PERIPHERAL COVERING PROPERTIES IMPLY COVERING PROPERTIES

E. E. GRACE

1. Introduction. Recently several papers (**11**; **12**; **13**; **14**) have been published in which it is shown that a Moore space (normal, in one case) is metrizable if it has the peripheral version (in the sense defined below) of a certain covering property that was known to imply metrizability of Moore spaces. Each of these metrization theorems can be proved more easily by using a slight variation of the appropriate standard proof to show that such a space is collectionwise normal and hence (**2**, Theorem 10) metrizable. But this approach, as well as that followed in (**11**; **12**; **13**; **14**), obscures the point that, in Moore spaces and in more general settings, the peripheral versions of these covering properties imply the covering properties. It is the purpose of this paper to point out the existence of a large class of covering properties, such as paracompactness (**3**; **6**; **7**; **8**; **9**; **10**) and strong screenability (**2**), which apply to a space as a whole if they apply (in the very restrictive sense given below) to each nowhere dense, closed sub*set* of the space.

In this paper, the collection $\mathscr{V}$ is a *refinement* of the collection $\mathscr{U}$ if each member of $\mathscr{V}$ is a subset of some member of $\mathscr{U}$, regardless of whether $\mathscr{U}$ and $\mathscr{V}$ cover the same set. The union of all the members of a collection $\mathscr{V}$ of sets is denoted by $\mathscr{V}^*$.

A collection $\mathscr{V}$ of sets *covers* a set K if $\mathscr{V}^* \supset K$. $\mathscr{V}$ is an *open refinement* of $\mathscr{U}$ if $\mathscr{V}$ is a refinement of $\mathscr{U}$ and each member of $\mathscr{V}$ is open in the topological space under consideration (rather than merely open relative to some subset which $\mathscr{V}$ covers). Much of the other terminology used here is defined in (**6**, **7**, **8**, or **9**).

Definition. A regular topological space T is said to be *peripherally paracompact* in the strong (weak) sense if, for each frontier set (i.e., each nowhere dense, closed set) F in T and each open cover $\mathscr{U}$ of T, there is an open refinement $\mathscr{V}$ of $\mathscr{U}$, covering F, which is locally finite at each point of T (of $\mathscr{V}^*$). Similarly, a collection V of open sets in a topological space T is said to *have property P in the strong* (*weak*) *sense* if $\mathscr{V}$ has the property as a collection of open sets in T (in the subspace $\mathscr{V}^*$ of T). For example, a collection $\mathscr{V}$ of open (in T) sets in a topological space T is said to be *locally finite in the strong sense* if each point of T is contained in an open set which intersects at most a finite number of elements of $\mathscr{V}$. The collection $\mathscr{V}$ of open (in T) sets is said to be *locally finite in the weak sense* if $\mathscr{V}$ is locally finite as a cover of the subspace $\mathscr{V}^*$ of T.

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It should be noted that, in this context, requiring local finiteness of $\mathscr{V}$ at each point of $\mathscr{V}^*$ is equivalent to requiring it only at each point of F. The form used here, however, is more nearly typical of what works for other covering properties.

Definition. A topological space T is said to be *peripherally strongly screenable* in the strong (weak) sense if, for each frontier set F and each open cover $\mathscr{U}$ of T, there is an open refinement $\mathscr{V} = \mathscr{V}_1 \cup \mathscr{V}_2 \cup \ldots$ of $\mathscr{U}$ such that $\mathscr{V}^*$ covers F and each $\mathscr{V}_i$ is discrete in T (in the subspace $\mathscr{V}^*$).

It is hoped that the definitions intended for other peripheral covering properties will be clear from the preceding definitions of peripheral paracompactness and peripheral strong screenability.

For certain covering properties, e.g., full normality and screenability, the strong and weak senses are equivalent, and in such a case the qualifying phrase may be omitted. Also note that if $\mathscr{V}^* = T$, then $\mathscr{V}$ has property P in the strong sense if and only if $\mathscr{V}$ has property P in the weak sense (no matter what P is).

2. Open covering properties.

THEOREM 1. *A regular topological space T is paracompact* (**6**, p. 156) *if (and only if) it is peripherally paracompact in the strong sense.*

Proof. Let $\mathscr{U}$ be an open cover of T and let $\mathscr{D}$ be a collection of mutually exclusive open sets, refining $\mathscr{U}$, such that $\mathscr{D}^*$ is dense in T. Further let $\mathscr{V}$ be a locally finite, open refinement of $\mathscr{U}$ covering the frontier set $T - \mathscr{D}^*$, and let $\mathscr{W}$ be a locally finite open refinement of $\mathscr{U}$ covering the boundary of $\mathscr{V}^*$.

Let $\mathscr{D}' = \{D \cap (T - \overline{\mathscr{V}^*}) \mid D \in \mathscr{D}\}$, which is a discrete collection of open sets. Then $\mathscr{D}' \cup \mathscr{V} \cup \mathscr{W}$ is a locally finite open refinement of $\mathscr{U}$ that covers T. Hence T is paracompact.

The characteristics of local finiteness (in the strong sense) that are crucial to the proof of Theorem 1 are (1) it is an additive property, i.e., the union of two collections has the property if each of the two collections has the property, and (2) each discrete collection of open sets has the property. From this comment it should be clear that the following theorem can be established by an argument similar to the proof of Theorem 1.

THEOREM 2. *A topological space is strongly screenable if (and only if) it is peripherally strongly screenable in the strong sense.*

THEOREM 3. *A regular topological space is peripherally paracompact in the strong sense if (and only if) it is peripherally paracompact in the weak sense.*

Proof. Let T be a regular topological space that is peripherally paracompact in the weak sense. First it will be shown that T is normal. Let A and B be two mutually exclusive closed sets and let A' and B', respectively, be their boundaries. Then $A' \cup B'$ is a frontier subset of T. Let $\mathscr{U}$ be an open cover of T such that the closure of no element of $\mathscr{U}$ intersects both A' and B'. Let $\mathscr{V}$ be

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an open refinement of $\mathscr{U}$ that covers $A' \cup B'$ and is locally finite at each point of $\mathscr{V}^*$. In addition, let $\mathscr{V}_A = \{V \in \mathscr{V} \mid V \cap A' \neq \emptyset\}$. The closure of $\mathscr{V}_A{}^*$ and B' do not intersect, since $\mathscr{V}_A$ is locally finite at each point of B' and the closure of each element of $\mathscr{V}_A$ fails to intersect B'. It follows that $[\mathscr{V}_A{}^* \cup \operatorname{Int}(A)] - B$ is an open set containing A, the closure of which does not intersect B, and consequently that T is normal.

Now, let F be any nowhere dense, closed subset of T and let $\mathscr{U}'$ be any open cover of T. Then by hypothesis there is an open refinement $\mathscr{V}'$ of $\mathscr{U}'$, covering F, which is locally finite at each point of $\mathscr{V}'^*$. Since T is normal, and F and the complement of $\mathscr{V}'^*$ are mutually exclusive closed sets, there is an open set D such that $\mathscr{V}'^* \supset \bar{D} \supset D \supset K$. Let $\mathscr{W} = \{V \cap D \mid V \in \mathscr{V}'\}$. This is the desired open refinement of $\mathscr{U}'$, which covers F and is locally finite in the strong sense.

Corollary 3.1. *A regular topological space that is peripherally paracompact in the weak sense is paracompact.*

Theorem 4. *Let T be a topological space and let P be any one of the following properties:*
(a) *locally finite in the strong sense,*
(b) *point finite,*
(c) *point countable,*
(d) *cushioned in $\mathscr{U}$ in the strong sense,*
(e) *closure-preserving in the strong sense,*
(f) *countable,*
(g) *σ-locally finite in the strong sense,*
(h) *σ-point finite,*
(i) *σ-cushioned in $\mathscr{U}$ in the strong sense,*
(j) *σ-closure-preserving in the strong sense,*
(k) *σ-disjoint,*
(l) *σ-point star refinement of $\mathscr{U}$.*
Then an open cover $\mathscr{U}$ of T has an open refinement covering T and having property P if (and only if), for each frontier set F in T, there is an open (in T) refinement of $\mathscr{U}$ covering F and having property P.

Proof. (a)–(l) all follow from essentially the proof of Theorem 1.

Corollary 4.1. *A topological space is countably paracompact if (and only if) it is either* (a) *peripherally countably paracompact in the strong sense or* (b) *peripherally pointwise countably paracompact and normal.*

Corollary 4.1 follows from Parts (a) and (b) of Theorem 4 and Dowker's characterization (**4**, Theorem 2) of countably paracompact normal spaces as pointwise countably paracompact normal spaces.

Theorem 5. *Let T be a normal topological space and let P be any one of the following properties:*

(a) *locally finite in the weak sense,*
(b) *cushioned in $\mathscr{U}$ in the weak sense,*
(c) *closure-preserving in the weak sense,*
(d) *point star refinement of $\mathscr{U}$,*
(e) *has a partition of unity subordinated to it in the weak sense,*
(f) *σ-locally finite in the weak sense,*
(g) *σ-cushioned in $\mathscr{U}$ in the weak sense,*
(h) *σ-closure-preserving in the weak sense.*

Then an open cover $\mathscr{U}$ of T has an open refinement covering T and having property P if (and only if), for each frontier set F in T, there is an open (in T) refinement of $\mathscr{U}$ covering F and having property P.

Proof. Parts (a), (b), (c), (f), (g), and (h): In each case a "cutting back" such as that in the proof of Theorem 3 yields an open refinement of $\mathscr{U}$ covering F and having the desired property in the strong sense. These parts of the theorem then follow from Theorem 4.

Part (d): Let $\mathscr{U}$ be an open cover of T and let $\mathscr{D}$ be a collection of disjoint open sets, refining $\mathscr{U}$, such that $\mathscr{D}^*$ is dense in T. In addition, let $\mathscr{V}$ be an open point star refinement of $\mathscr{U}$ covering $T - \mathscr{D}^*$. The problem here, as in the cases treated above, is that the property under consideration is not additive. But the treatment required here is somewhat different. Since T is normal, there is a continuous function g from T into the unit interval such that $g^{-1}(0) \supset (T - \mathscr{V}^*)$ and $g^{-1}(1) \supset (T - \mathscr{D}^*)$. For each D in $\mathscr{D}$, let $D' = D \cap [g^{-1}([0, \frac{1}{2}))$. For each V in $\mathscr{V}$ and each D in $\mathscr{D}$, let $V_D = V \cap D$ and let $V' = V \cap g^{-1}((\frac{1}{2}, 1])$. The desired open point star refinement of $\mathscr{U}$ which covers T is

$$\mathscr{V}' = \{D' \mid D \in \mathscr{D}\} \cup \{V_D \mid V \in \mathscr{V} \text{ and } D \in \mathscr{D}\} \cup \{V' \mid V \in \mathscr{V}\}.$$

Part (e): This could be dealt with by first proving that the property is additive and then relying on essentially the proof of Theorem 1. However, a proof in the spirit of the proof of part (d) seems more natural at this point.

Let $\mathscr{D}$ be defined as above and let $\mathscr{V}$ be an open refinement of $\mathscr{U}$ covering $T - \mathscr{D}^*$ and having a partition of Unity ϕ subordinated to it in the subspace $\mathscr{V}^*$. Let g be defined as in the proof of part (d). For each f in ϕ, let $f'(t) = f(t)\cdot g(t)$, for t in $\mathscr{V}^*$, and let $f'(t) = 0$, for t in $T - \mathscr{V}^*$. For each D in $\mathscr{D}$ there is a continuous function g_D from T into the unit interval such that $g_D^{-1}(0) \supset (T - D)$ and $g_D^{-1}(1) \supset D \cap g^{-1}(0)$. Let

$$h(t) = \begin{cases} \sum_{f\in\phi} f'(t), & \text{if } t \in T - \mathscr{D}^*, \\ \sum_{f\in\phi} f'(t) + g_D(t), & \text{if } t \in D \in \mathscr{D}^*. \end{cases}$$

Then $\mathscr{U}$ itself is an open refinement of $\mathscr{U}$ which covers T and has the partition of unity

$$\phi' = \{f'/h \mid f \in \phi\} \cup \{g_D/h \mid D \in \mathscr{D}\}$$

subordinated to it.

It can readily be seen that a topological space T is normal if, for each finite open cover $\mathscr{U}$ of T and each frontier set F, there is an open refinement of $\mathscr{U}$

having some one of the properties (b), (d), (e), or (g) of Theorem 5. For a regular topological space, the comment applies also to properties (a), (c), (f), and (h) of Theorem 5. For properties (a), (b), (c), (d), and, to a lesser extent, (e), the proof is similar to the normality part of the proof of Theorem 3. For properties (f), (g), and (h), the proof is similar to that of **(6**, Lemma 5.34**)**, but uses some of the ideas in the proof of Theorem 3.

COROLLARY 5.1. *If T is a topological space such that for each open cover $\mathscr{U}$ of T and each frontier set F in T, there is an open refinement $\mathscr{V}$ of $\mathscr{U}$, covering F, that has one of the properties*

(a) *cushioned in $\mathscr{U}$ in the weak sense,*

(b) *point star refinement of $\mathscr{U}$, or*

(c) *has a partition of unity subordinated to it in the weak sense,*

then (and only then) each open cover $\mathscr{U}$ of T has an open refinement which covers T and has that same property.

The author does not know whether the hypothesis that the space is regular can be omitted in the following corollaries.

COROLLARY 5.2. *If T is a regular topological space such that, for each open cover $\mathscr{U}$ of T and each frontier set F in T, there is an open refinement $\mathscr{V}$ of $\mathscr{U}$ covering F that is closure-preserving in the weak sense, then (and only then) each open cover of T has a closure-preserving open refinement which covers T.*

COROLLARY 5.3. *A regular topological space that is peripherally countably paracompact in the weak sense is countably paracompact.*

Definition. A cover $\mathscr{U}$ of a topological space T is said to be *peripherally even* if, for each frontier set F in T, there is a neighbourhood W of $\{(p, p) \mid p \in F\}$ in $T \times T$ such that $\{W(p) \mid (p, q) \in W$, for some q in $T\}$ refines $\mathscr{U}$, where $W(p) = \{q \mid (p, q) \in W\}$.

THEOREM 6. *If each open cover of a topological space T is peripherally even, then (and only then) each open cover of T is even.*

Proof. It suffices to show that, under the hypothesis of the theorem, T is peripherally fully normal. It then follows, by Corollary 5.1, that T is fully normal and hence, by **(6**, *p*. 171*U***)**, each open cover of T is even.

To show that T is peripherally fully normal, let $\mathscr{U}$ be an open cover of T and F be a frontier set in T. Let W be a neighbourhood of $\{(p, p) \mid p \in F\}$ in $T \times T$ such that $\{W(p) \mid (p, q) \in W$, for some q in $T\}$ refines $\mathscr{U}$. For each point p in F let D_p be an open set, containing p, such that $D_p \times D_p \subset W$. The collection $\{D_p \mid p \in F\}$ is an open point star refinement of $\mathscr{U}$ which covers F. Consequently, T is peripherally fully normal.

One might hope that Theorem 6 would remain true if the definition of peripheral evenness were changed by requiring only that $\{W(p) \mid p \in F\}$ refines $\mathscr{U}$. Example 2 of **(5)** eliminates that hope.

THEOREM 7. *There is a metric space that is peripherally strongly paracompact but not strongly paracompact* **(1**, p. 36**)**.

Proof. The space is the "hedgehog" of spininess c (**1**, p. 36), i.e. a cantor fan with the "path distance." One can use the fact that each nowhere dense set in this space is totally disconnected to prove the space is peripherally strongly paracompact. It is well known (**1**, p. 36) that the space is not strongly paracompact.

3. Covering properties on frontier subspaces. It is natural to ask whether a theory similar to the foregoing holds for closed coverings or (almost equivalently, in view of (**6**), (**7**), (**8**), and others) whether a space, in which each frontier set has a certain covering property as a subspace, must also have that property. The next four theorems (each of which has been known essentially for some time) give partial answers to such questions.

THEOREM 8. *If T is a T_1-space with at most countably many isolated points and each frontier set in T is a Lindelöf space as a subspace, then T is a Lindelöf space.*

Proof. It is easily seen that T is a peripherally Lindelöf space and that each discrete collection of open subsets of T is countable. Hence, by part (f) of Theorem 4, T is a Lindelöf space.

THEOREM 9. *If P is a property that is equivalent to strong screenability in collectionwise normal T_1-spaces and T is a collectionwise normal T_1-space in which each frontier set has property P as a subspace, then T has property P.*

Proof. It is easily seen that a collectionwise normal topological space is peripherally strongly screenable in the strong sense, and hence strongly screenable, if each frontier subset of it is strongly screenable as a subspace.

Note that the T_1-condition placed on property P and on the space T in Theorem 9 is not necessary to the proof. The only requirement is that the same class of spaces be used in both places.

THEOREM 10. *There exists a non-normal (and hence non-paracompact, etc.) Moore space in which each frontier set has the discrete topology as a subspace (and hence is paracompact, etc. as a subspace).*

Proof. (**5**, Example 2) is such a space. The frontier subsets are the subsets of the x-axis.

THEOREM 11. *There exists a perfectly normal, non-collectionwise normal (and hence non-paracompact, etc.) Hausdorff space in which each frontier set has the discrete topology as a subspace (and hence is paracompact, etc., as a subspace).*

Proof. (**2**, Example H) is such a space.

Definition. A property Q of topological spaces is said to be an *open covering property* if there is a property R such that Q is "each open covering of the space has an open refinement which covers the space and has property R."

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THEOREM 12. *If Q is an open covering property which is equivalent to paracompactness for Moore spaces and such that peripherally Q in the "strong sense" and in the "weak sense" are the same, then σ-Q is not equivalent to paracompactness for Moore spaces.*

Proof. The theorem follows from the existence of Example 2 of **(5)**, which is a non-paracompact Moore space that is the union of two open subsets each of which is paracompact as a subspace. The two open subsets are (1) the rationals on the x-axis together with the points above the x-axis and (2) the irrationals on the x-axis together with the points above the x-axis.

COROLLARY 12.1. *σ-full normality is not equivalent to paracompactness (and hence is not equivalent to full normality) even for Moore spaces.*

Questions. Must a *normal* Moore space be metrizable if each of its frontier subsets is metrizable? Can collectionwise normality in Theorem 9 be replaced by the condition that, for each discrete collection K of *points*, there is an indexed collection $\{D_k \mid k \in K\}$ of disjoint open sets such that $k \in D_k$ for each k in K? Do Theorems 10 and 11 remain true if the additional requirement of no isolated points is made?

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Arizona State University,
Tempe, Arizona