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ON THE BEHAVIOR OF ZEROS OF POLYNOMIALS OF BEST AND NEAR-BEST APPROXIMATION

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ABSTRACT. Assume f is continuous on the closed disk $D_1 : |z| \leq 1$, analytic in $|z| < 1$, but not analytic on D_1. Our concern is with the behavior of the zeros of the polynomials $\{P_n^*(f)\}_1^\infty$ of best uniform approximation to f on D_1. It is known that, for such f, every point of the circle $|z| = 1$ is a cluster point of the set of all zeros of $\{P_n^*(f)\}_1^\infty$. Here we show that this property need not hold for every subsequence of the $P_n^*(f)$. Specifically, there exists such an f for which the zeros of a suitable subsequence $\{P_{n_k}^*(f)\}$ all tend to infinity. Further, for near-best polynomial approximants, we show that this behavior can occur for the whole sequence. Our examples can be modified to apply to approximation in the L_q-norm on $|z| = 1$ and to uniform approximation on general planar sets (including real intervals).

1. **Introduction.** We investigate the behavior of best and near-best polynomial approximants in the complex plane $\mathbb{C}$. Let $V \subset \mathbb{C}$ be a compact set containing infinitely many points such that $\overline{\mathbb{C}} \setminus V$ is connected. By $\|\,.\,\|_V$ we denote the uniform norm on V, i.e.,

$$\|f\|_V := \sup\{\,|f(z)| : z \in V\}\,.$$

Let Π_n denote the set of all algebraic polynomials of degree $\leq n$. For any function f analytic on the interior V^o of V and continuous on V we denote by $P_n^*(f)$ the best uniform approximant to f on V with respect to Π_n, i.e.,

$$E_n(f)_V := \|f - P_n^*(f)\|_V \leq \|f - P_n\|_V$$

for all $P_n \in \Pi_n$. By Mergelyan's theorem we know that $E_n(f)_V \longrightarrow 0$ as $n \longrightarrow \infty$.

In this paper we shall be concerned with functions f that are continuous on V, analytic in V^o, but not analytic on V (that is, f has some singularity on the boundary of V). We denote the collection of all such functions f by $A_0(V)$.

Let $\{\,S_n\}$ be any sequence of functions holomorphic on a neighborhood U of V ($U^o \supset V$) such that $\|S_n - f\|_V \longrightarrow 0$ as $n \longrightarrow \infty$. By Montel's theorem (see eg. [5,§ 15.2]), $\{\,S_n\}$ will be a normal family in U if $\{\,S_n(z)\}$ omits two different values α and β in U. If

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this is the case, then an appropriate subsequence $\{S_{n_k}\}$ will converge to a function g holomorphic in U^o and g will be an analytic continuation of f to U^o. Thus if $f \in A_0(V)$, then any sequence of functions analytic in a neighborhood of V that approximates f uniformly on V can omit no more than one value in this neighborhood.

It was shown by Blatt and Saff [1] that if $\overline{\mathbb{C}} \setminus V$ is simply connected, then the sequence $\{P_n^*(f)\}_0^\infty$ of polynomials of best approximation to $f \in A_0(V)$ cannot omit any value in a neighborhood of V. More precisely, we have

THEOREM A ([1]). *Let $f \in A_0(V)$, where $\overline{\mathbb{C}} \setminus V$ is simply connected. Then there is a subsequence $\{n_k\}$ having the following property: given any boundary point z_0 of V, any ε-neighborhood $U_\varepsilon(z_0)$ of z_0, and any $\alpha \in \mathbb{C}$, the equation $P_{n_k}^*(f;z) = \alpha$ has a root in $U_\varepsilon(z_0)$ for all large k.*

In other words, every boundary point of V attracts α-points of the sequence $\{P_{n_k}^*(f)\}_{k=1}^\infty$. Actually, in [2], a stronger result is proved concerning the limiting distribution of these α-points.

Theorem A illustrates what Saff [8] has called the *principle of contamination,* which roughly states that the existence of one or more singularities of f on the boundary of V adversely affects the behavior over the *whole* boundary of V of some subsequence of the best polynomial approximants $P_n^*(f)$ to f on V. It is important to note that this principle as well as Theorem A refer only to some *subsequence* of the best approximants.

One goal of this paper is to show that Theorem A does not, in general, hold for the whole sequence $\{P_n^*(f)\}_1^\infty$. With the notation

$$D_r := \{z : |z| \leq r\},$$

we shall prove

THEOREM 1. *There exists a function $f \in A_0(D_1)$ and a sequence of integers N_k, $k = 1, 2, \dots$, such that the polynomial $P_{N_k}^*(f)$ of best uniform approximation to f on D_1 has no zeros in D_k for every k.*

In other words, the zeros of $P_{N_k}^*(f)$ diverge to infinity.

REMARK 1. Theorem 1 remains valid if we replace D_1 by any compact set V whose complement is connected and regular with respect to the Dirichlet problem. This is an improvement of a result of Grothmann and Saff [4, Theorem 2.1], which asserts that there exists an $f \in A_0(V)$ and a subsequence $\{n_k\}$ such that any bounded set contains $o(n_k)$ zeros of $P_{n_k}^*(f)$.

REMARK 2. It is not necessary to restrict our considerations to polynomials of best *uniform* approximation. In Theorem 1 we may replace $P_n^*(f,z)$ by $P_n^*(f,q,z)$—the polynomial of best L_q ($1 \leq q < \infty$) approximation to f defined by

$$E_n(f)_q := \|f - P_n^*(f,q)\|_q \leq \|f - P_n\|_q$$

for any $P_n \in \Pi_n$, where

$$\|g\|_q := \left(\frac{1}{2\pi}\int_0^{2\pi} |g(e^{i\theta})|^q d\theta\right)^{1/q}.$$

In the special case $q = 2$, the polynomial $P_n^*(f, 2)$ is the Taylor polynomial of f and we obtain that there is a function in $A_0(D_1)$ such that *all* zeros of a special subsequence of its Taylor polynomials about the origin diverge to infinity. A similar example was obtained by Jentzsch [7] who also showed (cf. [6]) that, for any $f \in A_0(D_1)$, every point of the unit circle is an accumulation point of the set of zeros of all Taylor polynomials.

Theorem 1 and Remarks 1 and 2 are proved in Section 2.

Let us now consider the behavior of polynomials of near-best approximation. We say that the sequence of polynomials $\{\hat{Q}_n(f)\}_0^\infty$ is of *near-best approximation* to f on V if $\hat{Q}_n(f) \in \Pi_n$, $n = 0, 1, \ldots$, and there is a constant $c \geq 1$ such that

$$\|f - \hat{Q}_n(f)\|_V \leq cE_n(f)_V$$

for any n.

It was asked in [4] if at least one point of the boundary of V must be a limit of zeros of near-best approximants to $f \in A_0(V)$. Our next theorem shows that the answer is no; that is, it may happen that no point of the boundary of V attracts zeros of the whole sequence of near-best approximants. In such a situation, we note, however, that for any value $\alpha \neq 0$, Montel's theorem implies that the α-points of this sequence must have at least one limit point on the boundary of V.

THEOREM 2. *There exists a function $f \in A_0(D_1)$ and a sequence of polynomials $\hat{Q}_n \in \Pi_n$ such that:*

(i) $\|f - \hat{Q}_n\|_{D_1} \leq cE_n(f)_{D_1}$, $n = 0, 1, \ldots$, *and*

(ii) for any $\rho > 1$ there is an N such that $\hat{Q}_n$ has no zeros in D_ρ for any $n \geq N$.

Theorem 2 should be compared to Theorem 1.3 in Grothmann and Saff [4] which says that if we require enough regularity for the error in best approximation of the function $f \in A_0(V)$, then at least one point of the boundary of V is a limit point of the zeros of $\hat{Q}_n(f)$.

REMARK 3. As in Remark 2, Theorem 2 also holds if $\hat{Q}_n$ is a suitable sequence of polynomials of near-best L_q $(1 \leq q < \infty)$ approximation to f.

Theorem 2 and Remark 3 are proved in Section 3.

2. **Proofs of Theorem 1 and Remarks 1 and 2.**

LEMMA 1. *For $N \geq 5|w|$ we have*

$$\left|e^w - \sum_{j=0}^{N} \frac{w^j}{j!}\right| < \frac{1}{2}e^{-|w|}.$$

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PROOF. For the remainder of the Taylor seies of e^w we have

$$e^w - \sum_{j=0}^{N} \frac{w^j}{j!} = \frac{1}{N!} \int_0^w (w-t)^N e^t \, dt.$$

Therefore,

$$\Big|e^w - \sum_{j=0}^{N} \frac{w^j}{j!}\Big| \le |w|^{N+1} e^{|w|} / N!, \tag{2.1}$$

and using the inequality $N! > N^N e^{-N}$ we get for $N \ge 5|w|$

$$\begin{aligned} \Big|e^w - \sum_{j=0}^{N} \frac{w^j}{j!}\Big| &\le \left(\frac{|w|}{N}\right)^N |w| e^{|w|+N} \\ &\le |w| e^{(1-\ln 5)N + 2|w|} e^{-|w|} \\ &\le |w| e^{-(5\ln 5 - 7)|w|} e^{-|w|} \\ &\le \frac{1}{e(5\ln 5 - 7)} e^{-|w|} < \frac{1}{2} e^{-|w|}. \end{aligned}$$ ∎

PROOF OF THEOREM 1. We set

$$g(z) := \sum_{j=1}^{\infty} \varepsilon_j z^{m_j}, \quad g_k(z) := \sum_{j=1}^{k} \varepsilon_j z^{m_j},$$

where ε_j and m_j are determined by induction in the following way. Set $\varepsilon_1 := \frac{1}{2}\ln 2$, $m_1 := 1, n_1 := 5$. If ε_k, m_k and n_k are chosen, then we first determine $\varepsilon_{k+1} > 0$ such that

$$\varepsilon_{k+1} \le \frac{1}{2}\varepsilon_k, \tag{2.2}$$

and

$$\varepsilon_{k+1} \le \frac{1}{64} k^{-m_k n_k} \exp(-\|g_k\|_{D_k}). \tag{2.3}$$

Then we set

$$m_{k+1} := [1/\varepsilon_{k+1}], \tag{2.4}$$

and finally we choose n_{k+1} so big that

$$n_{k+1} \ge 5(k+1)^{m_{k+1}} \tag{2.5}$$

and

$$\frac{(k+1)^{m_{k+1} n_{k+1}}}{n_{k+1}!} < \frac{1}{16} \exp(-\|g_{k+1}\|_{D_{k+1}}). \tag{2.6}$$

We note that inequalities (2.5) and (2.6) are also satisfied for $k = 0$ because

$$\|g_1\|_{D_1} = \varepsilon_1 = \frac{1}{2} \ln 2 \text{ and } n_1 = 5.$$

Next we set

$$f(z) := e^{g(z)}, \quad f_k(z) := e^{g_k(z)}, \quad N_k := m_k n_k.$$

From (2.4) we have

$$\lim_{j \to \infty} (\varepsilon_j)^{1/m_j} = 1,$$

and (2.2) gives $\|g - g_k\|_{D_1} \le \varepsilon_k$. Hence $g \in A_0(D_1)$ and the same is true for f.

Next we are going to prove that $P^*_{N_k}(f)$ has no zeros in D_k. We shall make use of the following simple observation:

For any $f \in C(D_1)$ and any $Q_\nu \in \Pi_\nu$ we have

$$\begin{aligned} \|P^*_\nu(f) - Q_\nu\|_{D_1} &\le \|P^*_\nu(f) - f\|_{D_1} + \|f - Q_\nu\|_{D_1} \\ &\le 2\|f - Q_\nu\|_{D_1}, \end{aligned} \tag{2.7}$$

because $P^*_\nu(f)$ is the polynomial of best approximation to f out of Π_ν.

From (2.2) we have $\|g_k\|_{D_1} \le 2\varepsilon_1 = \ln 2$, $\|g\|_{D_1} \le \ln 2$, which imply that $\|f_k\|_{D_1} \le 2$, $\|f\|_{D_1} \le 2$. Therefore (2.1) with $N = 0$ yields

$$\begin{aligned} \|f - f_k\|_{D_1} &\le \|f_k\|_{D_1} \left\| \exp\left(\sum_{j=k+1}^{\infty} \varepsilon_j z^{m_j} \right) - 1 \right\|_{D_1} \\ &\le 2 \exp\left(\sum_{j=k+1}^{\infty} \varepsilon_j \right) \cdot \sum_{j=k+1}^{\infty} \varepsilon_j \le 8\varepsilon_{k+1}. \end{aligned} \tag{2.8}$$

Using (2.2) once more we get

$$\|g_k(z)\|_{D_k} \le \sum_{j=1}^{k} \varepsilon_j k^{m_j} \le k^{m_k} \sum_{j=1}^{k} \varepsilon_j \le k^{m_k}. \tag{2.9}$$

Set $Q_{N_k}(z) := \sum_{j=0}^{n_k} g_k(z)^j / j! \in \Pi_{N_k}$. From Lemma 1 with $N = n_k$ and $w = g_k(z)$, (2.9) and (2.5) we get

$$|Q_{N_k}(z) - e^{g_k(z)}| < \frac{1}{2} e^{-|g_k(z)|} \text{ for any } z \in D_k. \tag{2.10}$$

From (2.1) with $N = n_k$, $w = g_k(z)$, we obtain for any $z \in D_1$,

$$|Q_{N_k}(z) - f_k(z)| \le \frac{|g_k(z)|^{n_k+1} e^{|g_k(z)|}}{n_k!} < \frac{2}{n_k!},$$

which together with (2.8) gives

$$\|Q_{N_k} - f\|_{D_1} \le 8\varepsilon_{k+1} + \frac{2}{n_k!}. \tag{2.11}$$

From Bernstein's lemma (cf. [10, §4.6]), (2.7) with $\nu = N_k$, and (2.11) we get

$$\|P^*_{N_k}(f) - Q_{N_k}\|_{D_k} \leq k^{N_k}\|P^*_{N_k}(f) - Q_{N_k}\|_{D_1} \leq 2k^{N_k}\|f - Q_{N_k}\|_{D_1} \leq (16\varepsilon_{k+1} + 4/n_k!)k^{N_k}.$$

Now using (2.6) (with k instead of $k+1$) and (2.3) we obtain

$$\|P^*_{N_k}(f) - Q_{N_k}\|_{D_k} \leq \frac{1}{2}\exp(-\|g_k\|_{D_k}). \tag{2.12}$$

Combining (2.10) and (2.12) we get

$$|P^*_{N_k}(f,z) - e^{g_k(z)}| < \frac{1}{2}e^{-|g_k(z)|} + \frac{1}{2}e^{-\|g_k(z)\|_{D_k}} \leq e^{-|g_k(z)|} \leq |e^{g_k(z)}|$$

for any z, $|z| = k$, which, in view of Rouché's theorem implies that $P^*_{N_k}(f)$ has no zeros in D_k. This proves Theorem 1. ■

PROOF OF REMARK 1. Let G be the Green's function for $\overline{\mathbb{C}} \setminus V$ with pole at ∞. Then, by assumption, G is continuous on $\overline{\mathbb{C}} \setminus V$ and takes the value 0 on the boundary of V. We set, for $\rho \geq 1$, $D^*_\rho := \{z \in \overline{\mathbb{C}} \setminus V : |G(z)| \leq \ln\rho\} \cup V$. Denote by $T_n(z) = z^n + \cdots$ the generalized Chebyshev polynomial of degree n for V, i.e.

$$\|T_n\|_V = \min\{\|z^n - p(z)\|_V : p \in \Pi_{n-1}\}$$

and let $\tilde{T}_n(z) := T_n(z)/\|T_n\|_V$. If we set $g(z) := \sum_{j=1}^{\infty}\varepsilon_j\tilde{T}_{m_j}(z)$ and $g_k(z) := \sum_{j=1}^{k}\varepsilon_j\tilde{T}_{m_j}(z)$, then with obvious modifications the proof of Theorem 1 will give us Remark 1, with D_k replaced by D^*_k. Notice that $\{D^*_k\}$ is an increasing sequence converging to the whole complex plane $\mathbb{C}$ in an obvious sense.

PROOF OF REMARK 2. The only changes in the proof of Theorem 1 are:

(a) Using Nikolskii's inequality [9, §4.9.2] one replaces (2.7) by

$$\begin{aligned}\|P^*_\nu(f,q) - Q_\nu\|_{D_1} &\leq c\nu^{1/q}\|P^*_\nu(f,q) - Q_\nu\|_q \\ &\leq c\nu^{1/q}(\|P^*_\nu(f,q) - f\|_q + \|f - Q_\nu\|_q) \\ &\leq 2c\nu^{1/q}\|f - Q_\nu\|_q \leq 2c\nu^{1/q}\|f - Q_\nu\|_{D_1}\end{aligned}$$

(c is an absolute constant).

(b) Therefore we have to replace (2.3) and (2.6) by

$$\varepsilon_{k+1} \leq \frac{1}{64c}e^{-\|g_k\|_{D_k}}(m_kn_k)^{-1/q}k^{-m_kn_k}$$

and

$$\frac{c(k+1)^{m_{k+1}n_{k+1}}(m_{k+1}n_{k+1})^{1/q}}{n_{k+1}!} < \frac{1}{16}e^{-\|g_{k+1}\|_{D_{k+1}}},$$

respectively.

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3. **Proofs of Theorem 2 and Remark 3.** For any 2π periodic function F we denote by

$$\omega(F,\delta) := \sup\{\,|F(t_1) - F(t_2)| : |t_1 - t_2| \le \delta\,\}$$

its modulus of continuity.

LEMMA 2. *Let G be a 2π periodic continuous complex-valued function and $\delta > 0$ be such that $\omega(G,\delta) \le 1$. If $F(t) := \exp\{G(t)\}$, then*

$$(3-e)^{-\|G\|}\omega(G,\delta) \le \omega(F,\delta) \le e^{\|G\|}\omega(G,\delta),$$

where $\|G\| := \|G\|_{[0,2\pi]}$.

PROOF. Let $w \in \mathbb{C}$, $|w| \le 1$. Then

$$\begin{aligned}|e^w - 1| &= |w|\left|1 + \frac{w}{2!} + \frac{w^2}{3!} + \cdots\right| \\ &\ge |w|\left|1 - \left|\frac{w}{2!} + \frac{w^2}{3!} + \cdots\right|\right| \\ &\ge |w|\big(1-(e-2)\big) = (3-e)|w|.\end{aligned}$$

Therefore, for any $a, b \in \mathbb{C}$, $|a-b| \le 1$ we have

$$|e^a - e^b| = |e^b|\,|e^{a-b} - 1| \ge e^{\operatorname{Re} b}(3-e)|a-b|.$$

Thus if t_1 and t_2 are two points in $[0, 2\pi)$ such that $|t_1 - t_2| \le \delta$ and $|G(t_1) - G(t_2)| = \omega(G,\delta)$, we have

$$\begin{aligned}e^{-\|G\|}(3-e)\omega(G,\delta) &\le e^{\operatorname{Re} G(t_2)}(3-e)\,|\,G(t_1) - G(t_2)| \\ &\le |e^{G(t_1)} - e^{G(t_2)}| = |F(t_1) - F(t_2)| \le \omega(F,\delta).\end{aligned}$$

This proves the first inequality. We get the second inequality in a similar way from

$$|e^a - e^b| \le |a-b|\,e^{\max\{|a|,|b|\}} \tag{3.1}$$

for any $a, b, \in \mathbb{C}$. ∎

Denote by $\mathcal{J}_n$ the set of all trigonometric polynomials of degree n. For any 2π periodic function F let

$$E_n^T(F) := \inf_{P\in\mathcal{J}_n} \sup_{0\le t<2\pi} |F(t) - P(t)|$$

denote the best approximation of F by trigonometric polynomials in $\mathcal{J}_n$.

LEMMA 3. *If f is continuous on D_1, analytic in $|z| < 1$, and $F(t) := f(e^{it})$, then*

$$E_n^T(F) \le E_n(f)_{D_1} \le 4E^T_{[n/2]}(F). \tag{3.2}$$

PROOF. If $P \in \Pi_n$, $P(z) = \sum_{k=0}^n b_k z^k$, then

$$Q(t) := P(e^{it}) = \sum_{k=0}^{n}(b_k \cos kt + i b_k \sin kt)$$

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belongs to $\mathcal{J}_n$. Therefore, the maximum principle gives

$$\|f-P\|_{D_1} = \max_{|z|=1} |f(z)-P(z)| = \|F-Q\|_{[0,2\pi)} \geq E_n^T(F).$$

This proves the left-hand inequality in (3.2).

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then the Fourier series of F is given by

$$\sum_{k=0}^{\infty} (a_k \cos kt + i a_k \sin kt)$$

and the corresponding de la Vallée-Poussin sums are $V_m(F,t) = Q_m(e^{it})$, where $Q_m \in \Pi_{2m-1}$,

$$Q_m(z) = \sum_{k=0}^{m} a_k z^k + \sum_{k=m+1}^{2m-1} \left(2-\frac{k}{m}\right) a_k z^k.$$

Therefore, with $m = [n/2]$,

$$\begin{aligned} E_n(f)_{D_1} &\leq \|f-Q_m\|_{D_1} = \max_{|z|=1} |f(z)-Q_m(z)| \\ &= \|F-V_m(F)\|_{[0,2\pi)} \leq 4E_m^T(F), \end{aligned}$$

where in the last inequality we used the well-known estimate for the de la Vallée-Poussin sums given in [3, § 6.1]. ∎

PROOF OF THEOREM 2. Let $m_1 := 1$ and $m_{j+1} := (j+1)^{m_j}$, $j = 1, 2, \ldots$. Set

$$g(z) := \sum_{j=1}^{\infty} 4^{-j} z^{m_j}, \quad G(t) := g(e^{it}),$$
$$f(z) := e^{g(z)}, \quad F(t) := f(e^{it}).$$

For $k = 1, 2, \ldots$, we further set

$$R_k(z) := \sum_{j=1}^{k} 4^{-j} z^{m_j}, \quad G_k(t) := 4^{-k} e^{i m_k t},$$
$$\tilde{G}_k(t) := R_{k-1}(e^{it}), \quad \tilde{\tilde{G}}_k(t) := G(t) - G_k(t) - \tilde{G}_k(t),$$
$$Q_k(z) := \sum_{j=0}^{m_k} \left(R_k(z)\right)^j / j!\,.$$

Note that $Q_k \in \Pi_{m_{k+1}}$. Finally, for $m_{k+1} \leq n < m_{k+2}$, we set $\hat{Q}_n := Q_k \in \Pi_n$, $k = 1, 2, \ldots$.

We claim that f and $\hat{Q}_n$ satisfy all the requirements of the theorem. To this end we shall prove the following:

(3.3) $$f \in A_0(D_1);$$

(3.4) $$\frac{1}{c_1} \leq 4^k E_n(f)_{D_1} \leq c_1, \text{ for } m_k \leq n < m_{k+1}$$

(here and below $c_1, c_2, \ldots$ denote possibly different absolute constants);

$$\|f - Q_k\|_{D_1} \le c_2 4^{-k}, \text{ for } k = 1, 2, \ldots; \tag{3.5}$$

$$Q_k \text{ has no zeros in } D_{(k+1)/2}. \tag{3.6}$$

Then (i) of Theorem 2 will follow from (3.4) and (3.5) and (ii) will follow from (3.6).

For any $j \ge 4$ we have $m_j \ge 4^j$, which implies that

$$\lim_{j\to\infty} (4^{-j})^{1/m_j} = 1.$$

Also the series for g is absolutely convergent in D_1 and hence $g \in A_0(D_1)$. This implies (3.3).

In order to prove (3.4) we first estimate the modulus of continuity of G. Let $\delta = \pi/m_k$. Then

$$\begin{aligned} \omega(\tilde{G}_k;\delta) &\le \sum_{j=1}^{k-1} 4^{-j}\omega(e^{im_jt},\delta) \le \sum_{j=1}^{k-1} 4^{-j} m_j \frac{\pi}{m_k} \\ &\le \frac{m_{k-1}}{m_k}\frac{\pi}{3} \le \frac{1}{3}4^{-k} \text{ for } k \ge 4, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \omega(\tilde{\tilde{G}}_k,\delta) &\le \sum_{j=k+1}^{\infty} 4^{-j}\omega(e^{im_jt},\delta) \\ &\le 2\sum_{j=k+1}^{\infty} 4^{-j} = \frac{2}{3}4^{-k}, \end{aligned} \tag{3.8}$$

and

$$\omega(G_k,\delta) = 4^{-k}|e^{im_k(0)} - e^{im_k(\pi/m_k)}| = 2 \cdot 4^{-k}. \tag{3.9}$$

From (3.7), (3.8) and (3.9) we easily obtain

$$\omega(G,\delta) \le \omega(G_k,\delta) + \omega(\tilde{G}_k,\delta) + \omega(\tilde{\tilde{G}}_k,\delta) \le 3 \cdot 4^{-k}$$

and

$$\begin{aligned} \omega(G,\delta) &\ge \omega(G_k,\delta) - \omega(\tilde{G}_k,\delta) - \omega(\tilde{\tilde{G}}_k,\delta) \\ &\ge 2 \cdot 4^{-k} - \frac{1}{3}4^{-k} - \frac{2}{3}4^{-k} = 4^{-k}, \end{aligned}$$

for $k \ge 4$. This implies

$$c_3^{-1} \le 4^k \omega(G, \pi/m_k) \le c_3 \text{ for any } k. \tag{3.10}$$

From the monotonicity of the modulus of continuity and (3.10) we get

$$c_4^{-1} \le 4^k \omega(G,\delta) \le c_4$$

for any $\delta \in [1/m_{k+1}, 1/m_k]$. Thus Lemma 2 gives

$$c_5^{-1} \le 4^k \omega(F,\delta) \le c_5 \tag{3.11}$$

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for any $\delta \in [1/m_{k+1}, 1/m_k]$.

Jackson's theorem (cf. [9, § 5.1.2]) and (3.11) imply that

$$E_n^T(F) \le c_6 4^{-k} \text{ for any } m_k \le n < m_{k+1}. \tag{3.12}$$

Using (3.11) together with the converse theorem for the best trigonometric approximation (see eg. [9, § 6.1.1]) we get

$$\begin{aligned} c_7\, 4^{-k} &\le c_8\, \omega(F, 1/m_{k+1}) \le m_{k+1}^{-1} \sum_{j=0}^{m_{k+1}} E_j^T(F) \\ &= m_{k+1}^{-1} \sum_{j=0}^{m_k-1} E_j^T(F) + m_{k+1}^{-1} \sum_{j=m_k}^{m_{k+1}} E_j^T(F) \\ &\le m_{k+1}^{-1}\left(m_k\|F\| + m_{k+1} E_{m_k}^T(F)\right) \\ &\le 2m_k(k+1)^{-m_k} + E_{m_k}^T(F), \end{aligned}$$

which, for large enough k and $m_k \le n < m_{k+1}$, yields

$$E_n^T(F) \ge E_{m_{k+1}}^T(F) \ge c_9\, 4^{-(k+1)} = c_{10}\, 4^{-k}. \tag{3.13}$$

Inequalities (3.12), (3.13) and Lemma 3 yield (3.4) for sufficiently large k's. Therefore (3.4) is valid for all k (with a possibly large constant c_1).

In order to prove (3.5) we observe that $|R_k(z)| \le 1/3$ for any $z \in D_1$. Hence from (2.1) with $N = m_k$, $w = R_k(z)$ we get

$$|Q_k(z) - e^{R_k(z)}| \le (1/3)^{m_k+1} e^{1/3}/m_k! \le 4^{-k}. \tag{3.14}$$

From (3.1) with $a = R_k(z)$, $b = g(z)$, we get for $z \in D_1$

$$\begin{aligned} |e^{R_k(z)} - e^{g(z)}| &\le e^{1/3}|R_k(z) - g(z)| \\ &= e^{1/3}\left|\sum_{j=k+1}^{\infty} 4^{-j} z^{m_j}\right| \le 4^{-k}. \end{aligned} \tag{3.15}$$

Combining (3.14) and (3.15) we obtain (3.5) with $c_2 = 2$.

Finally we prove (3.6). Let $|z| = (k+1)/2$. Then

$$|R_k(z)| \le \left(\frac{k+1}{2}\right)^{m_k} \sum_{j=1}^{k} 4^{-j} \le m_{k+1}/5,$$

for any k. By Lemma 1 with $N = m_{k+1}$, $w = R_k(z)$, we have

$$|e^{R_k(z)} - Q_k(z)| < e^{-|R_k(z)|} \le |e^{R_k(z)}|.$$

Thus Rouché's theorem asserts that Q_k has no zeros in $D_{(k+1)/2}$. This completes the proof. ∎

PROOF OF REMARK 3. The same function f and polynomials $\hat{Q}_n$ from the preceding proof are suitable. It is enough to evaluate from below the L_q modulus of G:

$$\omega(G,\delta)_q = \sup\left\{ \left(\frac{1}{2\pi} \int_0^{2\pi} |G(x+t) - G(x)|^q \, dx \right)^{1/q} : 0 < t \leq \delta \right\}.$$

To this end (3.9) should be replaced by

$$\begin{aligned} \omega\left(G_k, \frac{\pi}{m_k}\right) &\geq \left(\frac{1}{2\pi} \int_0^{2\pi} |G_k(x+\pi/m_k) - G_k(x)|^q \, dx \right)^{1/q} \\ &= 2\left(\frac{1}{2\pi} \int_0^{2\pi} |G_k(x)|^q \, dx \right)^{1/q} \\ &= 2 \cdot 4^{-k} \left(\frac{1}{2\pi} \int_0^{2\pi} dx \right)^{1/q} = 2 \cdot 4^{-k}, \end{aligned}$$

because $G_k(x+\pi/m_k) = -G_k(x)$ for any x. Inequalities (3.7) and (3.8) remain the same for L_q moduli and hence $\omega(G,\pi/m_k)_q \geq c_{10}\, 4^{-k}$, which implies an inequality similar to (3.4) for the best L_q approximation of f. ∎

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