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COUNTING HECKE EIGENFORMS WITH NONVANISHING *L*-VALUE

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Abstract

We develop some asymptotics for a kernel function introduced by Kohnen and use them to estimate the number of normalised Hecke eigenforms in $S_k(\Gamma_0(1))$ whose L-values are simultaneously nonvanishing at a given pair of points each of which lies inside the critical strip.

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1. Introduction

Let

$$f(z) = \sum_{n\geq 1} a_f(n)e^{2\pi i n z},$$

be a normalised cuspidal Hecke eigenform of weight k on $\Gamma_0(1) := \mathrm{SL}_2(\mathbb{Z})$. It is well known that its associated L-function $L(f, s)$, defined for $\mathrm{Re}(s) > (k + 1)/2$ by the absolutely convergent series

$$\sum_{n\geq 1} \frac{a_f(n)}{n^s},$$

can be analytically continued as an entire function for all s. Further, by virtue of the Euler product and functional equation, its nontrivial zeros lie inside the critical strip $(k - 1)/2 < \mathrm{Re}(s) < (k + 1)/2$. The generalised Riemann Hypothesis predicts that they all lie on the critical line $\mathrm{Re}(s) = k/2$.

For even integers $k \geq 12$, let S_k denote the space of cusp forms of weight k and level 1 and let $\mathcal{B}_k$ denote the (orthogonal) basis of arithmetically normalised (that is, $a_f(1) = 1$) Hecke eigenforms in S_k. Given a real number t_0 and a small positive real number δ, Kohnen in [6] proved the nonvanishing of the sum

$$\sum_{f\in\mathcal{B}_k}\frac{L^*(f,s)}{\langle f,f\rangle}$$

for any point s on the line segments

$$\Big\{\mathrm{Im}(s)=t_0,\frac{k-1}{2}<\mathrm{Re}(s)<\frac{k}{2}-\delta\Big\}\cup\Big\{\mathrm{Im}(s)=t_0,\frac{k}{2}+\delta<\mathrm{Re}(s)<\frac{k+1}{2}\Big\}$$

for large enough $k\gg_{t_0,\delta}1$, where $L^*(f,s)$ is the completed L-function (defined in Section 2.2). As a corollary, it follows that given such a point s, there is at least one form f in $\mathcal{B}_k$ such that $L(f,s)\neq 0$ for k large. Recently, in [2], the authors extended the above results to the simultaneous nonvanishing of L-values (on average) at two points inside the critical strip. More precisely, given positive real numbers T and δ, they proved the nonvanishing of the sum

$$\sum_{f\in\mathcal{B}_k}\frac{L^*(f,s_1)L^*(f,s_2)}{\langle f,f\rangle}$$

for large enough $k\gg_{T,\delta}1$ when $(s_1,s_2)\in R'_{T,\delta}$, where $R'_{T,\delta}$ is the set

$$\Big\{\Big(s_1=\frac{k}{2}+\epsilon_1+i\beta_1,s_2=\frac{k}{2}+\epsilon_2+i\beta_2\Big)\in\mathbb{C}^2\;\Big|\;-T\le\beta_1,\beta_2\le T,\ \delta\le|\epsilon_1|,|\epsilon_2|<\frac{1}{2}\Big\}.$$

Again, as a consequence, they observed that for a given $(s_1,s_2)\in R'_{T,\delta}$, there exists a Hecke eigenform f in S_k such that $L(f,s_1)L(f,s_2)\neq 0$, when k is sufficiently large.

In this context, it seems natural to ask the following question.

QUESTION 1.1. Given a weight k and complex points s,s_1,s_2 such that

$$\frac{k-1}{2}\le\mathrm{Re}(s),\mathrm{Re}(s_1),\mathrm{Re}(s_2)\le\frac{k+1}{2},$$

is it possible to quantify the numbers

$$\begin{aligned}N_k(s)&:=\#\{f\in\mathcal{B}_k\mid L(f,s)\neq 0\},\\ N_k(s_1,s_2)&:=\#\{f\in\mathcal{B}_k\mid L(f,s_1)\cdot L(f,s_2)\neq 0\},\end{aligned}\tag{1.1}$$

in terms of k?

We provide some partial estimates for (1.1). Given an arbitrary positive real number T and small positive reals δ and δ', we consider the subset of $\mathbb{C}^2$,

$$\begin{aligned}R_{T,\delta,\delta'}:=\Big\{\Big(s_1=\frac{k}{2}+\epsilon_1+i\beta_1,s_2=\frac{k}{2}+\epsilon_2+i\beta_2\Big)\in\mathbb{C}^2\;\Big|\\ -T\le\beta_1,\beta_2\le T,0<|\epsilon_1|<|\epsilon_1|+\delta'\le|\epsilon_2|\le\frac{1}{2},\ |\epsilon_1|+|\epsilon_2|\ge\frac{1}{2}+\delta\Big\}.\end{aligned}\tag{1.2}$$

If $(s_1,s_2)\in R_{T,\delta,\delta'}$, we prove the lower bound

$$N_k(s_1,s_2)\gg_{T,\delta',\delta''}k^{|\epsilon_1|+|\epsilon_2|-\delta''},\tag{1.3}$$

when k is sufficiently large, $k \gg_{T,\delta,\delta'} 1$. Here, δ'' is an arbitrarily small fixed positive number.

The lower bound for $N_k(s)$ is obtained as a special case of (1.3). For this, first we consider the strip

$$S_{T,\delta,\delta'} := \Big\{ -T \le \mathrm{Im}(s) \le T, \frac{k-1}{2} + \delta' \le \mathrm{Re}(s) \le \frac{k}{2} - \delta \Big\} \cup \Big\{ -T \le \mathrm{Im}(s) \le T, \frac{k}{2} + \delta \le \mathrm{Re}(s) \le \frac{k+1}{2} - \delta' \Big\}.$$

For any $s \in S_{T,\delta,\delta'}$, we obtain the explicit lower bound in terms of the weight k,

$$N_k(s) \gg_{T,\delta',\delta''} k^{(1/2)+|\mathrm{Re}(s)-(k/2)|-\delta''} \tag{1.4}$$

(see Corollary 3.5 below) for large enough $k \gg_{T,\delta,\delta'} 1$.

At the central critical point $s = k/2$, Luo [7, formula (4)] showed that for integers k divisible by 4, $N_k(k/2) \gg k$ as $k \to \infty$.

Our bound is of the order $k^{1/2}$ and so is weaker than this estimate of Luo. However, the method is considerably simpler and does not use the technique of mollified averages as in [7]. Further, we only assume k to be even. As for $N_k(s_1, s_2)$, currently, it seems that no such estimates are available.

1.1. The approach. As in our earlier paper [9], we consider a kernel function $f_{k,s}(z)$ studied by Kohnen [6] for suitable s (for its definition see Section 2.3) satisfying

$$f_{k,s}(z) = (\text{constant}) \sum_{f \in \mathcal{B}_k} \frac{L^*(f,s)}{\langle f, f\rangle} f(z), \tag{1.5}$$

and adapt a method of Rankin and Swinnerton-Dyer [13] to obtain an explicit expression for $f_{k,s}(it)$, valid for all $t \ge 1$, all even $k \ge 12$ and any s satisfying $(k-1)/2 \le \mathrm{Re}(s) \le (k+1)/2$ (see Theorem 3.1 below). As a consequence, we are able to provide the following applications.

1.2. Application 1. Using the above expression for a real parameter σ, we prove that $f_{k,\sigma}(ik) > 0$ when $k \gg 1$ and $4 \mid k$. Along with the fact (see Lemma 6.1) that $f(ik) > 0$ for any normalised Hecke eigenform f of weight k for $\mathrm{SL}_2(\mathbb{Z})$, this proves for a given σ inside the critical strip that there is an f satisfying $L(f, \sigma) > 0$. This leads us to a small improvement to [9, Corollary 2.2.1].

1.3. Application 2. Let T, δ, δ' be as above. For complex points s_1 and s_2 inside the critical strip such that the pair $(s_1, s_2) \in R_{T,\delta,\delta'}$, we study the Mellin transform of f_{k,s_1} with respect to s_2 by applying a Mellin transform on both sides of (1.5). Then we show (Theorem 3.3) that there is a constant $C = C(T, \delta, \delta') > 0$ such that

$$L^*(f_{k,s_1}, s_2) \gg k^{|\epsilon_1|+|\epsilon_2|}, \tag{1.6}$$

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for $k \geq C$. Here, we are using (1.5) and writing

$$L^*(f_{k,s_1}, s_2) \;=\; (\text{constant}) \sum_{f \in \mathcal{B}_k} \frac{L^*(f, s_1)}{\langle f, f\rangle} L^*(f, s_2).$$

Clearly C is independent of the chosen points s_1, s_2, and the implicit constant depends only on T, δ'. Using the lower bound obtained for $L^*(f_{k,s_1}, s_2)$ in (1.6) for large k, we prove (1.3). Finally, as a corollary, we obtain (1.4). The lower bound (1.4) is meant to illustrate an immediate application of (1.3) and these bounds may be known already.

In [14], the author proves the lower bound $N_k(k/2) \gg_{\delta''} k^{1-\delta''}$ when $4 \mid k$, assuming the Lindelöf hypothesis in the k-aspect for $L(f, s)$. The method also involves Kohnen's kernel function and is related to ours. However, unlike [2, 14], we do not use the explicit expressions for the Fourier coefficients of the respective kernel functions.

We also mention here that an identity for $L^*(f_{k,s_1}, s_2)$ in terms of ratios of Γ functions, ζ functions and hypergeometric functions is known [5, Theorem 1] when $s_1 + s_2 \in 2\mathbb{Z} + 1 \cap (1, k-1)$, $1 < \mathrm{Re}(s_j) < k-1$ and $\mathrm{Re}(s_1) > \mathrm{Re}(s_2) + 1$. The approach in [5] is to write the Mellin transform of f_{k,s_1} with respect to s_2 as a sum of certain term-wise integrals, obtained by splitting the series f_{k,s_1} in a suitable way. We also use similar methods to estimate $L^*(f_{k,s_1}, s_2)$, although we cover a wider range of points s_1, s_2 within the critical strip. But our main goal is to address the question posed regarding counts which differs from the aim of [5].

2. Notations and preliminaries

2.1. Notation. We mention the following asymptotic notation.

(1) $f(s) = O(g(s))$, $s \in S$, or equivalently, $f(s) \ll g(s)$, $s \in S$, means there exists a constant c such that $|f(s)| \leq c|g(s)|$ for all $s \in S$.
(2) $f(s) \sim g(s)$, $s \to s_0$, means $\lim_{s\to s_0} f(s)/tg(s) = 1$.

2.2. Preliminaries. Let $\mathbb{H}$ denote the upper half plane. For $z \neq 0$ and $s \in \mathbb{C}$, we set $z^s = \exp(s \log z)$ with $\log z = \log|z| + i \arg z$, where $-\pi < \arg z \leq \pi$. Let $k \geq 12$ be a positive even integer and $z \in \mathbb{H}$.

Let S_k denote the space of cusp forms of weight k with respect to $\mathrm{SL}_2(\mathbb{Z})$. For $g \in S_k$, the associated L-series,

$$L(g, s) := \sum_{n \geq 1} \frac{a_g(n)}{n^s},$$

is holomorphic on the half plane $\mathrm{Re}(s) > (k+1)/2$. We define the completed Hecke L-series associated to g as

$$L^*(g, s) := (2\pi)^{-s}\Gamma(s)L(g, s),$$

which is also equal to the Mellin transform of g, given by

$$\mathcal{M}(g)(s) := \int_0^\infty g(it)t^{s-1}\, dt.$$

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Since $\mathcal{M}(g)(s)$ is known to be entire, $L^*(g,s)$ (and also $L(g,s)$) can be uniquely extended as an entire function. Moreover, $L^*(g,s)$ satisfies the functional equation

$$L^*(g,k-s) = (-1)^{k/2} L^*(g,s). \tag{2.1}$$

For a normalised Hecke eigenform f in S_k, we define the *symmetric square L-function* of f by the formula

$$L(\mathrm{Sym}^2(f),s) := \prod_p \frac{1}{(1-\alpha_p^2 p^{-s})(1-\alpha_p\beta_p p^{-s})(1-\beta_p^2 p^{-s})},$$

for $\mathrm{Re}(s) > k$, where α_p and β_p are the roots of the polynomial $X^2 - a_f(p)X + p^{k-1}$. The function $L(\mathrm{Sym}^2(f),s)$ extends to an entire function [15] which is invariant under the map $s \mapsto 2k-1-s$.

For $x, y \in \mathbb{C}$ and $\mathrm{Re}(x), \mathrm{Re}(y) > 0$, we have the beta function

$$B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

and we recall that it satisfies

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^{\pi/2} (\sin\theta)^{2x-1}(\cos\theta)^{2y-1}\, d\theta. \tag{2.2}$$

2.3. Kohnen's cusp form. Let $s \in \mathbb{C}$ with $1 < \mathrm{Re}(s) < k-1$. The kernel function $R_{k,s}(z)$, introduced by Kohnen [6], is defined by

$$R_{k,s}(z) := \gamma_k(s) \sum{}' z^{-s}\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.3}$$

Here $|_k$ denotes the standard weight k (integer) action of $\mathrm{GL}_2^+(\mathbb{Q})$ on the functions $g : \mathbb{H} \to \mathbb{C}$, defined by

$$(g|_k\gamma)(\tau) := (ad-bc)^{k/2}(c\tau+d)^{-k} g\Big(\frac{a\tau+b}{c\tau+d}\Big), \quad \tau \in \mathbb{H}, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Q}),$$

and the sum $\sum'$ is taken over all matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Moreover,

$$\gamma_k(s) := \tfrac{1}{2} e^{\pi i s/2}\Gamma(s)\Gamma(k-s).$$

The function $R_{k,s}(z)$ is a cusp form of weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. If $\langle\, , \rangle$ denotes the usual Petersson inner product on S_k, then the essential property of $R_{k,s}$ is that for any $g \in S_k$, by [6, Lemma 1],

$$\langle g, R_{k,\bar{s}}\rangle = c_k L^*(g,s), \quad \text{where } c_k := \frac{(-1)^{k/2}\pi(k-2)!}{2^{k-2}}. \tag{2.4}$$

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3. Statement of results

For $1 < \sigma := \mathrm{Re}(s) < k-1$ and $z \in \mathbb{H}$, we define

$$f_{k,s}(z) := 2\frac{R_{k,s}(z)}{\Gamma(s)\Gamma(k-s)} = e^{i\pi s/2}\sum{}' z^{-s}\Big|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.1}$$

It follows from (2.1) and (3.1) that

$$f_{k,s}(z) = (-1)^{k/2} f_{k,k-s}(z). \tag{3.2}$$

Let $k \geq 12$, $2 \mid k$ and $k \neq 14$ throughout, unless further conditions are specified.

3.1. Asymptotics of $f_{k,s}(z)$ on the imaginary axis. In [9, Theorem 2.1], it was shown that for $\sigma \in [(k-1)/2, (k+1)/2)$,

$$f_{k,\sigma}(i) = 4 + O(2^{-k/2}). \tag{3.3}$$

Moreover, [9, Theorem 2.2] made explicit the constant implied in the O-term and (for $4 \mid k$) deduced a lower bound for $f_{k,\sigma}(i)$ and from there the lower bound [9, Corollary 2.2.1]

$$\max_{f\in\mathcal{B}_k} |L(f,\sigma)| \gg_\delta k^{-(\sigma-k/2)-1-\delta}$$

for any $\delta > 0$. Our first result extends (3.3) by replacing σ with a complex value of s, and $z = i$ by a general point $z = it$ on the imaginary axis.

THEOREM 3.1. *Let $s = \sigma + i\beta$ be a complex number such that $(k-1)/2 \leq \sigma \leq (k+1)/2$. Then, for a given $t \geq 1$, the cusp form $f_{k,s}(it)$ satisfies*

$$\Big| f_{k,s}(it) - \Big(2\frac{(2\pi)^s}{\Gamma(s)} \sum_{n\geq 1} n^{s-1}e^{-2\pi nt} + (-1)^{k/2} 2\frac{(2\pi)^{k-s}}{\Gamma(k-s)} \sum_{n\geq1} n^{k-s-1}e^{-2\pi nt}\Big)\Big| \leq \frac{300e^{\pi|\beta|/2}}{t^{k-2}}.$$

As a consequence of this result, we show that in fact $\max_{f\in\mathcal{B}_k} L(f,\sigma)$ is a positive quantity and satisfies the following lower bound.

COROLLARY 3.2. *Let $4 \mid k$ and $(k-1)/2 \leq \sigma = (k/2) + \epsilon \leq (k+1)/2$. For k large enough, there is a Hecke eigenform f in S_k such that $L(f,\sigma) > 0$. In particular, for any arbitrarily small $\delta' > 0$ and large enough $k \gg_{\delta'} 1$, we have*

$$\max_{f\in\mathcal{B}_k} L(f,\sigma) \gg_{\delta'} k^{-(\sigma-k/2)-1-\delta'}.$$

3.2. Simultaneous nonvanishing. In the next result, we replace σ by a complex point. For this purpose, let us set

$$L^*(f_{k,s}, w) = \frac{(-1)^{k/2}\pi\Gamma(k-1)}{2^{k-3}\Gamma(s)\Gamma(k-s)} \sum_{f\in\mathcal{B}_k} \frac{1}{\langle f,f\rangle} L^*(f,s)L^*(f,w).$$

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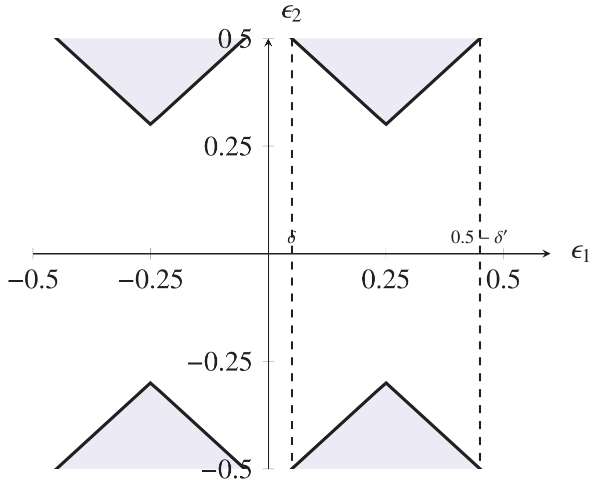

FIGURE 1. $\delta = 0.05$, $\delta' = 0.05$.

THEOREM 3.3. *Let T, δ, δ' be arbitrary but fixed positive real numbers such that $0 < \delta$, $\delta' \le 1/2$. Let $(s_1 = (k/2) + \epsilon_1 + i\beta_1,\ s_2 = (k/2) + \epsilon_2 + i\beta_2) \in R_{T,\delta,\delta'}$. Then there exists a constant $C = C(T, \delta, \delta') > 0$ depending only on T, δ, δ' such that*

$$L^*(f_{k,s_1}, s_2) \gg_{T,\delta'} k^{|\epsilon_1|+|\epsilon_2|} \quad \text{for } k \ge C(T, \delta, \delta').$$

In Figure 1, we illustrate the real points in $R_{T,\delta,\delta'}$ with $\delta = \delta' = 0.05$. Combining Theorem 3.3 with a straightforward estimate (see Section 8), we deduce the following result about simultaneous nonvanishing at two points.

COROLLARY 3.4. *Let $T, \delta, \delta', \delta''$ be arbitrary but fixed positive real numbers with $0 < \delta, \delta' \le 1/2$. Let $(s_1, s_2) \in R_{T,\delta,\delta'}$. Then, for $k \ge C(T, \delta, \delta')$,*

$$N_k(s_1, s_2) \gg_{T,\delta',\delta''} k^{|\epsilon_1|+|\epsilon_2|-\delta''}.$$

We remark here that since $N_k(s_1, s_2) = N_k(s_2, s_1)$, it suffices to assume that either (s_1, s_2) or $(s_2, s_1) \in R_{T,\delta,\delta'}$. It should also be noted that Corollary 3.4 does not imply [2, Corollary 3.2] since the region $R_{T,\delta,\delta'}$ that we are considering is a proper subset of $R'_{T,\delta}$ (except for points (s_1, s_2) in $R_{T,\delta,\delta'}$ with $\mathrm{Re}(s_2) = (k + 1)/2$).

Now, as a corollary to Corollary 3.4, we obtain an asymptotic lower bound for $N_k(s_1)$ in terms of k.

COROLLARY 3.5. *Let T be an arbitrary but fixed positive real number and let $\delta, \delta', \delta''$ be arbitrary small but fixed positive reals with $0 < \delta, \delta' \le 1/2$. Let $s_1 = (k/2) + \epsilon_1 + i\beta_1$ satisfy $|\beta_1| \le T$ and $\delta \le |\epsilon_1| \le \frac{1}{2} - \delta'$. Then, for $k \ge C(T, \delta, \delta')$,*

$$N_k(s_1) \gg_{T,\delta',\delta''} k^{(1/2)+|\epsilon_1|-\delta''}.$$

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4. Lemmas

We recall some preliminary lemmas which will be useful in the proofs of the results.

LEMMA 4.1.

(a) *For* $z_1, z_2, s \in \mathbb{C}$,

$$(z_1 z_2)^s = \begin{cases} z_1^s z_2^s e^{-2\pi i s} & \text{if } \arg(z_1) + \arg(z_2) \geq \pi, \\ z_1^s z_2^s & \text{if } -\pi \leq \arg(z_1) + \arg(z_2) < \pi, \\ z_1^s z_2^s e^{2\pi i s} & \text{if } \arg(z_1) + \arg(z_2) < -\pi. \end{cases}$$

(b) *If* $z, s = \sigma + i\beta \in \mathbb{C}$, *then* $|z^s| = |z|^\sigma e^{-\beta \arg(z)}$.

PROOF. The proof is straightforward and follows from the definitions of log and arg. Note that $\arg(z)$ denotes the principal argument of z as defined in Section 2.2. □

LEMMA 4.2 ([4], Gautschi's inequality). *For* $x > 0$ *and* $s \in (0, 1)$,

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}.$$

LEMMA 4.3 [1].

(a) *For* $a, b \in \mathbb{C}$,

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}$$

as $z \to \infty$, *along any curve joining* 0 *and* ∞, *provided* $z \notin -a - \mathbb{N} \cup -b - \mathbb{N}$.

(b) *(Real Stirling's formula). As* $x \to \infty$, *we have*

$$\Gamma(x) \sim \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}.$$

(c) [11]. *We have*

$$|\Gamma(x+iy)| \geq (\cosh \pi y)^{-1/2} \Gamma(x), \; x \geq \tfrac{1}{2}, \quad y \in \mathbb{R}. \tag{4.1}$$

PROOF. Part (a) follows from the asymptotic expansion

$$z^{b-a}\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim 1 + \frac{(a-b)(a+b-1)}{2}\frac{1}{z} + \frac{1}{12}\binom{a-b}{2}(3(a+b-1)^2 - a + b - 1)\frac{1}{z^2} + \cdots$$

(see [1, 6.1.47 on page 257]) valid for z as above. Part (b) is well known. For part (c), we refer to [11, 5.6.7 on page 138]. □

LEMMA 4.4.

(a) *For any fixed* $a > 0$,

$$\sum_{m \geq 1} m^a e^{-mz} = O_a\left(\frac{1}{z}\right)^{a+1} \quad \textit{for each } z > 0.$$

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(b)

$$\max_{z>0} z^{a+1} \sum_{m\geq 1} m^a e^{-mz} \ll \left(\frac{a+1}{e}\right)^{a+1} \quad (a \to \infty).$$

PROOF. We refer to [3, Lemma 9.3.13] for part (a) and [9, Lemma 6.1] for part (b). □

LEMMA 4.5. *For* $\sigma \geq 3$ *and* $\mathrm{Im}(z) = y \geq 1$,

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^\sigma} < \frac{7}{y^{\sigma-1}}. \tag{4.2}$$

PROOF. From [3, Lemma 3.5.9],

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^\sigma} \leq \frac{1}{y^\sigma} + \frac{4}{y^{\sigma-1}} \int_0^\infty \frac{1}{(u^2+1)^{\sigma/2}}\, du.$$

We estimate this explicitly when $y \geq 1$ and $\sigma \geq 3$, using (2.2) and Lemma 4.2:

$$\int_0^\infty \frac{1}{(u^2+1)^{\sigma/2}}\, du = \int_0^{\pi/2} (\cos\theta)^{\sigma-2}\, d\theta = \frac{\Gamma(1/2)}{2} \frac{\Gamma((\sigma-1)/2)}{\Gamma(\sigma/2)} \leq \sqrt{\frac{\pi}{2(\sigma-2)}}.$$

Note here that $\Gamma(1/2) = \sqrt{\pi}$. It follows that

$$\sum_{n\in\mathbb{Z}} \frac{1}{|z+n|^\sigma} \leq \frac{1}{y^\sigma} + \frac{4}{y^{\sigma-1}} \sqrt{\frac{\pi}{2(\sigma-2)}} < \frac{7}{y^{\sigma-1}}.$$

□

Finally, we mention an estimate for $\zeta(s)$ which is valid on $\mathrm{Re}(s) > 1$.

LEMMA 4.6. *For* $s = \sigma + it$, *where* $\sigma > 1$,

$$|\zeta(s)| \geq \left|\frac{\zeta(2\sigma)}{\zeta(\sigma)}\right|.$$

PROOF. For $\mathrm{Re}(s) > 1$,

$$\sum_{n\geq 1} \frac{1}{n^s} = \prod_{p\in\mathcal{P}} \frac{1}{1-p^{-s}}$$

where $\mathcal{P}$ denotes the set of primes. From this and the fact that $|p^s - 1| \leq p^\sigma + 1$,

$$\left|\frac{\zeta(2\sigma)}{\zeta(\sigma)\zeta(s)}\right| = \left|\prod_{p\in\mathcal{P}} \frac{p^{2\sigma}(p^\sigma-1)(p^s-1)}{(p^{2\sigma}-1)p^\sigma p^s}\right| \leq 1.$$

□

5. Proof of Theorem 3.1

Except in the proof of Corollary 3.2, we only assume $k \geq 12$ and $2 \mid k$.

PROOF OF THEOREM 3.1. The series for $R_{k,s}(z)$ in (2.3) runs over a set of integral matrices with determinant 1. As we have absolute convergence for z with positive imaginary part, we can rearrange this series in terms of the sum of the squares of the

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entries along each row (of the matrix) to get

$$e^{i\pi s/2}\sum_{\substack{M\geq 1\\ N\geq 1}}\sum_{a^2+b^2=M}\sum_{\substack{c^2+d^2=N\\ \det\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)=1}}(cz+d)^{-k}\Big(\frac{az+b}{cz+d}\Big)^{-s}. \tag{5.1}$$

For $M_0, N_0 \in \mathbb{N}$, we define

$$T_{M_0,N_0,s}(z) := e^{i\pi s/2}\sum_{a^2+b^2=M_0}\sum_{\substack{c^2+d^2=N_0\\ \det\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)=1}}(cz+d)^{-k}\Big(\frac{az+b}{cz+d}\Big)^{-s},$$

$$T_{M_0,\geq N_0,s}(z) := e^{i\pi s/2}\sum_{a^2+b^2=M_0}\sum_{\substack{N\in\mathbb{Z}\\ N\geq N_0}}\sum_{\substack{c^2+d^2=N\\ \det\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)=1}}(cz+d)^{-k}\Big(\frac{az+b}{cz+d}\Big)^{-s},$$

$$T_{\geq M_0,N_0,s}(z) := e^{i\pi s/2}\sum_{\substack{M\in\mathbb{Z}\\ M\geq M_0}}\sum_{a^2+b^2=M}\sum_{\substack{c^2+d^2=N_0\\ \det\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)=1}}(cz+d)^{-k}\Big(\frac{az+b}{cz+d}\Big)^{-s},$$

$$T_{\geq M_0,\geq N_0,s}(z) := e^{i\pi s/2}\sum_{\substack{M\in\mathbb{Z}\\ M\geq M_0}}\sum_{a^2+b^2=M}\sum_{\substack{N\in\mathbb{Z}\\ N\geq N_0}}\sum_{\substack{c^2+d^2=N\\ \det\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)=1}}(cz+d)^{-k}\Big(\frac{az+b}{cz+d}\Big)^{-s},$$

where $T_{-,-,s}(z)$ denotes the unique complex-valued holomorphic function associated to the respective series on the right-hand side for $1 < \sigma < k-1$. Further, let $s = \sigma + i\beta$ be a complex point inside the critical strip. Also, let $t > 0$. Then

$$f_{k,s}(it) = T_{1,1,s}(it) + T_{\geq 2,1,s}(it) + T_{1,\geq 2,s}(it) + T_{\geq 2,\geq 2,s}(it). \tag{5.2}$$

Now, $T_{1,1,s}(it)$ is formed by the matrices in $\{\pm\left(\begin{smallmatrix} 1 & 0\\ 0 & 1\end{smallmatrix}\right), \pm\left(\begin{smallmatrix} 0 & -1\\ 1 & 0\end{smallmatrix}\right)\}$. We therefore get

$$T_{1,1,s}(it) = 2e^{i(\pi/2)s}\Big\{(it)^{-s} + (it)^{-k}\Big(\frac{-1}{it}\Big)^{-s}\Big\} = \frac{2}{t^s} + \frac{2(-1)^{k/2}}{t^{k-s}}.$$

The last equality is true since $(it)^{-s} = e^{-i\pi s/2}t^{-s}$ and $(-1/it)^{-s} = e^{-i\pi s/2}t^{s}$ using Lemma 4.1. The term $T_{1,\geq 2,s}(it)$ is formed by the matrices of the form

$$\Big\{\pm\left(\begin{smallmatrix} 1 & 0\\ c & 1\end{smallmatrix}\right) \Big| \, c\in\mathbb{Z}, c^2+1\geq 2\Big\} \cup \Big\{\pm\left(\begin{smallmatrix} 0 & -1\\ 1 & c\end{smallmatrix}\right) \Big| \, c\in\mathbb{Z}, c^2+1\geq 2\Big\}.$$

Thus,

$$T_{1,\geq 2,s}(it) = 2e^{i\pi s/2}\sum_{|c|\geq 1}\Big\{(c+it)^{-k}\Big(\frac{-1}{c+it}\Big)^{-s} + (cit+1)^{-k}\Big(\frac{it}{cit+1}\Big)^{-s}\Big\}.$$

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Let

$$T^{\mathrm{main}}_{1,\geq 2,s}(it) := 2e^{i\pi s/2}\sum_{|c|\geq 1}(c+it)^{-k}\Big(\frac{-1}{c+it}\Big)^{-s},$$
$$T^{\mathrm{error}}_{1,\geq 2,s}(it) := 2e^{i\pi s/2}\sum_{|c|\geq 1}(cit+1)^{-k}\Big(\frac{it}{cit+1}\Big)^{-s}.$$

The first term above may be simplified to give

$$\begin{aligned}T^{\mathrm{main}}_{1,\geq 2,s}(it) &= 2e^{-i\pi s/2}\sum_{|c|\geq 1}(c+it)^{-(k-s)}\\ &= 2(-1)^{k/2}\Big\{e^{i\pi(k-s)/2}\sum_{c\in\mathbb{Z}}(c+it)^{-(k-s)} - e^{i\pi(k-s)/2}(it)^{-(k-s)}\Big\}\\ &= 2(-1)^{k/2}\Big\{\frac{(2\pi)^{k-s}}{\Gamma(k-s)}\sum_{n\geq 1}n^{k-s-1}e^{-2\pi nt} - \frac{1}{t^{k-s}}\Big\}.\end{aligned}$$

The last equality follows from the Lipschitz summation formula,

$$\sum_{n\in\mathbb{Z}}\frac{1}{(\tau+n)^s} = e^{-\pi is/2}\frac{(2\pi)^s}{\Gamma(s)}\sum_{n\geq 1}n^{s-1}e^{2\pi in\tau}\quad\text{for } \mathrm{Re}(s)>1, \tau\in\mathbb{H}.$$

The term $T^{\mathrm{error}}_{1,\geq 2,s}(it)$ is easily seen to be bounded above by $O_\beta(t^{-k})$. Indeed, Lemma 4.1 implies that

$$\Big|\Big(\frac{it}{cit+1}\Big)^{-s}\Big| = \Big|\Big(c+\frac{1}{it}\Big)^{s}\Big| = \Big(c^2+\frac{1}{t^2}\Big)^{\sigma/2}e^{-\beta\arg(c+1/it)}.$$

It follows that

$$\begin{aligned}|T^{\mathrm{error}}_{1,\geq 2,s}(it)| &\leq \frac{2}{t^k}e^{-\pi\beta/2}\sum_{|c|\geq 1}\Big|c+\frac{1}{it}\Big|^{-k}\Big|\Big(c+\frac{1}{it}\Big)^{s}\Big|\\ &\leq \frac{4}{t^k}e^{-\pi\beta/2}\sum_{c\in\mathbb{N}}\Big(c^2+\frac{1}{t^2}\Big)^{-(k-\sigma)/2}\exp(-\beta\arg(c-i/t))\\ &\leq e^{\pi|\beta|/2}\frac{4}{t^k}\zeta(k-\sigma),\end{aligned}\tag{5.3}$$

since $|(\pi/2)+\arg(c-i/t)| < (\pi/2)$. For later use, we also simplify $T^{\mathrm{error}}_{1,\geq 2,s}(it)$ as follows:

$$\begin{aligned}T^{\mathrm{error}}_{1,\geq 2,s}(it) &= \frac{2}{(it)^k}e^{i\pi s/2}\sum_{|c|\geq 1}(c-i/t)^{-(k-s)} = \frac{2}{t^k}e^{i\pi(k-s)/2}\sum_{|c|\geq 1}(c+i/t)^{-(k-s)}\\ &= \frac{2}{t^k}\frac{(2\pi)^{k-s}}{\Gamma(k-s)}\sum_{n\geq 1}n^{k-s-1}e^{-2\pi n/t} - \frac{2}{t^s}.\end{aligned}\tag{5.4}$$

For fixed $N \in \mathbb{N}$, $T_{N,1}(z)$ is formed precisely from the matrices in

$$\left\{\begin{pmatrix} a & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} a & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & a \\ 0 & -1 \end{pmatrix} \;\middle|\; a \in \mathbb{Z}, a^2 + 1 = N\right\}.$$

Performing similar computations to those above, one can see that

$$T_{\geq 2,1,s}(it) = \sum_{N\geq 2} T_{N,1}(it) = 2e^{i\pi s/2} \sum_{|a|\geq 1} \left\{(it)^{-k}\left(a + \frac{i}{t}\right)^{-s} + (a + it)^{-s}\right\}.$$

Let $T^{\text{error}}_{\geq 2,1,s}(it)$ denote the first sum $2e^{i\pi s/2} \sum_{|a|\geq 1} (it)^{-k}(a + i/t)^{-s}$. From its definition, we obtain the following equality and bound:

$$|T^{\text{error}}_{\geq 2,1,s}(it)| = \left|\frac{2}{(it)^k}\left\{\frac{(2\pi)^s}{\Gamma(s)} \sum_{n\geq 1} n^{s-1} e^{-2\pi n/t} - t^s\right\}\right| \leq 4e^{\pi|\beta|/2} \frac{\zeta(\sigma)}{t^k}. \tag{5.5}$$

The second sum, denoted by $T^{\text{main}}_{\geq 2,1,s}(it)$, may also be simplified using the Lipschitz summation formula as before to obtain

$$T^{\text{main}}_{\geq 2,1,s}(it) = 2\frac{(2\pi)^s}{\Gamma(s)} \sum_{n\geq 1} n^{s-1} e^{-2\pi nt} - \frac{2}{t^s}.$$

The main term (of $f_{k,s}(it)$) is

$$T_{\text{main},s}(it) := T_{1,1,s}(it) + T^{\text{main}}_{\geq 2,1,s}(it) + T^{\text{main}}_{1,\geq 2,s}(it).$$

The remaining terms in the summation (5.2) are brought into the error term (5.7). We therefore have

$$T_{\text{main},s}(it) = 2\frac{(2\pi)^s}{\Gamma(s)} \sum_{n\geq 1} n^{s-1} e^{-2\pi nt} + 2(-1)^{k/2} \frac{(2\pi)^{k-s}}{\Gamma(k-s)} \sum_{n\geq 1} n^{k-s-1} e^{-2\pi nt}. \tag{5.6}$$

We remark here that the term on the right-hand side is the contribution of the terms with $ac = 0$ in (5.1) as observed by Kohnen [6, page 186]. Also, we note here that the term above vanishes if $s = k/2$ and $k \equiv 2 \bmod 4$.

Next we bound the error term. Note that

$$T_{\text{error},s}(it) := T^{\text{error}}_{\geq 2,1,s}(it) + T^{\text{error}}_{1,\geq 2,s}(it) + T_{\geq 2,\geq 2,s}(it). \tag{5.7}$$

The final term in (5.2) and (5.7) is bounded above by

$$\begin{aligned} |T_{\geq 2,\geq 2,s}(it)| &\leq e^{-\pi\beta/2} \sum_{M\geq 2} \sum_{a^2+b^2=M} \sum_{N\geq 2} \sum_{\substack{c^2+d^2=N \\ \det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)=1}} |cit + d|^{-k} \left|\left(\frac{ait + b}{cit + d}\right)^{-s}\right| \\ &\leq \sum_{M\geq 2} \sum_{a^2+b^2=M} \sum_{N\geq 2} \sum_{\substack{c^2+d^2=N \\ ad-bc=1}} |cit + d|^{-k+\sigma} |ait + b|^{-\sigma} e^{\beta(\arg((ait+b)/(cit+d))-(\pi/2))} \\ &\leq e^{\pi|\beta|/2} \sum_{\substack{a^2+b^2\geq 2 \\ (a,b)=1}} \frac{1}{|ait + b|^\sigma} \sum_{\substack{c^2+d^2\geq 2 \\ (c,d)=1}} \frac{1}{|cit + d|^{k-\sigma}}. \end{aligned} \tag{5.8}$$

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Here $|ait+b|^\sigma = |-ait+b|^\sigma$ for any $a \in \mathbb{N}$ and $b \in \mathbb{Z}$. It follows that

$$\sum_{\substack{a^2+b^2\geq 2\\(a,b)=1}} \frac{1}{|ait+b|^\sigma} = 2\sum_{a=1}^{\infty}\frac{1}{a^\sigma}\sum_{\substack{b\in\mathbb{Z}\setminus\{0\}\\(b,a)=1}}\frac{1}{|it+b/a|^\sigma} = 2\sum_{a=1}^{\infty}\frac{1}{a^\sigma}\sum_{\substack{0<r<a\\(r,a)=1}}\sum_{q\in\mathbb{Z}}\frac{1}{|it+q+r/a|^\sigma}.$$

By Lemma 4.5, the inner sum on the right-hand side is $O(1/t^{\sigma-1})$ if $t \geq 1$. Note that up to this point, we have only assumed that $t > 0$. This follows since $\phi(a) < a$ and $\sum_{a=1}^{\infty} 1/a^{\sigma-1}$ converges. More precisely,

$$\sum_{\substack{a^2+b^2\geq 2\\(a,b)=1}} \frac{1}{|ait+b|^\sigma} < 14\frac{\zeta(\sigma-1)}{t^{\sigma-1}} \quad\text{and}\quad \sum_{\substack{c^2+d^2\geq 2\\(c,d)=1}} \frac{1}{|cit+d|^{k-\sigma}} < 14\frac{\zeta(k-\sigma-1)}{t^{k-\sigma-1}}.$$

Hence, by (5.8),

$$|T_{\geq 2,\geq 2,s}(it)| < e^{\pi|\beta|/2}\frac{14^2\times\zeta^2(4.5)}{t^{k-2}} < e^{\pi|\beta|/2}\frac{250}{t^{k-2}}.$$

(Note that the condition $k \geq 12$ implies that σ and $k-\sigma$ are greater than or equal to 5.5.) Now, (5.3) and (5.5), together with the above fact, imply that as long as $k \geq 12$ and $t \geq 1$,

$$|T_{\text{error},s}(it)| \leq e^{\pi\beta|/2|}\frac{300}{t^{k-2}}.$$

Thus, for all $t \geq 1$ and all even integers $k \geq 12$,

$$f_{k,s}(it) = 2\frac{(2\pi)^s}{\Gamma(s)}\sum_{n\geq 1} n^{s-1}e^{-2\pi nt} + 2(-1)^{k/2}\frac{(2\pi)^{k-s}}{\Gamma(k-s)}\sum_{n\geq 1} n^{k-s-1}e^{-2\pi nt} + C_\beta\Big(\frac{1}{t^{k-2}}\Big),$$

where $C_\beta \leq 300e^{\pi|\beta|/2}$. This completes the proof of Theorem 3.1. □

6. Proof of Corollary 3.2

We remark here that from (2.4) and (3.1) it follows that

$$f_{k,s_1} = \sum_{f\in\mathcal{B}_k}\langle f_{k,s_1}, f\rangle\frac{f}{\langle f,f\rangle} = \frac{(-1)^{k/2}\pi\Gamma(k-1)}{2^{k-3}\Gamma(s_1)\Gamma(k-s_1)}\sum_{f\in\mathcal{B}_k} L^*(f,s_1)\frac{f}{\langle f,f\rangle}. \tag{6.1}$$

To prove Corollary (3.2), we need the following fact.

Lemma 6.1. *Let $k \geq 12, k \neq 14$ be an even integer. Then $f(ki) > 0$ for any $f \in \mathcal{B}_k$.*

Proof. Note that $f(ki) = e^{-2\pi k} + \sum_{n\geq 2} a_f(n)e^{-2\pi nk}$. Multiplying by $e^{2\pi k}$ on both sides,

$$e^{2\pi k}f(ki) = 1 + \sum_{n\geq 1} a_f(n+1)e^{-2\pi nk}.$$

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It is enough to prove that the summation term on the right-hand side is bounded above by $\frac{1}{2}$. Using Hardy's estimate $|a_f(n)| \le 3n^{k/2}$, we have

$$\left|\sum_{n\ge1} a_f(n+1)e^{-2\pi nk}\right| \le 3\sum_{n\ge1}(n+1)^{k/2}e^{-2\pi nk} \le \frac{3}{e^{\pi k}}\sum_{n\ge1}\frac{1}{(n+1)^{k/2}} \le \frac{3}{e^{\pi k}}\zeta(k/2) < \frac{1}{2},$$

since $e^{2\pi nk} > (e^{\pi}(n+1))^k$ for all $n \ge 1$, which proves the claim. □

Proof of Corollary 3.2. For the remainder of this section, we assume $4 \mid k$. The functional equation (3.2) allows us to choose $\sigma \in [k/2, (k+1)/2]$ without loss of generality. From (6.1),

$$f_{k,\sigma}(ik) = \frac{\pi\Gamma(k-1)(2\pi)^{-\sigma}}{2^{k-3}\Gamma(k-\sigma)}\sum_{f\in\mathcal{B}_k}\frac{L(f,\sigma)}{\langle f,f\rangle}f(ik). \tag{6.2}$$

To produce a Hecke eigenform whose L-value is positive at σ, we use Theorem 3.1:

$$f_{k,\sigma}(ik) = 2\frac{(2\pi)^{\sigma}}{\Gamma(\sigma)}\sum_{n\ge1}n^{\sigma-1}e^{-2\pi nk} + 2\frac{(2\pi)^{k-\sigma}}{\Gamma(k-\sigma)}\sum_{n\ge1}n^{k-\sigma-1}e^{-2\pi nk} + T_{\mathrm{error},\sigma}(ik),$$

where $T_{\mathrm{error},\sigma}(ik) \ll 1/k^{k-2}$ for large enough k. From Lemma 4.3(b), the main term is bounded below by

$$T_{\mathrm{main},\sigma}(ik) > 2\frac{(2\pi)^{k-\sigma}}{\Gamma(k-\sigma)}e^{-2\pi k} \gg \left(\frac{\sqrt{4\pi e}}{e^{2\pi}}\right)^k k^{-(k-1)/2},$$

as $k \to \infty$. Hence, $f_{k,\sigma}(ik) > 0$ for $k \gg 1$, $4 \mid k$. Combining this with (6.2) and Lemma 6.1, we deduce that there is a Hecke eigenform f in S_k such that $L(f,\sigma) > 0$. Now, Corollary 3.2 easily follows using the same method as in [9, Corollary 2.2.1]. □

7. Proof of Theorem 3.3

Let T, δ, δ' be fixed constants as in the statement of Theorem 3.3. Let $(s_1, s_2) \in R_{T,\delta,\delta'}$, where $s_j = \sigma_j + i\beta_j$ with $\sigma_j = k/2 + \epsilon_j$ for $j = 1, 2$. Recall (from (3.2)) that f_{k,s_1} satisfies the functional equation $f_{k,s_1} = (-1)^{k/2} f_{k,k-s_1}$. Combining this with the functional equation of the L-function (2.1) gives

$$L^*(f_{k,s_1}, s_2) = (-1)^{k/2}L^*(f_{k,k-s_1}, s_2) = L^*(f_{k,k-s_1}, k-s_2) = (-1)^{k/2}L^*(f_{k,s_1}, k-s_2). \tag{7.1}$$

We can therefore assume s_1 and s_2 are on the right half of the critical strip, that is, from (1.2), $(k+1)/2 \ge \sigma_2 \ge \sigma_1 + \delta' > \sigma_1 > k/2$. From (5.2) and (5.6),

$$\begin{aligned}
L^*(f_{k,s_1}, s_2) &= \int_0^\infty f_{k,s_1}(it)t^{s_2-1}\,dt \\
&= \int_0^\infty\bigg(\frac{2(-1)^{k/2}(2\pi)^{k-s_1}}{\Gamma(k-s_1)}\sum_{n\ge1}n^{k-s_1-1}e^{-2\pi nt} \\
&\qquad + \frac{2(2\pi)^{s_1}}{\Gamma(s_1)}\sum_{n\ge1}n^{s_1-1}e^{-2\pi nt} + T_{\mathrm{error},s_1}(it)\bigg)t^{s_2-1}\,dt
\end{aligned}$$

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$$
\begin{aligned}
&= 2(-1)^{k/2}\frac{(2\pi)^{k-s_1}}{\Gamma(k-s_1)}\sum_{n\geq 1} n^{k-s_1-1}\int_0^\infty e^{-2\pi n t}t^{s_2-1}\,dt\\
&\quad + 2\frac{(2\pi)^{s_1}}{\Gamma(s_1)}\sum_{n\geq 1} n^{s_1-1}\int_0^\infty e^{-2\pi n t}t^{s_2-1}\,dt + L^*(T_{\mathrm{error},s_1}, s_2)\\
&= 2\frac{(2\pi)^{s_1-s_2}}{\Gamma(s_1)}\Gamma(s_2)\zeta(s_2-s_1+1)\\
&\quad + 2(-1)^{k/2}\frac{(2\pi)^{k-s_1-s_2}}{\Gamma(k-s_1)}\Gamma(s_2)\zeta(s_2-(k-s_1)+1) + L^*(T_{\mathrm{error},s_1}, s_2). \qquad (7.2)
\end{aligned}
$$

Interchanging the summation and integration in the second last step is justified by the Fubini–Tonelli theorem, noting that $\sum_{n\geq 1}\int_0^\infty |n^{s_1-1}e^{-2\pi n t}t^{s_2-1}|\,dt < \infty$ since $\mathrm{Re}(s_2) > \mathrm{Re}(s_1)$ and $\sum_{n\geq 1}\int_0^\infty |n^{k-s_1-1}e^{-2\pi n t}t^{s_2-1}|\,dt < \infty$ since $\mathrm{Re}(s_2) > \mathrm{Re}(k-s_1)$. (The notation $L^*(T_{\mathrm{error},s_1}, s_2)$ makes sense as the Mellin transform of T_{error,s_1}. It is well defined as all the other integrals in the equality are finite.) Next, we note that

$$
\begin{aligned}
L^*(T_{\mathrm{main},s_1}, s_2) := {}& 2\frac{(2\pi)^{s_1-s_2}}{\Gamma(s_1)}\Gamma(s_2)\zeta(s_2-s_1+1)\\
&+ 2(-1)^{k/2}\frac{(2\pi)^{k-s_1-s_2}}{\Gamma(k-s_1)}\Gamma(s_2)\zeta(s_2-(k-s_1)+1).
\end{aligned}
$$

7.1. Estimating the main term. We have $0 < \epsilon_1 < \epsilon_1 + \delta' \leq \epsilon_2 \leq \frac{1}{2}$, $|\beta_j| \leq T$ and $\sum_{j=1}^{2}\epsilon_j \geq \frac{1}{2} + \delta$. (Note that this forces $\epsilon_1 \geq \delta$.) From (4.1),

$$
\left|\frac{\Gamma(s_2)}{\Gamma(k-s_1)}\right| \geq \frac{1}{\sqrt{\cosh \pi\beta_2}}\frac{\Gamma(\frac{k}{2}+\epsilon_2)}{\Gamma(\frac{k}{2}-\epsilon_1)} \geq e^{-\pi|\beta_2|/2}\left(\frac{k}{2}+\epsilon_2-1\right)^{\epsilon_1+\epsilon_2} \geq \frac{k^{\epsilon_1+\epsilon_2}}{3}e^{-\pi T/2}.
$$

Similarly, one can see that

$$
\left|\frac{\Gamma(s_2)}{\Gamma(s_1)}\right| \leq e^{\pi T/2}k^{\epsilon_2-\epsilon_1}.
$$

Next, we find a lower bound for $\zeta(s_1+s_2-k+1)$. By Lemma (4.6),

$$
|\zeta(s_1+s_2-k+1)| \geq \frac{\zeta(2(1+\epsilon_2+\epsilon_1))}{\zeta(1+\epsilon_2+\epsilon_1)} > \frac{\zeta(4)}{\zeta(1.5)} = c_1.
$$

Now, the function $(x-1)\zeta(x)$ is bounded in the interval $1 \leq x \leq 2$. So, set

$$
c_2 := \max_{1\leq x\leq 2}(x-1)\zeta(x).
$$

Then $|\zeta(s_2-s_1+1)| \leq \zeta(\epsilon_2-\epsilon_1+1) \leq \zeta(\delta'+1) \leq c_2/\delta'$, since $\epsilon_2-\epsilon_1 \geq \delta'$.

One therefore sees that

$$
\begin{aligned}
&|L^*(T_{\mathrm{main},s_1}, s_2)| \\
&\geq 2\Big\{(2\pi)^{-(\epsilon_1+\epsilon_2)}\Big|\frac{\Gamma(s_2)}{\Gamma(k-s_1)}\zeta(s_1+s_2-k+1)\Big| - (2\pi)^{\epsilon_1-\epsilon_2}\Big|\frac{\Gamma(s_2)}{\Gamma(s_1)}\zeta(\epsilon_2-\epsilon_1+1)\Big|\Big\} \\
&\geq \frac{c_1}{3\pi}k^{\epsilon_1+\epsilon_2}e^{-\pi T/2} - \frac{2c_2}{\delta'}e^{\pi T/2}k^{\epsilon_2-\epsilon_1}. \qquad (7.3)
\end{aligned}
$$

7.2. Estimating the error term. Next, we consider the term $T_{\mathrm{error},s_1}(it)$. We claim that the component $T_{\geq 2,\geq 2,s_1}(it)$ is modular under the action of $S := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, that is, for all $t > 0$,

$$T_{\geq 2,\geq 2,s_1}(it) = (it)^{-k}T_{\geq 2,\geq 2,s_1}(i/t).$$

This follows since

$$
\begin{aligned}
T_{\geq 2,\geq 2,s_1}(it) &= e^{i\pi s_1/2}\sum_{a^2+b^2\geq 2}\ \sum_{\substack{c^2+d^2\geq 2\\ \det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)=1}} (cit+d)^{-k}\Big(\frac{ait+b}{cit+d}\Big)^{-s_1} \\
&= e^{i\pi s_1/2}\sum_{a^2+b^2\geq 2}\ \sum_{\substack{c^2+d^2\geq 2\\ \det\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)=1}} (it)^{-k}(-d(i/t)+c)^{-k}\Big(\frac{-b(i/t)+a}{-d(i/t)+c}\Big)^{-s} \\
&= (it)^{-k}T_{\geq 2,\geq 2,s_1}(i/t). \qquad (7.4)
\end{aligned}
$$

Thus,

$$L^*(T_{\geq 2,\geq 2,s_1}, s_2) = \int_1^\infty T_{\geq 2,\geq 2,s_1}(it)(t^{s_2} + (-1)^{k/2}t^{k-s_2})\,\frac{dt}{t}$$

which implies that

$$
\begin{aligned}
|L^*(T_{\geq 2,\geq 2,s_1}, s_2)| &\leq 250e^{\pi T/2}\Big(\int_1^\infty \frac{1}{t^{k-2}}(t^{\sigma_2} + t^{k-\sigma_2})\,\frac{dt}{t}\Big) \\
&\leq 250e^{\pi T/2}\Big(\frac{1}{k-\sigma_2-2} + \frac{1}{\sigma_2-2}\Big) \leq \frac{2000e^{\pi T/2}}{k} \qquad (7.5)
\end{aligned}
$$

for all $k \geq 12$. However, the other terms $T^{\mathrm{error}}_{\geq 2,1,s}(it)$ and $T^{\mathrm{error}}_{1,\geq 2,s}(it)$ in $T_{\mathrm{error},s}(it)$ are not invariant under S, so we estimate them individually. Note that

$$L^*(T_{\mathrm{error},s_1}, s_2) = L^*(T_{\geq 2,\geq 2,s_1}, s_2) + \Big(\int_0^1 + \int_1^\infty\Big)(T^{\mathrm{error}}_{\geq 2,1,s_1}(it) + T^{\mathrm{error}}_{1,\geq 2,s_1}(it))t^{s_2-1}\,dt.$$

Here, using the bounds in (5.3) and (5.5), we get

$$\Big|\int_1^\infty (T^{\mathrm{error}}_{\geq 2,1,s_1}(it) + T^{\mathrm{error}}_{1,\geq 2,s_1}(it))t^{s_2-1}\,dt\Big| \leq 10e^{\pi T/2}\Big(\int_1^\infty t^{\sigma_2-1-k}\,dt\Big) \leq \frac{40e^{\pi T/2}}{k}, \qquad (7.6)$$

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again for all $k \geq 12$. However, these estimates are not useful when $0 < t < 1$. Therefore, we proceed as follows. From (4.1) and (5.4), we see that

$$|T^{\text{error}}_{1,\geq 2,s_1}(it)| \leq \frac{2}{t^k}(\cosh \pi\beta_1)^{1/2} \frac{(2\pi)^{k-\sigma_1}}{\Gamma(k-\sigma_1)} \sum_{n\geq 1} n^{k-\sigma_1-1} e^{-2\pi n/t} + \frac{2}{t^{\sigma_1}}.$$

From Lemma 4.4, it follows that there exist absolute constants M_0 and K_0 such that

$$\sum_{n\geq 1} n^{k-\sigma_1-1} e^{-2\pi nu} \leq M_0 \Big(\frac{k-\sigma_1}{e}\Big)^{k-\sigma_1} \frac{1}{(2\pi u)^{k-\sigma_1}},$$

for $k \geq K_0$ uniformly for $u \geq 1$. (Here we have replaced $1/t$ by u.) Thus,

$$\begin{aligned}
&\Big|\int_0^1 T^{\text{error}}_{1,\geq 2,s_1}(it) t^{s_2-1}\, dt\Big| \\
&\quad \leq 2(\cosh \pi\beta_1)^{1/2} \frac{(2\pi)^{k-\sigma_1}}{\Gamma(k-\sigma_1)} \int_1^\infty \sum_{n\geq 1} n^{k-\sigma_1-1} e^{-2\pi nu} u^{k-\sigma_2} \frac{du}{u} + \frac{2}{\sigma_2-\sigma_1} \\
&\quad \ll e^{\pi T/2} \frac{1}{\Gamma(k-\sigma_1)} \Big(\frac{k-\sigma_1}{e}\Big)^{k-\sigma_1} \frac{1}{\sigma_2-\sigma_1} + \frac{2}{\sigma_2-\sigma_1} \\
&\quad \ll e^{\pi T/2} \sqrt{(k-\sigma_1)}\ \frac{1}{\delta'} \leq \frac{e^{\pi T/2}}{\delta'} \sqrt{k}
\end{aligned}$$

where we have used $\sigma_2 - \sigma_1 \geq \delta'$ and the fact that

$$\frac{1}{\Gamma(k-\sigma_1)} \Big(\frac{k-\sigma_1}{e}\Big)^{k-\sigma_1} \ll \sqrt{k-\sigma_1}.$$

The last inequality follows from Stirling's estimates in Lemma 4.3(b). Thus,

$$\int_0^1 T^{\text{error}}_{1,\geq 2,s_1}(it) t^{s_2-1}\, dt = O_{T,\delta'}(k^{1/2}). \tag{7.7}$$

Similarly, it follows that

$$\int_0^1 T^{\text{error}}_{\geq 2,1,s_1}(it) t^{s_2-1} dt \ll e^{\pi T/2} \sqrt{\sigma_1}\ \frac{1}{\sigma_2+\sigma_1-k} = O_T(k^{1/2}). \tag{7.8}$$

The implicit constant in (7.8) has no dependency on δ' since we assume $\sigma_1 + \sigma_2 - k > \frac{1}{2}$. From (5.7), (7.4), (7.6), (7.7) and (7.8), it follows that $L^*(T_{\text{error},s_1}, s_2) = O_{T,\delta'}(\sqrt{k})$. Along with the above and (7.2) and (7.3), we have

$$|L^*(f_{k,s_1}, s_2)| \geq \Big|\frac{c_1}{3\pi} e^{-\pi T/2} k^{\epsilon_1+\epsilon_2} - \frac{2c_2}{\delta'} e^{\pi T/2} k^{\epsilon_2-\epsilon_1} - C(T,\delta') k^{1/2}\Big|.$$

Thus, it follows that

$$L^*(f_{k,s_1}, s_2) \gg_{T,\delta'} k^{\epsilon_2+\epsilon_1}, \tag{7.9}$$

for sufficiently large $k \gg_{T,\delta,\delta'} 1$ since $\epsilon_2 + \epsilon_1 \geq \frac{1}{2} + \delta$.

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Given an arbitrary pair of points s_1, s_2 satisfying $\epsilon_1 + \epsilon_2 \geq \frac{1}{2} + \delta$ on the strict right-hand side of the critical line, that is, $\{k/2 < \mathrm{Re}(s_j) \leq (k+1)/2\}$, one can always choose the point with smaller real part as the kernel parameter and the other one to be the Mellin transform parameter (unless both points lie on the same vertical line, in which case, clearly, our result does not hold). And, even in situations where either (or both) s_1, s_2 assume values strictly to the left of the critical line, by virtue of (7.1), one can perform the appropriate reflection $s_j \mapsto k - s_j$, thus effectively reducing it to a question about a pair of points on the right half of the critical strip. This proves the theorem.

From (6.1),

$$L^*(f_{k,s_1}, s_2) = \int_1^\infty f_{k,s_1}(it)(t^{s_2} + (-1)^{k/2} t^{k-s_2}) \frac{dt}{t}$$
$$= \frac{(-1)^{k/2}\pi\Gamma(k-1)}{2^{k-3}\Gamma(s_1)\Gamma(k-s_1)} \sum_{f\in\mathcal{B}_k} \frac{L^*(f,s_1)}{\langle f,f\rangle} L^*(f,s_2). \tag{7.10}$$

From (7.9) and (7.10) we immediately observe that, given a pair of points s_1, s_2 in the critical strip such that ϵ_j satisfies the conditions in (1.2), there is at least one eigenform $f \in \mathcal{B}_k$ whose L-function is simultaneously nonvanishing at both s_1 and s_2 for $k \gg_{T,\delta,\delta'} 1$. However, one may observe that

$$\sum_{f\in\mathcal{B}_k} \frac{L^*(f,s_1)}{\langle f,f\rangle} L^*(f,s_2)$$

is the first Fourier coefficient of $E^*_{s_1,k-s_1}(z,s_2)$ (as a function of $z \in \mathbb{H}$), and this sum is already known to be nonzero for sufficiently large k (see [2, formula (3.1)]). This already guarantees the existence of such a Hecke eigenform as above. However, we improve, albeit partially, the result in [2] by explicitly determining a lower bound for the number $N_k(s_1, s_2)$ in Corollary 3.4.

8. Proof of Corollary 3.4

Let $(s_1, s_2) \in R_{T,\delta,\delta'}$. Without loss of generality, as earlier, we assume s_1 and s_2 on the right half of the critical strip. From (7.10),

$$k^{\epsilon_2+\epsilon_1} \ll_{T,\delta'} \frac{1}{k2^{k-3}} \frac{\Gamma(k)}{|\Gamma(s_1)\Gamma(k-s_1)|} \sum_{f\in\mathcal{B}_k} \left| \frac{L^*(f,s_1)L^*(f,s_2)}{\langle f,f\rangle} \right|, \tag{8.1}$$

for large enough k (as mentioned in (7.9)). By the Phragmén–Lindelöf theorem [12],

$$L(f,s_1) \ll_{\delta''} k^{1/2-\epsilon_1+\delta''} \quad \text{for an arbitrarily small } \delta'' > 0. \tag{8.2}$$

It is well known that for any normalised Hecke eigenform $f \in S_k$,

$$k^{-\delta'''} \ll_{\delta'''} L(\mathrm{Sym}^2(f), k) \ll_{\delta'''} k^{\delta'''}$$

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holds for arbitrarily small $\delta''' > 0$ [8, page 4]. Thus

$$\langle f, f\rangle = \frac{2\Gamma(k)}{\pi(4\pi)^k} L(\mathrm{Sym}^2(f), k) \gg_{\delta'''} \frac{\Gamma(k)}{(4\pi)^k k^{\delta'''}},$$

where the first relation is from [3, Corollary 11.12.7(b)]. Using this estimate and (8.2),

$$\sum_{f\in\mathcal{B}_k} \left|\frac{L^*(f,s_1)L^*(f,s_2)}{\langle f, f\rangle}\right| \ll_{\delta'''} N_k(s_1,s_2)(2\pi)^{-(\sigma_1+\sigma_2)}|\Gamma(s_1)\Gamma(s_2)|k^{1-(\epsilon_2+\epsilon_1)+2\delta''} \frac{(4\pi)^k k^{\delta'''}}{\Gamma(k)}.$$

We estimate $\Gamma(s_2)/\Gamma(k-s_1)$ by (4.1) to rewrite (8.1) as

$$k^{\epsilon_2+\epsilon_1} \ll_{T,\delta',\delta'',\delta'''} N_k(s_1,s_2)k^{-(\epsilon_2+\epsilon_1)+2\delta''+\delta'''}\left|\frac{\Gamma(s_2)}{\Gamma(k-s_1)}\right| \ll_T k^{2\delta''+\delta'''} N_k(s_1,s_2).$$

Thus, by first putting $\delta''' = \delta''$ and then replacing δ'' by $\delta''/3$,

$$N_k(s_1,s_2) \gg k^{\epsilon_2+\epsilon_1-\delta''},$$

where the implied constant depends only on T, δ', δ''. Since

$$N_k(s_1,s_2) = N_k(s_1,k-s_2) = N_k(k-s_1,k-s_2) = N_k(k-s_1,s_2),$$

we conclude that there is a constant $C(T,\delta,\delta') > 0$ such that, for $k \geq C(T,\delta,\delta')$, the number of Hecke eigenforms in S_k whose L^*-value is simultaneously nonvanishing at any two points s_1, s_2 such that $(s_1,s_2) \in R_{T,\delta,\delta'}$ is at least $k^{|\epsilon_1|+|\epsilon_2|-\delta''}$.

9. Proof of Corollary 3.5

Points on the (right) edge of the critical strip lie inside the known nonvanishing region of L-functions of Hecke eigenforms [10], given by

$$\mathrm{Re}(s) \geq \frac{k+1}{2} - \frac{c}{\log(k+|t|+3)},$$

where $c > 0$ is an absolute constant. By substituting $s_2 = (k+1)/2$ in Corollary 3.4, we have $N_k(s_1) \gg_{T,\delta',\delta''} k^{1/2+|\epsilon_1|-\delta''}$ for $k \gg_{T,\delta,\delta'} 1$, when $\delta \leq \epsilon_1 \leq \frac{1}{2} - \delta'$.

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