* = P(B_0^p = S_N^p)$ and that $\tau^* = 0$ is an optimal selling time. Finally, we want to show that 0 is the unique optimal selling time; indeed, if a stopping time τ is such that $P(\tau > 0) = P(\tau \ge 1) > 0$ then

$$\begin{aligned} E[f(\tau, X_\tau^p)\mathbf{1}_{\{\tau>0\}}] &\le E[f(1, X_1^p)\mathbf{1}_{\{\tau>0\}}] \\ &= E[E[f(1, X_1^p) \mid \mathcal{F}_0]\mathbf{1}_{\{\tau>0\}}] \\ &< E[f(0, 0)\mathbf{1}_{\{\tau>0\}}], \end{aligned}$$

where the last strict inequality comes from (3.3). Therefore,

$$\begin{aligned} P(B_\tau^p = S_N^p) - P(B_0^p = S_N^p) &\le E[f(\tau, X_\tau^p)] - f(0, X_0^p) \\ &= E[f(\tau, X_\tau^p)\mathbf{1}_{\{\tau>0\}}] - E[f(0, 0)\mathbf{1}_{\{\tau>0\}}] \\ &< 0. \end{aligned}$$

4. Problem (1.2): maximizing the expected ratio

In this section we shall tackle problem (1.2), which was shown in Lemma 2.1 to be equivalent to

$$V_2^* = \sup_{0 \le \tau \le N} E[G(\tau, X_\tau^p)],$$

where the payoff function G is as given in Lemma 2.1. For any stopping time τ, we again define ρ_τ as in (3.2). In the following we shall prove that, for any $p \ge \frac{1}{2}$, ρ_τ dominates τ and is dominated by N, i.e.

$$E[\exp(\delta(B_N^p - S_N^p))] \ge E[\exp(\delta(B_{\rho_\tau}^p - S_N^p))] \ge E[\exp(\delta(B_\tau^p - S_N^p))],$$

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while, for $p < \frac{1}{2}$, ρ_τ does not dominate τ anymore, and we will prove, following Peskir's approach (see Du Toit and Peskir (2008)), that every stopping time $\tau \le N$ is dominated by 0, i.e. $\mathrm{E}[\exp(\delta(B_0^p - S_N^p))] \ge \mathrm{E}[\exp(\delta(B_\tau^p - S_N^p))]$. To state our first proposition, we need to define, for the $p < \frac{1}{2}$ case, another process, $(\widehat{B}_n)_{0\le n\le N}$, on the same probability space as $(B_n^p)_{0\le n\le N}$ that equals a q-random walk in law. Suppose that $p < \frac{1}{2}$. Let $(Z_n)_{1\le n\le N}$ be a sequence of independent Bernoulli random variables that are defined on the same probability space as $(B_n^p)_{0\le n\le N}$ and are independent of $(B_n^p)_{0\le n\le N}$, such that

$$\mathrm{P}(Z_n = 1) = \frac{q-p}{q} \quad \text{and} \quad \mathrm{P}(Z_n = -1) = \frac{p}{q} \quad \text{for each } 1 \le n \le N.$$

Define the new random walk $\widehat{B}_n$ as

$$\widehat{B}_0 = 0, \qquad \widehat{B}_{n+1} = \begin{cases} \widehat{B}_n + 1 & \text{if } B_{n+1}^p = B_n^p + 1, \\ \widehat{B}_n + 1 & \text{if } B_{n+1}^p = B_n^p - 1 \text{ and } Z_{n+1} = 1, \\ \widehat{B}_n - 1 & \text{if } B_{n+1}^p = B_n^p - 1 \text{ and } Z_{n+1} = -1, \end{cases}$$

and define $\widehat{\mathcal{F}}_n$ to be the smallest σ-algebra containing both $\mathcal{F}_n$ and $\sigma(Z_1, \dots, Z_N)$; then $\widehat{B}_n$ is $\widehat{\mathcal{F}}_n$-measurable. It is evident that $\widehat{B}_n$ equals a q-random walk in law with the property that, for any $k \le n$,

$$B_n^p - B_k^p \le \widehat{B}_n - \widehat{B}_k.$$

Therefore, if we define $\widehat{X}_n := \widehat{S}_n - \widehat{B}_n$, where $\widehat{S}_n := \max_{0\le k\le n} \widehat{B}_k$, then we have

$$X_n^p = S_n^p - B_n^p = \max_{0\le k\le n}(B_k^p - B_n^p) \ge \max_{0\le k\le n}(\widehat{B}_k - \widehat{B}_n) = \widehat{X}_n.$$

Proposition 4.1. *For any stopping time τ of $(\mathcal{F}_n)_{0\le n\le N}$,*

$$\mathrm{E}[\exp(\delta(B_{\rho_\tau}^p - S_N^p))] = \mathrm{E}[F(\tau, X_\tau^p)],$$

and, for any stopping time τ of $(\widehat{\mathcal{F}}_n)_{0\le n\le N}$ (and, hence, for any stopping time τ of $(\mathcal{F}_n)_{0\le n\le N}$),

$$\mathrm{E}[\exp(\delta(-S_N^p))] = \mathrm{E}[D(\tau, \widehat{X}_\tau)],$$

where

$$F(n,i) = \mathrm{E}[\exp(\delta(i - S_{N-n}^p))\mathbf{1}_{\{S_{N-n}^p \ge i\}} + \exp(\delta(B_{N-n}^p - i))\mathbf{1}_{\{S_{N-n}^p < i\}}] \tag{4.1}$$

and

$$\begin{aligned} D(n,i) &= \mathrm{E}[\exp(\delta(B_{N-n}^q - i \vee S_{N-n}^q))] \\ &= \mathrm{E}[\exp(\delta(B_{N-n}^q - S_{N-n}^q))\mathbf{1}_{\{S_{N-n}^q \ge i\}} + \exp(\delta(B_{N-n}^q - i))\mathbf{1}_{\{S_{N-n}^q < i\}}]. \end{aligned}$$

Proof. Denote $\mathrm{E}[Z\mathbf{1}_A]$ by $\mathrm{E}[Z; A]$, where $\mathbf{1}_A$ is an indicator function. Using the fact that, for any stopping time $\tau \le T$, the post-τ process of a p-random walk is still a p-random walk

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itself and is also independent of the σ-algebra $\mathcal{F}_\tau$, and the definition of ρ_τ, we then have

$$\begin{aligned}
&\mathrm{E}[\exp(\delta(B^p_{\rho_\tau} - S^p_N)) \mid \mathcal{F}_\tau] \\
&\quad = \mathrm{E}\Big[\exp\Big(\delta\Big(B^p_{\rho_\tau} - \Big(S^p_\tau \vee \max_{\tau\le n\le N} B^p_n\Big)\Big)\Big);\ \max_{\tau\le n\le N} B^p_n \ge S^p_\tau \;\Big|\; \mathcal{F}_\tau\Big] \\
&\qquad + \mathrm{E}\Big[\exp\Big(\delta\Big(B^p_{\rho_\tau} - \Big(S^p_\tau \vee \max_{\tau\le n\le N} B^p_n\Big)\Big)\Big);\ \max_{\tau\le n\le N} B^p_n < S^p_\tau \;\Big|\; \mathcal{F}_\tau\Big] \\
&\quad = \mathrm{E}\Big[\exp\Big(\delta\Big(S^p_\tau - \max_{\tau\le n\le N} B^p_n\Big)\Big);\ \max_{\tau\le n\le N} B^p_n \ge S^p_\tau \;\Big|\; \mathcal{F}_\tau\Big] \\
&\qquad + \mathrm{E}\Big[\exp(\delta(B^p_N - S^p_\tau));\ \max_{\tau\le n\le N} B^p_n < S^p_\tau \;\Big|\; \mathcal{F}_\tau\Big] \\
&\quad = \mathrm{E}\Big[\exp\Big(\delta\Big((S^p_\tau - B^p_\tau) - \Big(\max_{\tau\le n\le N} B^p_n - B^p_\tau\Big)\Big)\Big);\ \max_{\tau\le n\le N} B^p_n - B^p_\tau \ge S^p_\tau - B^p_\tau \;\Big|\; \mathcal{F}_\tau\Big] \\
&\qquad + \mathrm{E}\Big[\exp(\delta((B^p_N - B^p_\tau) - (S^p_\tau - B^p_\tau)));\ \max_{\tau\le n\le N} B^p_t - B^p_\tau < S^p_\tau - B^p_\tau \;\Big|\; \mathcal{F}_\tau\Big] \\
&\quad = \mathrm{E}[\exp(\delta(i - \widetilde{S}^p_{N-n}));\ \widetilde{S}^p_{N-n} \ge i]|_{i=S^p_\tau - B^p_\tau,\, n=\tau} \\
&\qquad + \mathrm{E}[\exp(\delta(\widetilde{B}^p_{N-n} - i));\ \widetilde{S}^p_{N-n} < i]|_{i=S^p_\tau - B^p_\tau,\, t=\tau} \\
&\quad = F(\tau, X^p_\tau).
\end{aligned}$$

Therefore, using the tower property of conditional expectation, we deduce that

$$\mathrm{E}[\exp(\delta(B^p_{\rho_\tau} - S^p_N))] = \mathrm{E}[F(\tau, X^p_\tau)].$$

Finally, since both S^p_N and X^q_N are equal in law as random variables, we have

$$\mathrm{E}[\exp(-\delta S^p_N)] = \mathrm{E}[\exp(-\delta X^q_n)] = \mathrm{E}[\exp(-\delta \widehat{X}_n)].$$

So, for any stopping time τ of $(\widehat{\mathcal{F}}_n)_{0\le n\le N}$,

$$\begin{aligned}
\mathrm{E}[\exp(-\delta \widehat{X}_n) \mid \widehat{\mathcal{F}}_\tau] &= \mathrm{E}\Big[\exp\Big(\delta\Big((\widehat{B}_N - \widehat{B}_\tau) - (\widehat{S}_\tau - \widehat{B}_\tau) \vee \Big(\max_{\tau\le n\le N} \widehat{B}_n - \widehat{B}_\tau\Big)\Big)\Big) \;\Big|\; \widehat{\mathcal{F}}_\tau\Big] \\
&= \mathrm{E}[\exp(\delta(\widehat{B}_{N-n} - i \vee \widehat{S}_{N-n}))]|_{i=\widehat{X}_\tau,\, n=\tau} \\
&= D(\tau, \widehat{X}_\tau).
\end{aligned}$$

Hence, we also have

$$\mathrm{E}[\exp(-\delta S^p_N)] = \mathrm{E}[D(\tau, \widehat{X}_\tau)].$$

As a consequence of Proposition 4.1, in order to establish the claim that $\mathrm{E}[\exp(\delta(B^p_{\rho_\tau} - S^p_N))] \ge \mathrm{E}[\exp(\delta(B^p_\tau - S^p_N))]$ when $p \ge \frac{1}{2}$, it is sufficient to show that $F(n, i) \ge G(n, i)$ when $p \ge \frac{1}{2}$. On the other hand, for the $p < \frac{1}{2}$ case, since $i \mapsto G(n, i)$ (as defined in Lemma 2.1) is a decreasing function (see (4.5), below) and $X^p_n \ge \widehat{X}_n$, as mentioned above Proposition 4.1, we have $G(\tau, \widehat{X}_\tau) \ge G(\tau, X^p_\tau)$ for any stopping time τ of $(\mathcal{F}_n)_{0\le n\le N}$; moreover, as long as $\mathrm{P}(\tau > 0) > 0$ with $\tau \le N$, then we must have $\mathrm{P}(X^p_\tau > 0) > 0$. To see this, suppose that $\mathrm{P}(X^p_\tau = 0) = 1$ and let σ_0 denote the first time after 0 that the process X_n equals 0. Then $\mathrm{P}(\sigma_0 > N) > 0$. Since $\{\tau > 0\} \in \mathcal{F}_0$, $\mathrm{P}(\tau > 0) > 0$ implies that $\mathrm{P}(\tau > 0) = 1$. Our assumption that $\mathrm{P}(X^p_\tau = 0) = 1$ implies that $\mathrm{P}(\tau \ge \sigma_0) = 1$. Therefore, $\mathrm{P}(\tau > N) > 0$, contradicting the fact that $\tau \le N$. Thus, we have

$$\mathrm{P}(\widehat{X}_\tau < X^p_\tau) = \mathrm{P}(\widehat{X}_\tau < X^p_\tau \mid X^p_\tau > 0)\, \mathrm{P}(X^p_\tau > 0) > 0;$$

therefore, we also have $\mathrm{P}(G(\tau, \widehat{X}_\tau) > G(\tau, X^p_\tau)) > 0$. As a result, we arrive at the claim that, for any stopping time τ of $(\mathcal{F}_n)_{0\le n\le N}$,

$$\mathrm{E}[G(\tau, \widehat{X}_\tau)] \ge \mathrm{E}[G(\tau, X^p_\tau)] \quad \text{with equality if and only if } \tau = 0. \tag{4.2}$$

Therefore, if we can establish the inequality $D(n,i) \ge G(n,i)$ when $p < \frac{1}{2}$, we can deduce that, for any stopping time τ of $(\mathcal{F}_n)_{0\le n\le N}$,

$$\mathrm{E}[D(\tau, \widehat{X}_\tau)] \ge \mathrm{E}[G(\tau, \widehat{X}_\tau)] \ge \mathrm{E}[G(\tau, X^p_\tau)],$$

which, by Proposition 4.1, is the same as $\mathrm{E}[\exp(\delta(B^p_0 - S^p_N))] \ge \mathrm{E}[\exp(\delta(B^p_\tau - S^p_N))]$. The comparisons among these deterministic functions will be established in the following proposition.

Proposition 4.2. 1. *If $p \ge \frac{1}{2}$, $F(n,i) \ge G(n,i)$ for any $i \ge 0$ and $0 \le n \le N$ with equality if and only if either $i = 0$ or $n = N$.*

2. *If $p < \frac{1}{2}$, $D(n,i) \ge G(n,i)$ for any $i \ge 0$ and $0 \le n \le N$ with equality if and only if $i = 0$.*

Proof. 1. We first consider the case in which $p \ge \frac{1}{2}$. Write $\Delta(n,i) = G(n,i) - F(n,i)$. We aim to show that $\Delta(n,i) \le 0$ when $p \ge \frac{1}{2}$. From the expression in Proposition 4.1, for any $0 \le n \le N$, we have

$$\Delta(n,0) = G(n,0) - F(n,0) = \mathrm{E}[\exp(-\delta S^p_{N-n})] - \mathrm{E}[\exp(-\delta S^p_{N-n})] = 0,$$

while, for any $0 \le i \le N$, we also have

$$\Delta(N,i) = G(N,i) - F(N,i) = \mathrm{E}[\mathrm{e}^{-\delta i}] - \mathrm{E}[\mathrm{e}^{-\delta i}] = 0.$$

So the two functions agree on the boundary. Now suppose, on the contrary, that $\Delta(n_0, i_0) > 0$ for some n_0 and i_0 with $n_0 < N$ and $i_0 > 0$. For this n_0, since both G and F are bounded above by 1, we can define i^* such that

$$\Delta(n_0, i^*) = \max_{0\le i\le n_0} \Delta(n_0, i).$$

Then $i^* > 0$ and

$$\Delta(n_0, i^*) \ge \Delta(n_0, i^* - 1), \qquad \Delta(n_0, i^*) > 0. \tag{4.3}$$

On the other hand, for any $i > 0$, $\Delta(n,i) > 0$ means that $G(n,i) > F(n,i)$. Upon substituting the expressions for G and F into this equation, we have

$$\begin{aligned}\mathrm{E}[\exp(-\delta S^p_{N-n})\mathbf{1}_{\{S^p_{N-n}\ge i\}} &+ \mathrm{e}^{-\delta i}\mathbf{1}_{\{S^p_{N-n}<i\}}] \\ &> \mathrm{E}[\exp(\delta(i - S^p_{N-n}))\mathbf{1}_{\{S^p_{N-n}\ge i\}} + \exp(\delta(B^p_{N-n} - i))\mathbf{1}_{\{S^p_{N-n}<i\}}].\end{aligned}$$

Since $\mathrm{E}[\exp(-\delta S^p_{N-n})\mathbf{1}_{\{S^p_{N-n}\ge i\}}] < \mathrm{E}[\exp(\delta(i - S^p_{N-n}))\mathbf{1}_{\{S^p_{N-n}\ge i\}}]$ for $i > 0$, the above inequality implies that

$$\mathrm{E}[\mathrm{e}^{-\delta i}\mathbf{1}_{\{S^p_{N-n}<i\}}] > \mathrm{E}[\exp(\delta(B^p_{N-n} - i))\mathbf{1}_{\{S^p_{N-n}<i\}}].$$

After multiplying both sides by $(u-1)$, this inequality becomes

$$(u-1)\mathrm{e}^{-\delta i}\,\mathrm{P}(S^p_{N-n} < i) > (u-1)\,\mathrm{E}[\exp(\delta(B^p_{N-n} - i))\mathbf{1}_{\{S^p_{N-n}<i\}}]. \tag{4.4}$$

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Explicit computation yields, for any n and any $i > 0$,

$$G(n, i-1) - G(n, i) = (\mathrm{e}^{-\delta(i-1)} - \mathrm{e}^{-\delta i})\,\mathrm{P}(S^p_{N-n} < i)$$
$$= (u-1)\mathrm{e}^{-\delta i}\,\mathrm{P}(S^p_{N-n} < i), \tag{4.5}$$

which says that the left-hand side of (4.4) is equal to $G(n, i-1) - G(n, i)$. If we can prove that the right-hand side of (4.4) is larger or equal to $F(n, i-1) - F(n, i)$, that is, if we can show that

$$(u-1)\,\mathrm{E}[\exp(\delta(B^p_{N-n} - i))\mathbf{1}_{\{S^p_{N-n} < i\}}] \geq F(n, i-1) - F(n, i),$$

then (4.4) will imply that

$$G(n, i-1) - G(n, i) > F(n, i-1) - F(n, i),$$

which is the same as

$$\Delta(n, i-1) > \Delta(n, i).$$

Therefore, we have proved that

$$\Delta(n, i) > 0 \quad \text{implies that} \quad \Delta(n, i-1) > \Delta(n, i) \quad \text{for any } i > 0,$$

which contradicts (4.3) at $i = i^*$, and, hence, our claim follows. Upon using the expression of F in Proposition 4.1, direct calculation yields, for any n and any $i > 0$,

$$(u-1)\,\mathrm{E}[\exp(\delta(B^p_{N-n} - i))\mathbf{1}_{\{S^p_{N-n} < i\}}] - F(n, i-1) - F(n, i)$$
$$= (u-1)\,\mathrm{E}[\exp((\delta(i-1-S^p_{N-n}))); S^p_{N-n} \geq i-1]$$
$$- u\,\mathrm{E}[(1 - \exp((\delta(B_{N-n} - i)))); S^p_{N-n} = i-1].$$

Define

$$\varphi(n, i) = (u-1)\,\mathrm{E}[\exp((\delta(i-1-S^p_n))); S^p_n \geq i-1]$$
$$- u\,\mathrm{E}[(1 - \exp((\delta(B^p_n - i)))); S^p_n = i-1], \tag{4.6}$$

so all we need is to prove that, when $p \geq \frac{1}{2}$,

$$\varphi(n, i) \geq 0 \quad \text{for any } 0 \leq n \leq N. \tag{4.7}$$

The proof of inequality (4.7) is a simple but tedious computation. We defer this computation to Appendix A.

2. For the case in which $p < \frac{1}{2}$, we aim to show that, for any $0 \leq n \leq N$,

$$\mathrm{e}^{\delta i} D(n, i) \geq \mathrm{e}^{\delta i} G(n, i). \tag{4.8}$$

We note that, for any $0 \leq n \leq N$,

$$D(n, 0) = \mathrm{E}[\exp(\delta(B^q_{N-n} - S^q_{N-n}))] = \mathrm{E}[\exp(-\delta S^p_{N-n})] = G(n, 0),$$

since $X^q_n = S^q_n - B^q_n$ and S^p_n are equal in law. Therefore, by telescoping both sides of (4.8) and applying the fact that $D(n, 0) = G(n, 0)$, in order to prove (4.8), it is sufficient to first establish the inequality

$$\mathrm{e}^{\delta(i+1)} D(n, i+1) - \mathrm{e}^{\delta i} D(n, i) > \mathrm{e}^{\delta(i+1)} G(n, i+1) - \mathrm{e}^{\delta i} G(n, i) \quad \text{for any } 0 \leq n \leq N.$$

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By substituting the expressions for D and G, as defined in Lemma 2.1 and Proposition 4.1, the above inequality can be reduced to

$$\begin{aligned}(u-1)\,\mathrm{E}[\exp(\delta(B^q_{N-n}-S^q_{N-n}+i));\ S^q_{N-n}\ge i+1]\\ > (u-1)\,\mathrm{E}[\exp(\delta(i-S^p_{N-n}));\ S^p_{N-n}\ge i+1] \quad \text{for any } 0\le n\le N,\end{aligned}$$

which is equivalent to

$$\mathrm{E}[\exp(\delta(B^q_n-S^q_n));\ S^q_n\ge i+1] > \mathrm{E}[\exp(-\delta S^p_n);\ S^p_n\ge i+1] \quad \text{for any } 0\le n\le N. \tag{4.9}$$

The proof of (4.9) again follows from a simple but tedious computation which we defer to Appendix A.

According to the discussion immediately after Proposition 4.1, Proposition 4.2 concludes that, when $p<\frac{1}{2}$, for any stopping time τ of $(\mathcal{F}_n)_{0\le n\le N}$,

$$\mathrm{E}[\exp(\delta(B^p_\tau-S^p_N))]=\mathrm{E}[G(\tau,X^p_\tau)]\le\mathrm{E}[G(\tau,\widehat{X}_\tau)]\le\mathrm{E}[D(\tau,\widehat{X}_\tau)]=\mathrm{E}[\exp(-\delta S^p_N)].$$

This suggests that, using $\tau^*=0$, we can obtain the optimal value $V_2^*=\mathrm{E}[\exp(-\delta S^p_N)]$. On the other hand, for any τ with $\mathrm{P}(\tau>0)>0$, applying (4.2) we obtain the strict inequality $\mathrm{E}[G(\tau,X^p_\tau)]<\mathrm{E}[G(\tau,X^q_\tau)]$, which in turn implies that $\mathrm{E}[\exp(\delta(B^p_\tau-S^p_N))]<\mathrm{E}[\exp(-\delta S^p_N)]$, and, hence, $\tau^*=0$ is the unique optimal selling time when $p<\frac{1}{2}$. For the case in which $p\ge\frac{1}{2}$, as an immediate consequence of Proposition 4.1 and Proposition 4.2, we have $\mathrm{E}[\exp(\delta(B^p_{\rho_t}-S^p_N))]\ge\mathrm{E}[\exp(\delta(B^p_\tau-S^p_N))]$, which suggests that the optimal selling time should be in the form of ρ_τ for some τ. Therefore, we are motivated to study ρ_τ for different stopping times τ, and this can be achieved via the following proposition.

Proposition 4.3. *The process $F(n,X^p_n)$ is* (i) *a submartingale if $p>\frac{1}{2}$;* (ii) *a martingale if $p=\frac{1}{2}$; and* (iii) *a supermartingale if $p<\frac{1}{2}$.*

Proof. We first show that

$$\mathrm{E}[F(n+1,\ X^p_{n+1})\mid\mathcal{F}_n]=F(n,X^p_n), \tag{4.10}$$

provided that $X^p_n>0$. Recall that

$$F(\tau,X^p_\tau)=\mathrm{E}[\exp(\delta(B^p_{\rho_\tau}-S^p_N))\mid\mathcal{F}_\tau],$$

and, hence, by the tower property of conditional expectations we have

$$\begin{aligned}F(\tau,X^p_\tau)&=\mathrm{E}[\mathrm{E}[\exp(\delta(B^p_{\rho_\tau}-S^p_N))\mid\mathcal{F}_{\rho_\tau}]\mid\mathcal{F}_\tau]\\&=\mathrm{E}[G(\rho_\tau,X^p_{\rho_\tau})\mid\mathcal{F}_\tau].\end{aligned} \tag{4.11}$$

As pointed out in Proposition 3.1, $\rho_\tau=\tau+\tau_0\circ\theta_\tau$. Then (4.11) can be expressed through the strong Markov property as

$$F(\tau,X^p_\tau)=\mathrm{E}_{\tau,X^p_\tau}[G(\tau_0\wedge(N-\tau),X^p_{\tau_0\wedge(N-\tau)})].$$

In particular, since (τ,X^p_τ) can attain all possible states with positive probability, for any feasible pair (i,n), we have

$$F(n,i)=\mathrm{E}_{n,i}[G(\tau_0\wedge(N-n),X^p_{\tau_0\wedge(N-n)})],$$

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and, hence, provided that $X_n^p > 0$, using a first-step analysis and the Markov property of (n, X_n^p), we can also obtain

$$\begin{aligned} F(n, X_n^p) &= \mathrm{E}_{n,X_n^p}[G(\tau_0 \wedge (N-n),\ X^p_{\tau_0\wedge(N-n)})] \\ &= p\,\mathrm{E}_{n+1,X_n^p-1}[G(\tau_0 \wedge (N-n-1), X^p_{\tau_0\wedge(N-n-1)})] \\ &\quad + q\,\mathrm{E}_{n+1,X_n^p+1}[G(\tau_0 \wedge (N-n-1),\ X^p_{\tau_0\wedge(N-n-1)})] \\ &= \mathrm{E}[F(n+1,\ X^p_{n+1}) \mid \mathcal{F}_n]. \end{aligned}$$

Thus, we establish (4.10). For the case when $X_n^p = 0$, we would like to show that

$$\begin{aligned} &\mathrm{E}[F(n+1, X^p_{n+1}) \mid \mathcal{F}_n] - F(n, X_n^p) \\ &\quad = pF(n+1, 0) + qF(n+1, 1) - F(n, 0) \\ &\quad \begin{cases} > 0 & \text{if } p > \frac{1}{2}, \\ = 0 & \text{if } p = \frac{1}{2}, \\ < 0 & \text{if } p < \frac{1}{2}. \end{cases} \end{aligned} \tag{4.12}$$

The proof of (4.12) constitutes a straightforward computation which we defer to Appendix A. We complete the proof by combining (4.10) and (4.12).

Finally, by combining all previous results, we can now conclude with our main theorem in this section.

Theorem 4.1. *Suppose that the dynamics of a stock price are modeled by the CRR model.*

1. *If $p > \frac{1}{2}$ then $\tau^* = N$ is an optimal selling time for problem (1.2), and*

$$V_2^* = \mathrm{E}[\exp(\delta(B_N^p - S_N^p))].$$

2. *If $p = \frac{1}{2}$ then any τ satisfying $\tau = \rho_\tau$ almost surely is an optimal selling time for problem (1.2), and*

$$V_2^* = \mathrm{E}[\exp(\delta(B_\tau^{1/2} - S_N^{1/2}))]$$

for any stopping time τ satisfying $\tau = \rho_\tau$. In particular,

$$V_2^* = \mathrm{E}[\exp(-\delta S_N^{1/2})] = \mathrm{E}[\exp(\delta(B_N^{1/2} - S_N^{1/2}))].$$

3. *If $p < \frac{1}{2}$ then $\tau^* = 0$ is the unique optimal selling time for problem (1.2), and*

$$V_2^* = \mathrm{E}[\exp(-\delta S_N^p)].$$

Proof. The third claim has already been proved in Proposition 4.2 together with the discussion that followed. It remains to show the first two claims. In accordance with Proposition 4.1 and Proposition 4.2, we know that

$$\mathrm{E}[\exp(\delta(B^p_{\rho_\tau} - S_N^p))] \geq \mathrm{E}[\exp(\delta(B_\tau^p - S_N^p))], \quad \text{whenever } p \geq \tfrac{1}{2}.$$

1. When $p > \frac{1}{2}$, since $F(n, X_n^p)$ is a submartingale, for any stopping time τ, by using the optional stopping theorem and the fact that $N = \rho_N$, we have

$$\begin{aligned} \mathrm{E}[\exp(\delta(B_\tau^p - S_N^p))] &\le \mathrm{E}[\exp(\delta(B_{\rho_\tau}^p - S_N^p))] \\ &= \mathrm{E}[F(\tau, X_\tau^p)] \\ &\le \mathrm{E}[F(N, X_N^p)] \\ &= \mathrm{E}[\exp(\delta(B_{\rho_N}^p - S_N^p))] \\ &= \mathrm{E}[\exp(\delta(B_N^p - S_N^p))], \end{aligned}$$

which implies that $V_2^* = \mathrm{E}[\exp(\delta(B_N^p - S_N^p))]$ and that $\tau^* = N$ is an optimal selling time

2. When $p = \frac{1}{2}$, since $F(n, X_n^{1/2})$ is a martingale, the above chain of inequalities remains valid, except that the inequality

$$\mathrm{E}[F(\tau, X_\tau^p)] \le \mathrm{E}[F(N, X_N^p)]$$

can now be replaced by an equality. As a consequence, the optimal value $V_2^* = \mathrm{E}[\exp(\delta(B_N^{1/2} - S_N^{1/2}))]$ can be achieved by any stopping time τ satisfying $\tau = \rho_\tau$; in particular, both 0 and N are optimal stopping times.

5. Conclusions and future work

We conclude by pointing out two possible extensions of our present work. Firstly, as both problems (1.1) and (1.2) are solved in the CRR framework, it is natural to consider the same problems under a trinomial tree model or even a multinomial tree model (see the work of Boyle (1988), in which a trinomial tree model was used in the context of option pricing). Secondly, our simple binomial tree model has neglected the possible long-range dependence of stock price movement; instead of using binomial tree processes, which is an exponential of a simple random walk, we may consider modeling a stock price by an exponential of a correlated random walk. In this case, as long as this new 'correlated' binomial tree process is directionally reinforced, i.e. if the last step is going up or going down then at the present step it is more likely to go up or, respectively, go down, it is interesting to ask whether the 'buy-and-hold or sell-at-once' rule would still be optimal for problems (1.1) and (1.2).

Appendix A. Proofs of inequalities (4.7) and (4.9), and statement (4.12)

Proof of (4.7). We shall prove that the function φ defined in (4.6) satisfies

$$\varphi(n, i) \begin{cases} > 0 & \text{if } p > \frac{1}{2}, \\ = 0 & \text{if } p = \frac{1}{2}, \\ < 0 & \text{if } p < \frac{1}{2}, \end{cases}$$

which will in turn imply inequality (4.7). For this, let us write

$$\varphi(n, i) = \varphi_1(n, i) - \varphi_2(n, i),$$

where

$$\varphi_1(n, i) := (u - 1)\,\mathrm{E}[\exp((\delta(i - 1 - S_n^p))); S_n^p \ge i - 1], \tag{A.1}$$

$$\varphi_2(n, i) := u\,\mathrm{E}[(1 - \exp((\delta(B_n^p - i)))); S_n^p = i - 1]. \tag{A.2}$$

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Let $d = \mathrm{e}^{-\delta}$. Then

$$\begin{aligned}\varphi_1(n,i) &:= (u-1)\,\mathrm{E}[d^{\delta(S_n^p - i+1)};\ S_n^p \geq i-1]\\ &= (u-1)\sum_{j=0}^{n-i+1} d^j\,\mathrm{P}(S_n^p = i-1+j)\\ &= (u-1)\sum_{j=0}^{n-i+1} d^j \sum_{l=2(j+i-1)-n}^{j+i-1} \mathrm{P}(S_n^p = i-1+j,\ B_n^p = l)\\ &= (u-1)\sum_{j=0}^{n-i+1} d^j \sum_{l=2(i-1)-n}^{i-1-j} \mathrm{P}(S_n^p = i-1+j,\ B_n^p = l+2j),\end{aligned} \tag{A.3}$$

and, by using $1-d^l = (1-d)(1+d+\cdots+d^{l-1})$ and $\mathrm{P}(S_n^p = k,\ B_n^p \leq l) = 0$ if $l < 2k-n$,

$$\begin{aligned}\varphi_2(n,i) &:= u\,\mathrm{E}[(1-d^{i-B_n^p});\ S_n^p = i-1]\\ &= (u-1)\,\mathrm{E}\Bigg[\sum_{j=0}^{i-1-B_n^p} d^j \mathbf{1}_{\{S_n^p = i-1\}}\Bigg]\\ &= (u-1)\sum_{j=0}^{n} d^j\,\mathrm{P}(S_n^p = i-1,\ B_n^p \leq i-1-j)\\ &= (u-1)\sum_{j=0}^{n-i+1} d^j \sum_{l=2(i-1)-n}^{i-1-j} \mathrm{P}(S_n^p = i-1,\ B_n^p = l).\end{aligned} \tag{A.4}$$

Then applying the joint density of (B_n^p, S_n^p) as given in (2.4), we see that

$$\mathrm{P}(S_n^p = i-1+j,\ B_n^p = l+2j) = \mathrm{P}(S_n^p = i-1,\ B_n^p = l)\left(\frac{p}{q}\right)^j;$$

therefore, $\mathrm{P}(S_n^p = i-1+j,\ B_n^p = l+2j) = \mathrm{P}(S_n^p = i-1,\ B_n^p = l)$ if $p = \frac{1}{2}$, and the equality becomes '$>$' if $p > \frac{1}{2}$ and '$<$' if $p < \frac{1}{2}$, which, together with (A.3) and (A.4), implies our initial claim.

Proof of (4.9). We first note that, since the joint distributions of (X_n^q, S_n^q) and (S_n^p, X_n^p) are the same, (4.9) is the same as

$$\mathrm{E}[\exp(-\delta S_n^p);\ S_n^p - B_n^p \geq i+1] > \mathrm{E}[\exp(-\delta S_n^p);\ S_n^p \geq i+1] \quad \text{for any } 0 \leq n \leq N. \tag{A.5}$$

Observe that on the event $\{S_n^p - B_n^p \geq i+1\}$, S_n^p cannot be larger than $n-i-1$. So, (A.5) can be expanded as

$$\sum_{k=0}^{n-i-1} \sum_{l=2k-n}^{k-i-1} \mathrm{e}^{-\delta k}\,\mathrm{P}(B_n^p = l,\ S_n^p = k) > \sum_{k=i+1}^{n} \sum_{l=2k-n}^{k} \mathrm{e}^{-\delta k}\,\mathrm{P}(B_n^p = l,\ S_n^p = k). \tag{A.6}$$

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Setting $k' = k - i - 1$ and $l' = l - 2(i+1)$ on the right-hand side of (A.6), we obtain

$$\sum_{k=0}^{n-i-1} \sum_{l=2k-n}^{k-i-1} \mathrm{e}^{-\delta k}\, \mathrm{P}(B_n^p = l,\ S_n^p = k)$$
$$> \sum_{k'=0}^{n-i-1} \sum_{l'=2k'-n}^{k'-i-1} \mathrm{e}^{-\delta k'}\, \mathrm{P}(B_n^p = l' + 2(i+1),\ S_n^p = k' + i + 1)\mathrm{e}^{-\delta(i+1)}. \tag{A.7}$$

Then applying the joint density of (B_n^p, S_n^p) as given in (2.4), we see that, if $p \le \frac{1}{2}$,

$$\begin{aligned}\mathrm{P}(B_n^p = l,\ S_n^p = k) &= \mathrm{P}(B_n^p = l + 2(i+1),\ S_n^p = k + (i+1))\left(\frac{p}{q}\right)^{-(i+1)} \\ &> \mathrm{P}(B_n^p = l + 2(i+1),\ S_n^p = k + (i+1))\mathrm{e}^{-\delta(i+1)},\end{aligned}$$

which proves inequality (A.7) and, hence, inequality (4.9).

Proof of statement (4.12). We first use expression (4.1) to obtain

$$\begin{aligned}&\mathrm{E}[F(n+1, X_{n+1}^p) \mid \mathcal{F}_n] - F(n, X_n^p) \\ &\quad = pF(n+1, 0) + qF(n+1, 1) - F(n, 0) \\ &\quad = p\,\mathrm{E}[\exp(-S_{N-n-1}^p)] + q\,\mathrm{E}[\exp(\delta(1 - S_{N-n-1}^p));\ S_{N-n-1}^p \ge 1] \\ &\qquad + q\,\mathrm{E}[\exp(\delta(B_{N-n-1}^p - 1));\ S_{N-n-1}^p = 0] - \mathrm{E}[\exp(-\delta S_{N-n}^p)].\end{aligned} \tag{A.8}$$

Let us write (A.8) in a more explicit form. Applying the first claim in Lemma 2.2, and then considering the two possible changes of B_{N-n-1}^q after one step, we have

$$\begin{aligned}\mathrm{E}[\exp(-\delta S_{N-n}^p)] &= \mathrm{E}[\exp(-\delta X_{N-n}^q)] \\ &= p\mathrm{e}^{-\delta}\,\mathrm{E}[\exp(-X_{N-n}^q)] + q\,\mathrm{E}[\exp(\delta(1 - X_{N-n-1}^q));\ X_{N-n-1}^q \ge 1] \\ &\quad + q\,\mathrm{P}(X_{N-n-1}^q = 0) \\ &= p\mathrm{e}^{-\delta}\,\mathrm{E}[\exp(-S_{N-n-1}^p)] + q\,\mathrm{E}[\exp(\delta(1 - S_{N-n-1}^p));\ S_{N-n-1}^p \ge 1] \\ &\quad + q\,\mathrm{P}(S_{N-n-1}^p = 0);\end{aligned}$$

hence, (A.8) becomes

$$\begin{aligned}&\mathrm{E}[F(n+1,\ X_{n+1}^p) \mid \mathcal{F}_n] - F(n,\ X_n^p) \\ &\quad = p(1 - \mathrm{e}^{-\delta})\,\mathrm{E}[\exp(-S_{N-n-1}^p)] + q\,\mathrm{E}[\exp(\delta(B_{N-n-1}^p - 1));\ S_{N-n-1}^p = 0] \\ &\qquad - q\,\mathrm{P}(S_{N-n-1} = 0) \\ &\quad = p(1 - \mathrm{e}^{-\delta})\,\mathrm{E}[\exp(-S_{N-n-1}^p)] - q\,\mathrm{E}[(1 - \exp(\delta(B_{N-n-1}^p - 1)));\ S_{N-n-1}^p = 0].\end{aligned}$$

Define

$$\psi(n) = \psi_1(n) - \psi_2(n),$$

where

$$\psi_1(n) = p(1 - \mathrm{e}^{-\delta})\,\mathrm{E}[\exp(-S_n^p)] \tag{A.9}$$

and

$$\psi_2(n) = q\,\mathrm{E}[(1 - \exp(\delta(B_n^p - 1)));\ S_n^p = 0]. \tag{A.10}$$

All we need to prove is that $\psi_1(n) = \psi_2(n)$ if $p = \frac{1}{2}$, and the equality becomes '>' if $p > \frac{1}{2}$ and '<' if $p < \frac{1}{2}$. But comparing (A.1) and (A.2) with (A.9) and (A.10), respectively, we see that

$$\psi_1(n) = \frac{p}{u}\varphi_1(n, 1), \qquad \psi_2(n) = \frac{q}{u}\varphi_2(n, 1).$$

Therefore, statement (4.12) follows from the proof of inequality (4.7).

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