3

Tensor algebras

In this chapter we study various constructions related to the tensor product of vector spaces. In particular, we introduce *symmetric* and *anti-symmetric tensor algebras*, whose Hilbert space versions are called *bosonic* and *fermionic Fock spaces*. Fock spaces are fundamental tools used to describe quantum field theories in terms of particles.

We also discuss the notions of *determinants*, *volume forms* and *Pfaffians*, which are closely related to anti-symmetric tensors.

3.1 Direct sums and tensor products

There are several non-equivalent versions of the tensor product of two infinite-dimensional vector spaces. We will introduce two of them, which are especially useful: the *algebraic tensor product* and the *tensor product in the sense of Hilbert spaces*. The former will be denoted with $\overset{\mathrm{al}}{\otimes}$ and the latter with $\otimes$.

There is a similar problem with the direct sum of an infinite number of vector spaces, where we will introduce the *algebraic direct sum* $\overset{\mathrm{al}}{\oplus}$ and the *direct sum in the sense of Hilbert spaces* $\oplus$.

3.1.1 Direct sums

Recall that if $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ is a finite family of vector spaces, then

$$\underset{1\leq i\leq n}{\oplus} \mathcal{Y}_i$$

stands for the direct sum of the spaces $\mathcal{Y}_i$, $i = 1, \dots, n$; see Def. 1.2. It is equal to the Cartesian product $\prod\limits_{1\leq i\leq n} \mathcal{Y}_i$ with the obvious operations.

The notion of the direct sum can be generalized in several ways to the case of an infinite family of vector spaces. One of the most useful is described below.

Let $\{\mathcal{Y}_i\}_{i\in I}$ be a family of vector spaces.

Definition 3.1 *The* algebraic direct sum of vector spaces $\{\mathcal{Y}_i\}_{i\in I}$, *denoted*

$$\underset{i\in I}{\overset{\mathrm{al}}{\oplus}} \mathcal{Y}_i, \tag{3.1}$$

is the subspace of the Cartesian product $\prod\limits_{i\in I} \mathcal{Y}_i$ *consisting of families with all but a finite number of terms equal to zero.*

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Note that for a finite family of spaces the symbols $\oplus$ and $\overset{\mathrm{al}}{\oplus}$ can be used interchangeably.

If $\{\mathcal{Y}_i\}_{i\in I}$ is a family of Hilbert spaces, then $\underset{i\in I}{\overset{\mathrm{al}}{\oplus}} \mathcal{Y}_i$ has a natural scalar product

$$(\{y_i\}_{i\in I}|\{w_i\}_{i\in I}) = \sum_{i\in I}(y_i|w_i),$$

where $\{y_i\}_{i\in I}, \{w_i\}_{i\in I}$ are elements of $\underset{i\in I}{\overset{\mathrm{al}}{\oplus}} \mathcal{Y}_i$.

Definition 3.2 *The* direct sum in the sense of Hilbert spaces *is defined as*

$$\underset{i\in I}{\oplus} \mathcal{Y}_i := \left(\underset{i\in I}{\overset{\mathrm{al}}{\oplus}} \mathcal{Y}_i\right)^{\mathrm{cpl}}.$$

3.1.2 Direct sums of operators

Let $\{\mathcal{Y}_i\}_{i\in I}$, $\{\mathcal{W}_i\}_{i\in I}$ be families of vector spaces.

Definition 3.3 *If $a_i \in L(\mathcal{Y}_i, \mathcal{W}_i)$, $i \in I$, then their* direct sum *is defined as the unique operator $\underset{i\in I}{\oplus} a_i$ in $L\left(\underset{i\in I}{\overset{\mathrm{al}}{\oplus}} \mathcal{Y}_i, \underset{i\in I}{\overset{\mathrm{al}}{\oplus}} \mathcal{W}_i\right)$ satisfying*

$$\left(\underset{i\in I}{\oplus} a_i\right)\{y_i\}_{i\in I} := \{a_i y_i\}_{i\in I}.$$

Let $\{\mathcal{Y}_i\}_{i\in I}$, $\{\mathcal{W}_i\}_{i\in I}$ be families of Hilbert spaces, and a_i, $i \in I$, be closable operators from $\mathcal{Y}_i$ to $\mathcal{W}_i$ with domains Dom a_i. Then the operator $\underset{i\in I}{\oplus} a_i$ with the domain $\underset{i\in I}{\overset{\mathrm{al}}{\oplus}} \mathrm{Dom}\, a_i$ is closable since

$$\underset{i\in I}{\oplus} a_i^* \subset (\underset{i\in I}{\oplus} a_i)^*.$$

Definition 3.4 *The closure of $\underset{i\in I}{\oplus} a_i \in L\left(\underset{i\in I}{\overset{\mathrm{al}}{\oplus}} \mathcal{Y}_i, \underset{i\in I}{\overset{\mathrm{al}}{\oplus}} \mathcal{W}_i\right)$ is denoted by the same symbol $\underset{i\in I}{\oplus} a_i \in Cl\left(\underset{i\in I}{\oplus} \mathcal{Y}_i, \underset{i\in I}{\oplus} \mathcal{W}_i\right)$.*

Clearly, $\underset{i\in I}{\oplus} a_i$ is bounded iff a_i are bounded and $\sup_{i\in I} \|a_i\| < \infty$, and then

$$\|\underset{i\in I}{\oplus} a_i\| = \sup_{i\in I} \|a_i\|.$$

Similarly, $\underset{i\in I}{\oplus} a_i$ is essentially self-adjoint on $\underset{i\in I}{\overset{\mathrm{al}}{\oplus}} \mathrm{Dom}\, a_i$ iff a_i are essentially self-adjoint.

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3.1.3 Algebraic tensor product

Let $\mathcal{Y}, \mathcal{W}$ be vector spaces over $\mathbb{K}$. Let $\mathcal{Z} = c_c(\mathcal{Y} \times \mathcal{W}, \mathbb{K})$, that is, the space of finite linear combinations of $(y, w) \in \mathcal{Y} \times \mathcal{W}$ with coefficients in $\mathbb{K}$ (see Def 2.6). Let $\mathcal{Z}_0$ be the subspace of $\mathcal{Z}$ spanned by elements of the form

$$(y, w_1 + w_2) - (y, w_1) - (y, w_2),\ (y_1 + y_2, w) - (y_1, w) - (y_2, w),$$
$$(\lambda y, w) - \lambda(y, w),\quad (y, \lambda w) - \lambda(y, w),\quad \lambda \in \mathbb{K},\ y, y_1, y_2 \in \mathcal{Y},\ w, w_1, w_2 \in \mathcal{W}.$$

Definition 3.5 *The* algebraic tensor product of $\mathcal{Y}$ and $\mathcal{W}$ *is defined as*

$$\mathcal{Y} \overset{\mathrm{al}}{\otimes} \mathcal{W} := \mathcal{Z}/\mathcal{Z}_0.$$

The formula $y \otimes w := (y, w) + \mathcal{Z}_0$ defines the bilinear map

$$\mathcal{Y} \times \mathcal{W} \ni (y, w) \mapsto y \otimes w \in \mathcal{Y} \overset{\mathrm{al}}{\otimes} \mathcal{W},$$

called the tensor multiplication.

We have natural isomorphisms

$$\mathcal{Y} \simeq \mathbb{K} \overset{\mathrm{al}}{\otimes} \mathcal{Y} \simeq \mathcal{Y} \overset{\mathrm{al}}{\otimes} \mathbb{K}.$$

More generally, let $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ be a finite family of vector spaces. Let $\mathcal{Z} := c_c(\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n, \mathbb{K})$, that is, the vector space over $\mathbb{K}$ of finite linear combinations of $(y_1, \dots, y_n) \in \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n$. Let $\mathcal{Z}_0$ be the subspace of $\mathcal{Z}$ spanned by elements of the form

$$(\dots, y_j + y_j', \dots) - (\dots, y_j, \dots) - (\dots, y_j', \dots),$$
$$(\dots, \lambda y_j, \dots) - \lambda(\dots, y_j, \dots),\quad \lambda \in \mathbb{K},\quad y_i, y_i' \in \mathcal{Y}_i,\ i = 1, \dots, n.$$

Definition 3.6 *The* algebraic tensor product of $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ *is defined as*

$$\mathcal{Y}_1 \overset{\mathrm{al}}{\otimes} \cdots \overset{\mathrm{al}}{\otimes} \mathcal{Y}_n := \mathcal{Z}/\mathcal{Z}_0.$$

The formula $y_1 \otimes \cdots \otimes y_n := (y_1, \dots, y_n) + \mathcal{Z}_0$ defines the n-linear map

$$\mathcal{Y}_1 \times \cdots \times \mathcal{Y}_n \ni (y_1, \dots, y_n) \mapsto y_1 \otimes \cdots \otimes y_n \in \mathcal{Y}_1 \overset{\mathrm{al}}{\otimes} \cdots \overset{\mathrm{al}}{\otimes} \mathcal{Y}_n,$$

called the tensor multiplication.

We have a natural identification

$$\mathcal{Y}_1 \overset{\mathrm{al}}{\otimes} (\mathcal{Y}_2 \overset{\mathrm{al}}{\otimes} \mathcal{Y}_3) \simeq (\mathcal{Y}_1 \overset{\mathrm{al}}{\otimes} \mathcal{Y}_2) \overset{\mathrm{al}}{\otimes} \mathcal{Y}_3 \simeq \mathcal{Y}_1 \overset{\mathrm{al}}{\otimes} \mathcal{Y}_2 \overset{\mathrm{al}}{\otimes} \mathcal{Y}_3. \tag{3.2}$$

The tensor multiplication $\otimes$ is associative.

Remark 3.7 *Note that we can replace the set $\{1, \dots, n\}$, labeling the spaces $\mathcal{Y}_i$ in Def. 3.6, by any finite set I. Then we obtain the definition of $\underset{i \in I}{\overset{\mathrm{al}}{\otimes}} \mathcal{Y}_i$.*

If $\mathcal{Y}$, $\mathcal{W}$ are real vector spaces, then we have the identification

$$\mathbb{C}(\mathcal{Y} \overset{\mathrm{al}}{\otimes} \mathcal{W}) \simeq \mathbb{C}\mathcal{Y} \overset{\mathrm{al}}{\otimes} \mathbb{C}\mathcal{W}. \tag{3.3}$$

Clearly, if $\mathcal{Y}$ and $\mathcal{W}$ are complex spaces, then $\overline{\mathcal{Y} \overset{\mathrm{al}}{\otimes} \mathcal{W}}$ can be identified with $\overline{\mathcal{Y}} \overset{\mathrm{al}}{\otimes} \overline{\mathcal{W}}$.

If one of the spaces $\mathcal{Y}$ or $\mathcal{W}$ is finite-dimensional then we will often write $\mathcal{Y} \otimes \mathcal{W}$ instead of $\mathcal{Y} \overset{\mathrm{al}}{\otimes} \mathcal{W}$.

If $\mathcal{Y}$ and $\mathcal{W}$ are finite-dimensional, then $(\mathcal{Y} \otimes \mathcal{W})^{\#}$ will be identified with $\mathcal{W}^{\#} \otimes \mathcal{Y}^{\#}$ using the following convention: if $\xi \in \mathcal{Y}^{\#}$, $\theta \in \mathcal{W}^{\#}$ then

$$\langle \theta \otimes \xi | y \otimes w \rangle := \langle \xi | y \rangle \langle \theta | w \rangle. \tag{3.4}$$

(Note the reversal of the order.)

3.1.4 Tensor product in the sense of Hilbert spaces

If $\mathcal{Y}$, $\mathcal{W}$ are Hilbert spaces, then $\mathcal{Y} \overset{\mathrm{al}}{\otimes} \mathcal{W}$ has a unique scalar product such that

$$(y_1 \otimes w_1 | y_2 \otimes w_2) := (y_1 | y_2)(w_1 | w_2), \quad y_1, y_2 \in \mathcal{Y}, \quad w_1, w_2 \in \mathcal{W}.$$

Definition 3.8 *We set*

$$\mathcal{Y} \otimes \mathcal{W} := (\mathcal{Y} \overset{\mathrm{al}}{\otimes} \mathcal{W})^{\mathrm{cpl}}, \tag{3.5}$$

and call it the tensor product of $\mathcal{Y}$ and $\mathcal{W}$ in the sense of Hilbert spaces.

If one of the spaces $\mathcal{Y}$ or $\mathcal{W}$ is finite-dimensional, then (3.5) coincides with $\mathcal{Y} \overset{\mathrm{al}}{\otimes} \mathcal{W}$.

The remaining part of the basic theory of the tensor product in the sense of Hilbert spaces is analogous to that of the algebraic tensor product described in the previous subsection.

3.1.5 Bases of tensor products

Let $\mathcal{Y}, \mathcal{W}$ be finite-dimensional vector spaces. If $\{e_i\}_{i \in I}$ is a basis of $\mathcal{Y}$ and $\{f_j\}_{j \in J}$ is a basis of $\mathcal{W}$, then

$$\{e_i \otimes f_j\}_{(i,j) \in I \times J}$$

is a basis of $\mathcal{Y} \otimes \mathcal{W}$.

If $\{e^i\}_{i \in I}$ is the dual basis in $\mathcal{Y}^{\#}$ and $\{f^j\}_{j \in J}$ is the dual basis in $\mathcal{W}^{\#}$ then

$$\{f^j \otimes e^i\}_{(j,i) \in J \times I}$$

is the dual basis in $(\mathcal{Y} \otimes \mathcal{W})^{\#} \simeq \mathcal{W}^{\#} \otimes \mathcal{Y}^{\#}$.

Suppose now that $\mathcal{Y}$ and $\mathcal{W}$ are Hilbert spaces. If $\{e_i\}_{i \in I}$ is an o.n. basis of $\mathcal{Y}$ and $\{f_j\}_{j \in J}$ is an o.n. basis of $\mathcal{W}$, then $\{e_i \otimes f_j\}_{(i,j) \in I \times J}$ is an o.n. basis of $\mathcal{Y} \otimes \mathcal{W}$.

3.1.6 Operators in tensor products

Let $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{W}_1, \mathcal{W}_2$ be vector spaces.

Definition 3.9 *If $a_1 \in L(\mathcal{Y}_1, \mathcal{W}_1)$ and $a_2 \in L(\mathcal{Y}_2, \mathcal{W}_2)$, then $a_1 \otimes a_2$ is defined as the unique operator in $L(\mathcal{Y}_1 \overset{\mathrm{al}}{\otimes} \mathcal{Y}_2, \mathcal{W}_1 \overset{\mathrm{al}}{\otimes} \mathcal{W}_2)$ such that*

$$(a_1 \otimes a_2)(y_1 \otimes y_2) := a_1 y_1 \otimes a_2 y_2.$$

If $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{W}_1, \mathcal{W}_2$ are Hilbert spaces and a_1, resp. a_2, are closable operators from $\mathcal{Y}_1$ to $\mathcal{W}_1$, resp. from $\mathcal{Y}_2$ to $\mathcal{W}_2$, then $a_1 \otimes a_2$ with the domain $\mathrm{Dom}\, a_1 \overset{\mathrm{al}}{\otimes} \mathrm{Dom}\, a_2$ is closable, since

$$a_1^* \otimes a_2^* \subset (a_1 \otimes a_2)^*.$$

Definition 3.10 *The closure of $a_1 \otimes a_2 \in L(\mathcal{Y}_1 \otimes \mathcal{Y}_2, \mathcal{W}_1 \otimes \mathcal{W}_2)$ will be denoted by the same symbol $a_1 \otimes a_2 \in Cl(\mathcal{Y}_1 \otimes \mathcal{Y}_2, \mathcal{W}_1 \otimes \mathcal{W}_2)$.*

If both a_1 and a_2 are non-zero, then $a_1 \otimes a_2$ is bounded iff both a_1 and a_2 are bounded, and then $\|a_1 \otimes a_2\| = \|a_1\|\|a_2\|$.

If both a_1 and a_2 are essentially self-adjoint, then $a_1 \otimes a_2$ is essentially self-adjoint on $\mathrm{Dom}\, a_1 \overset{\mathrm{al}}{\otimes} \mathrm{Dom}\, a_2$.

3.1.7 Permutations

Let $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ be vector spaces.

Definition 3.11 *Let S_n denote the permutation group of n elements and $\sigma \in S_n$. $\Theta(\sigma)$ is defined as the unique operator in $L(\mathcal{Y}_1 \overset{\mathrm{al}}{\otimes} \cdots \overset{\mathrm{al}}{\otimes} \mathcal{Y}_n, \mathcal{Y}_{\sigma^{-1}(1)} \overset{\mathrm{al}}{\otimes} \cdots \overset{\mathrm{al}}{\otimes} \mathcal{Y}_{\sigma^{-1}(n)})$ such that*

$$\Theta(\sigma) y_1 \otimes \cdots \otimes y_n = y_{\sigma^{-1}(1)} \otimes \cdots \otimes y_{\sigma^{-1}(n)}.$$

If $\mathcal{Y}_1, \dots, \mathcal{Y}_n$ are Hilbert spaces, then $\Theta(\sigma)$ is unitary.

3.1.8 Identifications

Let $\mathcal{Y}, \mathcal{W}$ be vector spaces, with $\mathcal{W}$ finite-dimensional. Then there exists a unique linear map $L(\mathcal{W}, \mathcal{Y}) \to \mathcal{Y} \otimes \mathcal{W}^{\#}$ such that

$$|y\rangle\langle\xi| \mapsto y \otimes \xi.$$

If $\mathcal{Y}, \mathcal{W}$ are Hilbert spaces, then there exists a unique unitary map $B^2(\mathcal{W}, \mathcal{Y}) \to \mathcal{Y} \otimes \overline{\mathcal{W}}$ such that

$$|y)(w| \mapsto y \otimes \overline{w}.$$

Note the identity that uses the above identification, valid for $y \in \mathcal{Y}$, $w \in \mathcal{W}$, $B \in B^2(\mathcal{W}, \mathcal{Y})$:

$$(y|Bw) = (y \otimes \overline{w}|B).$$

3.1.9 Infinite tensor product of grounded Hilbert spaces

It is well known that there are problems with the definition of the tensor product of an infinite family of Hilbert spaces. The most useful definition of such a tensor product depends on the choice of a normalized vector in each of these spaces.

Definition 3.12 *A pair $(\mathcal{H}, \Omega)$ consisting of a Hilbert space and a vector $\Omega \in \mathcal{H}$ of norm 1 is called a* grounded Hilbert space*.*

Let $\{(\mathcal{H}_i, \Omega_i)\}_{i \in I}$ be a family of grounded Hilbert spaces. If $J_1 \subset J_2 \subset I$ are two finite sets, we introduce the isometric identification

$$\underset{i \in J_1}{\otimes} \mathcal{H}_i \ni \Psi \mapsto \Psi \otimes \underset{i \in J_2 \setminus J_1}{\otimes} \Omega_i \in \underset{i \in J_2}{\otimes} \mathcal{H}_i.$$

Definition 3.13 *The* tensor product of grounded Hilbert spaces $\{(\mathcal{H}_i, \Omega_i)\}_{i \in I}$ *is defined as*

$$\underset{i \in I}{\otimes} (\mathcal{H}_i, \Omega_i) := \left(\bigcup_{J \in 2^I_{\mathrm{fin}}} \underset{i \in J}{\otimes} \mathcal{H}_i \right)^{\mathrm{cpl}}.$$

The image of $\Psi \in \underset{i \in J}{\otimes} \mathcal{H}_i$ will be denoted by

$$\Psi \otimes \underset{i \in I \setminus J}{\otimes} \Omega_i.$$

Such vectors are called finite vectors*. Similarly, if $B \in B(\underset{i \in J}{\otimes} \mathcal{H}_i)$, we will use the obvious notation*

$$B \otimes \underset{i \in I \setminus J}{\otimes} 1\!\!1_{\mathcal{H}_i} \in B\left(\underset{i \in I}{\otimes} \mathcal{H}_i \right).$$

Clearly, if I is a finite set, then $\underset{i \in I}{\otimes} (\mathcal{H}_i, \Omega_i) = \underset{i \in I}{\otimes} \mathcal{H}_i$ for any family of normalized vectors Ω_i. Moreover, for $I_1 \cap I_2 = \emptyset$ we have

$$\underset{i \in I_1}{\otimes} (\mathcal{H}_i, \Omega_i) \otimes \underset{i \in I_2}{\otimes} (\mathcal{H}_i, \Omega_i) \simeq \underset{i \in I_1 \cup I_2}{\otimes} (\mathcal{H}_i, \Omega_i).$$

3.1.10 Infinite tensor product of vectors and operators

Theorem 3.14 *Let $\Phi_i \in \mathcal{H}_i$, $i \in I$, have norm 1. Set*

$$\Psi_J := \underset{i \in J}{\otimes} \Phi_i \otimes \underset{i \in I \setminus J}{\otimes} \Omega_i \tag{3.6}$$

for $J \in 2^I_{\mathrm{fin}}$. Then the net $\{\Psi_J\}_{J \in 2^I_{\mathrm{fin}}}$ is convergent iff the infinite product $\prod_{i \in I} (\Omega_i|\Phi_i)$ is convergent.

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Definition 3.15 *The vector* $\lim_J \Psi_J$ *will be denoted by* $\underset{i\in I}{\otimes} \Phi_i$.

Proof of Thm. 3.14. Assume first that the net $\{\Psi_J\}_{J\in 2^I_{\rm fin}}$ is convergent in $\underset{i\in I}{\otimes}(\mathcal{H}_i, \Omega_i)$. If $I_0 = \{i \in I \;:\; (\Phi_i|\Omega_i) = 0\}$ is infinite, then clearly $\lim_J(\Psi_J|\Psi) = 0$ for all finite vectors Ψ. Since finite vectors are dense this is a contradiction, since $\lim_J \|\Psi_J\| = 1$. Therefore, I_0 is finite.

It remains to prove that the net $\left\{\prod_{i\in J\backslash I_0}(\Phi_i|\Omega_i)\right\}_{J\in 2^I_{\rm fin}}$ has a non-zero limit. Clearly,

$$\lim_J \Psi_J = \underset{i\in I_0}{\otimes} \Phi_i \otimes \underset{i\in I\backslash I_0}{\otimes} \Phi_i. \tag{3.7}$$

If $I_0 \subset J$, then

$$\left(\Psi_J \Big| \underset{i\in I_0}{\otimes} \Phi_i \otimes \underset{i\in I\backslash I_0}{\otimes} \Omega_i\right) = \prod_{i\in J\backslash I_0}(\Phi_i|\Omega_i),$$

which proves that the net $\left\{\prod_{i\in J\backslash I_0}(\Phi_i|\Omega_i)\right\}_{J\in 2^I_{\rm fin}}$ is convergent in $\mathbb{C}$. If the limit is 0, then, since $(\Phi_i|\Omega_i) \neq 0$ for $i \in I\backslash I_0$, we obtain that the vector $\underset{i\in I\backslash I_0}{\otimes} \Phi_i$ is orthogonal to all finite vectors in $\underset{i\in I\backslash I_0}{\otimes}(\mathcal{H}_i, \Omega_i)$, which using (3.7) yields a contradiction, since $\lim_J \|\Psi_J\| = 1$. Therefore, the infinite product $\prod_{i\in I}(\Phi_i|\Omega_i)$ is convergent.

Conversely, assume that the infinite product $\prod_{i\in I}(\Phi_i|\Omega_i)$ is convergent. Then

$$\sum_{i\in I} |1 - (\Phi_i|\Omega_i)| < \infty.$$

Note that if $J_1 \subset J_2$, then

$$\|\Psi_{J_1} - \Psi_{J_2}\|^2 = 2 - 2\,{\rm Re} \prod_{i\in J_2\backslash J_2}(\Phi_i|\Omega_i).$$

Therefore, the net $\{\Psi_J\}_{J\in 2^I_{\rm fin}}$ is Cauchy, and hence converges in $\underset{i\in I}{\otimes}(\mathcal{H}_i, \Omega_i)$. □

Using Thm. 3.14, we immediately obtain the following theorem.

Theorem 3.16 *Let* $A_i \in B(\mathcal{H}_i)$ *be contractions. Then there exists the strong limit of*

$$B_J := \underset{i\in J}{\otimes} A_i \otimes \underset{i\in I\backslash J}{\otimes} 1\!\!1_{\mathcal{H}_i} \tag{3.8}$$

iff the infinite product $\prod_{i\in I}(\Omega_i|A_i\Omega_i)$ *is convergent.*

Definition 3.17 *The operator* $\lim_J B_J$ *will be denoted by* $\underset{i\in I}{\otimes} A_i$.

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3.2 Tensor algebra

In this section we introduce the *tensor algebra over a vector space*. This concept has two basic versions: we can consider the *algebraic tensor algebra*, or if the vector space has the structure of a Hilbert space, the *complete tensor algebra* (which is also a Hilbert space), called sometimes the *full Fock space*. Full Fock spaces play the central role in the so-called *free probability*. For us, they are mainly intermediate constructions to be used in the discussion of *bosonic* and *fermionic Fock spaces*.

3.2.1 Full Fock space

Let $\mathcal{Y}$ be a vector space.

Definition 3.18 *Let* $\overset{\mathrm{al}}{\otimes}{}^n\mathcal{Y}$ *(or* $\mathcal{Y}^{\overset{\mathrm{al}}{\otimes} n}$*) denote the* n-th algebraic tensor power of $\mathcal{Y}$. *We will write* $\overset{\mathrm{al}}{\otimes}{}^0\mathcal{Y} := \mathbb{K}$. *The* algebraic tensor algebra over $\mathcal{Y}$ *is defined as*

$$\overset{\mathrm{al}}{\otimes}\mathcal{Y} := \overset{\mathrm{al}}{\underset{0\leq n<\infty}{\oplus}} \overset{\mathrm{al}}{\otimes}{}^n\mathcal{Y}.$$

The element $1 \in \overset{\mathrm{al}}{\otimes}{}^0\mathcal{Y}$ *is called the* vacuum *and denoted by* Ω. *If* $\mathcal{Y}$ *is a finite-dimensional space, we will often write* $\otimes^n\mathcal{Y}$ *instead of* $\overset{\mathrm{al}}{\otimes}{}^n\mathcal{Y}$.

$\overset{\mathrm{al}}{\otimes}\mathcal{Y}$ is an associative algebra with the operation $\otimes$ and the identity Ω.

Assume now that $\mathcal{Y}$ is a Hilbert space,

Definition 3.19 *We will write* $\otimes^n\mathcal{Y}$ *(or* $\mathcal{Y}^{\otimes n}$*) for the* n-th tensor power of $\mathcal{Y}$ in the sense of Hilbert spaces*. Clearly, it is equal to* $\left(\overset{\mathrm{al}}{\otimes}{}^n\mathcal{Y}\right)^{\mathrm{cpl}}$. *We set*

$$\otimes\mathcal{Y} := \underset{0\leq n<\infty}{\oplus} \otimes^n\mathcal{Y} = \left(\overset{\mathrm{al}}{\otimes}\mathcal{Y}\right)^{\mathrm{cpl}}.$$

$\otimes\mathcal{Y}$ *is called the* complete tensor algebra *or the* full Fock space*.*

We will also need notation for the finite particle full Fock space

$$\otimes^{\mathrm{fin}}\mathcal{Y} := \overset{\mathrm{al}}{\underset{0\leq n<\infty}{\oplus}} \otimes^n\mathcal{Y}.$$

$\otimes\mathcal{Y}$ and $\otimes^{\mathrm{fin}}\mathcal{Y}$ are associative algebras with the operation $\otimes$ and the identity Ω.

3.2.2 Operators $\mathrm{d}\Gamma$ and Γ in full Fock spaces

The definitions of this subsection have obvious algebraic counterparts. For simplicity, we restrict ourselves to the Hilbert space case and assume that $\mathcal{Y}, \mathcal{Y}_1, \mathcal{Y}_2$ are Hilbert spaces.

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Definition 3.20 *Let p be a linear operator from $\mathcal{Y}_1$ to $\mathcal{Y}_2$. Then we define $\Gamma^n(p) := p^{\otimes n}$ with domain $\overset{\mathrm{al}}{\otimes}{}^n \mathrm{Dom}\, p$, and the operator $\Gamma(p)$ from $\otimes\mathcal{Y}_1$ to $\otimes\mathcal{Y}_2$*

$$\Gamma(p) := \mathop{\oplus}_{n=0}^{\infty} \Gamma^n(p)$$

with domain $\overset{\mathrm{al}}{\otimes} \mathrm{Dom}\, p$.

By Subsects. 3.1.2 and 3.1.6 we see that if p is closable, resp. essentially self-adjoint, then so is $\Gamma(p)$. $\Gamma(p)$ is bounded iff $\|p\| \leq 1$. $\Gamma(p)$ is unitary iff p is.

Definition 3.21 *If h is a linear operator on $\mathcal{Y}$, we set*

$$\mathrm{d}\Gamma^n(h) := \sum_{j=1}^{n} 1\!\!1_{\mathcal{Y}}^{\otimes j-1} \otimes h \otimes 1\!\!1_{\mathcal{Y}}^{\otimes(n-j)}$$

with domain $\overset{\mathrm{al}}{\otimes}{}^n \mathrm{Dom}\, h$, and

$$\mathrm{d}\Gamma(h) := \mathop{\oplus}_{n=0}^{\infty} \mathrm{d}\Gamma^n(h)$$

with domain $\overset{\mathrm{al}}{\otimes} \mathrm{Dom}\, h$.

Again, if h is closable, resp. essentially self-adjoint, then so is $\mathrm{d}\Gamma(h)$.

Definition 3.22 *The* number operator *and the* parity operator *are defined respectively as*

$$N := \mathrm{d}\Gamma(1\!\!1), \tag{3.9}$$

$$I := (-1)^N = \Gamma(-1\!\!1). \tag{3.10}$$

Proposition 3.23 (1) *Let $h, h_1, h_2 \in B(\mathcal{Y})$, $p_1 \in B(\mathcal{Y}, \mathcal{Y}_1)$, $p_2 \in B(\mathcal{Y}_1, \mathcal{Y}_2)$, $\|p_1\|, \|p_2\| \leq 1$. We then have*

$$\Gamma(\mathrm{e}^h) = \mathrm{e}^{\mathrm{d}\Gamma(h)},$$

$$\Gamma(p_2)\Gamma(p_1) = \Gamma(p_2 p_1),$$

$$[\mathrm{d}\Gamma(h_1), \mathrm{d}\Gamma(h_2)] = \mathrm{d}\Gamma([h_1, h_2]).$$

(2) *Let $\Phi, \Psi \in \otimes^{\mathrm{fin}}\mathcal{Y}$, $h \in B(\mathcal{Y})$, $p \in B(\mathcal{Y}, \mathcal{Y}_1)$. Then*

$$\Gamma(p)\ \Phi \otimes \Psi = (\Gamma(p)\Phi) \otimes (\Gamma(p)\Psi),$$

$$\mathrm{d}\Gamma(h)\ \Phi\otimes\Psi = (\mathrm{d}\Gamma(h)\Phi) \otimes \Psi + \Phi \otimes (\mathrm{d}\Gamma(h)\Psi).$$

3.3 Symmetric and anti-symmetric tensors

In this section we describe *symmetric*, resp. *anti-symmetric tensor algebras*. Their Hilbert space versions are also called *bosonic*, resp. *fermionic Fock spaces*.

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Unfortunately, there seems to be no uniform terminology, and especially notation, in this context in the literature. We try to introduce a coherent notation, which in particular stresses parallel properties of the symmetric and anti-symmetric cases.

3.3.1 Fock spaces

Let $\mathcal{Y}$ be a vector space. Recall that in Subsect. 3.1.7, for $\sigma \in S_n$ we defined the operators $\Theta(\sigma) \in L(\overset{\mathrm{al}}{\otimes}{}^n)$. Clearly,

$$S_n \ni \sigma \mapsto \Theta(\sigma) \in L(\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y})$$

is a representation of the permutation group.

Definition 3.24 *We define the following operators on* $\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$*:*

$$\Theta_{\mathrm{s}}^n := \frac{1}{n!} \sum_{\sigma \in S_n} \Theta(\sigma),$$
$$\Theta_{\mathrm{a}}^n := \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\Theta(\sigma).$$

We will write s/a *as a subscript which can mean either* s *or* a.

It is easy to check that $\Theta_{\mathrm{s/a}}^n$ is a projection.

Definition 3.25 *Introduce the following projections acting on* $\overset{\mathrm{al}}{\otimes} \mathcal{Y}$*:*

$$\Theta_{\mathrm{s/a}} := \underset{0 \le n < \infty}{\oplus} \Theta_{\mathrm{s/a}}^n .$$

We set

$$\overset{\mathrm{al}}{\Gamma}{}^n_{\mathrm{s/a}}(\mathcal{Y}) := \Theta_{\mathrm{s/a}}^n \overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y},$$
$$\overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathcal{Y}) := \underset{0 \le n < \infty}{\overset{\mathrm{al}}{\oplus}} \overset{\mathrm{al}}{\Gamma}{}^n_{\mathrm{s/a}}(\mathcal{Y}) = \Theta_{\mathrm{s/a}} \overset{\mathrm{al}}{\otimes} \mathcal{Y}.$$

$\overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathcal{Y})$ *are called the* algebraic symmetric, resp. anti-symmetric tensor algebras *or* algebraic bosonic, resp. fermionic Fock spaces*.*

If $\mathcal{Y}$ *is a finite-dimensional space, we can write* $\Gamma^n_{\mathrm{s/a}}(\mathcal{Y})$ *instead of* $\overset{\mathrm{al}}{\Gamma}{}^n_{\mathrm{s/a}}(\mathcal{Y})$*.*

Elements of $\overset{\mathrm{al}}{\Gamma}{}^n_{\mathrm{s/a}}(\mathcal{Y})$ consist of symmetric, resp. anti-symmetric tensors, as expressed in the following proposition:

Proposition 3.26 *Let* $\Psi \in \overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$*. Then*

(1) $\Psi \in \overset{\mathrm{al}}{\Gamma}{}^n_{\mathrm{s}}(\mathcal{Y})$ *iff* $\Theta(\sigma)\Psi = \Psi$*,* $\sigma \in S_n$*;*
(2) $\Psi \in \overset{\mathrm{al}}{\Gamma}{}^n_{\mathrm{a}}(\mathcal{Y})$ *iff* $\Theta(\sigma)\Psi = \mathrm{sgn}(\sigma)\Psi$*,* $\sigma \in S_n$*.*

Assume now that $\mathcal{Y}$ is a Hilbert space. Then $\Theta_{\mathrm{s/a}}^n$ and $\Theta_{\mathrm{s/a}}$ are orthogonal projections.

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Definition 3.27 *We define*

$$\Gamma^n_{\mathrm{s/a}}(\mathcal{Y}) := \Theta^n_{\mathrm{s/a}} \otimes^n \mathcal{Y} = \left(\overset{\mathrm{al}}{\Gamma}{}^n_{\mathrm{s/a}}(\mathcal{Y})\right)^{\mathrm{cpl}},$$

$$\Gamma_{\mathrm{s/a}}(\mathcal{Y}) := \overset{\infty}{\underset{n=0}{\oplus}} \Gamma^n_{\mathrm{s/a}}(\mathcal{Y}) = \Theta_{\mathrm{s/a}} \otimes \mathcal{Y} = \left(\overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathcal{Y})\right)^{\mathrm{cpl}}.$$

$\Gamma_{\mathrm{s/a}}(\mathcal{Y})$ *is called the* bosonic, *resp.* fermionic Fock space.

Note that $\Gamma_{\mathrm{s/a}}(\mathcal{Y})$ itself is a Hilbert space (as a closed subspace of $\otimes\mathcal{Y}$).

Definition 3.28 *We will need notation for the* finite particle bosonic*, resp.* fermionic Fock space*:*

$$\Gamma^{\mathrm{fin}}_{\mathrm{s/a}}(\mathcal{Y}) := \overset{\mathrm{al}}{\underset{0\leq n<\infty}{\oplus}} \Gamma^n_{\mathrm{s/a}}(\mathcal{Y}).$$

3.3.2 Symmetric and anti-symmetric tensor products

Let $\Psi, \Phi \in \overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathcal{Y})$.

Definition 3.29 *We define the* symmetric, *resp.* anti-symmetric tensor product of Φ and Ψ*:*

$$\Psi \otimes_{\mathrm{s/a}} \Phi := \Theta_{\mathrm{s/a}} \Psi \otimes \Phi.$$

$\overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathcal{Y})$ is an associative algebra with the operation $\otimes_{\mathrm{s/a}}$ and the identity Ω. Note that the set of vectors of the form

$$\underbrace{y \otimes \cdots \otimes y}_{n \text{ times}} = \underbrace{y \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} y}_{n \text{ times}}, \tag{3.11}$$

for $y \in \mathcal{Y}$, spans $\overset{\mathrm{al}}{\Gamma}{}^n_{\mathrm{s}}(\mathcal{Y})$.

Definition 3.30 *For brevity we will denote (3.11) by* $y^{\otimes n}$.

The notation $\otimes_{\mathrm{a}}$ that we introduced is not common in the literature. Instead, one usually prefers a different closely related operation:

Definition 3.31 *The* wedge product of vectors Φ and Ψ *is defined as*

$$\Psi \wedge \Phi := \frac{(p+q)!}{p!q!} \Psi \otimes_{\mathrm{a}} \Phi, \ \ \textit{for } \Psi \in \overset{\mathrm{al}}{\Gamma}{}^p_{\mathrm{a}}(\mathcal{Y}), \ \Phi \in \overset{\mathrm{al}}{\Gamma}{}^q_{a}(\mathcal{Y}). \tag{3.12}$$

The advantage of the wedge product over $\otimes_{\mathrm{a}}$ is visible if we compare the following identities:

$$y_1 \wedge \cdots \wedge y_n = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\; y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(n)},$$

$$y_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} y_n = \tfrac{1}{n!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\; y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(n)}, \quad y_1, \cdots, y_n \in \mathcal{Y}.$$

Note that $\wedge$ is also associative.

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Definition 3.32 *One often writes* $\wedge^n \mathcal{Y}$ *and* $\wedge \mathcal{Y}$ *for* $\overset{\mathrm{al}}{\Gamma}{}^n_{\mathrm{a}}(\mathcal{Y})$ *and* $\overset{\mathrm{al}}{\Gamma}{}_{\mathrm{a}}(\mathcal{Y})$*.*

Definition 3.33 *If* $\mathcal{Y}$ *is a Hilbert space, we can define* $\otimes_{\mathrm{s/a}}$ *and* $\wedge$ *in* $\Gamma_{\mathrm{s/a}}(\mathcal{Y})$ *in the same way, with the same properties.*

3.3.3 $\mathrm{d}\Gamma$ *and* Γ *operators*

For brevity we restrict ourselves to the case of Hilbert spaces.

Let p be a closable operator from $\mathcal{Y}$ to $\mathcal{W}$. Then $\Gamma^n(p)$ maps $\Gamma^n_{\mathrm{s/a}}(\mathcal{Y})$ into $\Gamma^n_{\mathrm{s/a}}(\mathcal{W})$. Hence $\Gamma(p)$ maps $\Gamma_{\mathrm{s/a}}(\mathcal{Y})$ into $\Gamma_{\mathrm{s/a}}(\mathcal{W})$.

Definition 3.34 *We will use the same symbols* $\Gamma^n(p)$ *and* $\Gamma(p)$ *to denote the corresponding restricted operators.* $\Gamma(p)$ *is sometimes called the* second quantization of p*.*

Let h be a closable operator on $\mathcal{Y}$. Then $\mathrm{d}\Gamma^n(h)$ maps $\Gamma^n_{\mathrm{s/a}}(\mathcal{Y})$ into itself. Hence, $\mathrm{d}\Gamma(h)$ maps $\Gamma_{\mathrm{s/a}}(\mathcal{Y})$ into itself.

Definition 3.35 *We will use the same symbols* $\mathrm{d}\Gamma^n(h)$ *and* $\mathrm{d}\Gamma(h)$ *to denote the corresponding restricted operators. Perhaps the correct name of* $\mathrm{d}\Gamma(h)$ *should be the* infinitesimal second quantization of h*.*

Note that in the context of bosonic, resp. fermionic Fock spaces the operators $\Gamma(\cdot)$ and $\mathrm{d}\Gamma(\cdot)$ still have the properties described in Prop. 3.23 (1). Prop. 3.23 (2) needs to be replaced by the following statement:

Proposition 3.36 *Let* $p \in B(\mathcal{Y}, \mathcal{Y}_1)$*,* $h \in B(\mathcal{Y})$*,* $\Psi, \Phi \in \Gamma^{\mathrm{fin}}_{\mathrm{s/a}}(\mathcal{Y})$*. Then*

$$\Gamma(p)\ \Psi \otimes_{\mathrm{s/a}} \Phi = (\Gamma(p)\Psi) \otimes_{\mathrm{s/a}} (\Gamma(p)\Phi),$$
$$\mathrm{d}\Gamma(h)\ \Psi \otimes_{\mathrm{s/a}} \Phi = (\mathrm{d}\Gamma(h)\Psi) \otimes_{\mathrm{s/a}} \Phi + \Psi \otimes_{\mathrm{s/a}} (\mathrm{d}\Gamma(h)\Phi).$$

3.3.4 Identifications

Let $\mathcal{Y}$ be a finite-dimensional vector space. Then $\Gamma^2_{\mathrm{s/a}}(\mathcal{Y})$ can be identified with $L_{\mathrm{s/a}}(\mathcal{Y}^{\#}, \mathcal{Y})$, which were defined in Defs. 1.18 and 1.29.

Let $\mathcal{Y}$ be a Hilbert space. Recall that $B^2(\mathcal{Y}, \mathcal{W})$ denotes the space of Hilbert–Schmidt operators from $\mathcal{Y}$ to $\mathcal{W}$. We introduce the following symbols for the spaces of symmetric and anti-symmetric Hilbert–Schmidt operators:

$$B^2_{\mathrm{s}}(\overline{\mathcal{Y}}, \mathcal{Y}) := \left\{ a \in B^2(\overline{\mathcal{Y}}, \mathcal{Y}) \ :\ a^{\#} = a \right\},$$
$$B^2_{\mathrm{a}}(\overline{\mathcal{Y}}, \mathcal{Y}) := \left\{ a \in B^2(\overline{\mathcal{Y}}, \mathcal{Y}) \ :\ a^{\#} = -a \right\},$$

where as usual one identifies $\mathcal{Y}^{\#}$ with $\overline{\mathcal{Y}}$ using the Hilbert structure of $\mathcal{Y}$. Then the unitary map of Subsect. 3.1.8 allows us to unitarily identify $\Gamma^2_{\mathrm{s/a}}(\mathcal{Y})$ with $B^2_{\mathrm{s/a}}(\overline{\mathcal{Y}}, \mathcal{Y})$.

3.3.5 Bases in bosonic Fock spaces

Let $\mathcal{Y}$ be a finite-dimensional vector space and $\{e_i \;:\; i = 1, \ldots, d\}$ a basis of $\mathcal{Y}$.

Definition 3.37 *For $\vec{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d$, we set*

$$|\vec{k}| := k_1 + \cdots k_d, \quad \vec{k}! := k_1! \ldots k_d!,$$

$$e_{\vec{k}} := e_1^{\otimes k_1} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} e_d^{\otimes k_d}, \quad e_{\vec{0}} := \Omega.$$

Then

$$\{e_{\vec{k}} \;:\; \vec{k} \in \mathbb{N}^d, \; |\vec{k}| = n\} \tag{3.13}$$

is a basis of $\Gamma_{\mathrm{s}}^n(\mathcal{Y})$.

The dual of $\Gamma_{\mathrm{s}}^n(\mathcal{Y})$ can be identified with $\Gamma_{\mathrm{s}}^n(\mathcal{Y}^{\#})$. Let $\{e^i \;:\; i = 1, \ldots, d\}$ be the dual basis of $\mathcal{Y}^{\#}$.

Definition 3.38 *We set $e^{\vec{k}} := (e^1)^{\otimes k_d} \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} (e^d)^{\otimes k_1}$, for $k \in \mathbb{N}^d$.*

Then

$$\left\{ \frac{|\vec{k}|!}{\vec{k}!} e^{\vec{k}} \;:\; \vec{k} \in \mathbb{N}^d, \; |\vec{k}| = n \right\}$$

is the basis of $\Gamma_{\mathrm{s}}^n(\mathcal{Y}^{\#})$ dual to (3.13).

Let $\mathcal{Y}$ be now a Hilbert space with an o.n. basis $\{e_i\}_{i \in I}$.

Definition 3.39 *Recall that $c_{\mathrm{c}}(I, \mathbb{N})$ denotes the set of functions $I \to \mathbb{N}$ with all but a finite number of values equal to zero. If $\vec{k} \in c_{\mathrm{c}}(I, \mathbb{N})$, then the definitions of $|\vec{k}|$, $\vec{k}!$ and $e_{\vec{k}}$ have obvious versions in the present context.*

Then

$$\left\{ \frac{\sqrt{|\vec{k}|!}}{\sqrt{\vec{k}!}} e_{\vec{k}} \;:\; \vec{k} \in c_{\mathrm{c}}(I, \mathbb{N}), \; |\vec{k}| = n \right\}$$

is an o.n. basis of $\Gamma_{\mathrm{s}}^n(\mathcal{Y})$.

3.3.6 Bases in fermionic Fock spaces

Let $\mathcal{Y}$ be a finite-dimensional vector space and $\{e_i \;:\; i = 1, \ldots, d\}$ a basis of $\mathcal{Y}$.

Definition 3.40 *For $J = \{i_1, \cdots, i_n\} \subset \{1, \ldots, d\}$ with $1 \le i_1 < \cdots < i_n \le d$, set*

$$e_J := e_{i_1} \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} e_{i_n}.$$

Then

$$\{e_J \;:\; J \subset \{1, \ldots, d\}, \; \#J = n\} \tag{3.14}$$

is a basis of $\Gamma_{\mathrm{a}}^n(\mathcal{Y})$.

The dual of $\Gamma_{\mathrm{a}}^n(\mathcal{Y})$ can be identified as above with $\Gamma_{\mathrm{a}}^n(\mathcal{Y}^{\#})$.

Let $\{e^i \ : \ i = 1, \dots, d\}$ be the dual basis in $\mathcal{Y}^{\#}$.

Definition 3.41 *For $J = \{i_1, \cdots, i_n\} \subset \{1, \dots, d\}$ with $1 \le i_1 < \cdots < i_n \le d$ put $e^J := e^{i_n} \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} e^{i_1}$.*

Then

$$\left\{\#J! e^J \ : \ J \subset \{1, \dots, d\}, \ \#J = n\right\}$$

is the basis of $\Gamma_{\mathrm{a}}^n(\mathcal{Y}^{\#})$ dual to (3.14).

Let $\mathcal{Y}$ be now a Hilbert space with an o.n. basis $\{e_i \ : \ i \in I\}$. Let us choose a total order in the set I.

Definition 3.42 *For a finite subset J of I, we define e_J in an obvious way.*

Then

$$\left\{\sqrt{\#J!} e_J \ : \ J \subset I, \ , \#J = n\right\}$$

is an o.n. basis of $\Gamma_{\mathrm{a}}^n(\mathcal{Y})$.

3.3.7 Exponential law for Fock spaces

For brevity we restrict ourselves again to the case of Hilbert spaces. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be Hilbert spaces and let $j_i : \mathcal{Y}_i \to \mathcal{Y}_1 \oplus \mathcal{Y}_2$ be the canonical embeddings.

We introduce an identification

$$U : \Gamma_{\mathrm{s/a}}^{\mathrm{fin}}(\mathcal{Y}_1) \overset{\mathrm{al}}{\otimes} \Gamma_{\mathrm{s/a}}^{\mathrm{fin}}(\mathcal{Y}_2) \to \Gamma_{\mathrm{s/a}}^{\mathrm{fin}}(\mathcal{Y}_1 \oplus \mathcal{Y}_2)$$

as follows. Let $\Psi_1 \in \Gamma_{\mathrm{s/a}}^{n_1}(\mathcal{Y}_1)$, $\Psi_2 \in \Gamma_{\mathrm{s/a}}^{n_2}(\mathcal{Y}_2)$. Then

$$U \Psi_1 \otimes \Psi_2 := \sqrt{\tfrac{(n_1+n_2)!}{n_1! n_2!}} (\Gamma(j_1)\Psi_1) \otimes_{\mathrm{s/a}} (\Gamma(j_2)\Psi_2). \tag{3.15}$$

Theorem 3.43 (1) *U extends to a unitary operator from $\Gamma_{\mathrm{s/a}}(\mathcal{Y}_1) \otimes \Gamma_{\mathrm{s/a}}(\mathcal{Y}_2)$ to $\Gamma_{\mathrm{s/a}}(\mathcal{Y}_1 \oplus \mathcal{Y}_2)$.*

(2) *$U \Omega_1 \otimes \Omega_2 = \Omega$.*

(3) *If $h_i \in B(\mathcal{Y}_i)$, then*

$$\mathrm{d}\Gamma(h_1 \oplus h_2) U = U\big(\mathrm{d}\Gamma(h_1) \otimes 1\!\!1 + 1\!\!1 \otimes \mathrm{d}\Gamma(h_2)\big). \tag{3.16}$$

(4) *If $p_i \in B(\mathcal{Y}_i)$, then*

$$\Gamma(p_1 \oplus p_2) U = U \Gamma(p_1) \otimes \Gamma(p_2). \tag{3.17}$$

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Proof Let us prove (1). To simplify the notation let us restrict ourselves to the symmetric case. Let $\Psi_1 \in \Gamma_{\mathrm{s}}^{n_1}(\mathcal{Y}_1)$, $\Psi_2 \in \Gamma_{\mathrm{s}}^{n_2}(\mathcal{Y}_2)$. Then

$$\begin{aligned}\Gamma(j_1)\Psi_1 \otimes_{\mathrm{s}} \Gamma(j_2)\Psi_2 &= \tfrac{1}{(n_1+n_2)!} \sum_{\sigma \in S_{n_1+n_2}} \Theta(\sigma)\Gamma(j_1)\Psi_1 \otimes \Gamma(j_2)\Psi_2 \\ &= \tfrac{n_1!n_2!}{(n_1+n_2)!} \sum_{[\sigma] \in S_{n_1+n_2}/S_{n_1}\times S_{n_2}} \Theta(\sigma)\Gamma(j_1)\Psi_1 \otimes \Gamma(j_2)\Psi_2.\end{aligned}$$

Now the elements of the sum on the right are mutually orthogonal. Hence

$$\begin{aligned}\|\Gamma(j_1)\Psi_1 \otimes_{\mathrm{s}} \Gamma(j_2)\Psi_2\|^2 &= \left(\tfrac{n_1!n_2!}{(n_1+n_2)!}\right)^2 \sum_{[\sigma] \in S_{n_1+n_2}/S_{n_1}\times S_{n_2}} \|\Theta(\sigma)\Psi_1 \otimes \Psi_2\|^2 \\ &= \tfrac{n_1!n_2!}{(n_1+n_2)!}\|\Psi_1 \otimes \Psi_2\|^2.\end{aligned}$$ □

Using the concept of the tensor product of grounded Hilbert spaces, one can easily generalize the exponential law to the case of an infinite number of Fock spaces. In fact, let $\mathcal{Y}_i$, $i \in I$ be a family of Hilbert spaces and denote by $j_i : \mathcal{Y}_i \to \underset{i\in I}{\oplus} \mathcal{Y}_i$ the canonical embeddings. Let Ω_i denote the vacuum in $\Gamma_{\mathrm{s/a}}(\mathcal{Y}_i)$. Then

$$\begin{aligned}&U\Psi_{i_1} \otimes \cdots \otimes \Psi_{i_n} \otimes \underset{i\in I\setminus\{i_1,\dots,i_n\}}{\otimes} \Omega \\ &:= \frac{\sqrt{(i_1+\cdots+i_n)!}}{\sqrt{i_1!}\cdots\sqrt{i_n!}}\Gamma(j_{i_1})\Psi_1 \otimes_{\mathrm{s}} \cdots \otimes_{\mathrm{s}} \Gamma(j_{i_n})\Psi_{i_n}\end{aligned}$$

extends to a unitary map

$$U : \underset{i\in I}{\otimes} \left(\Gamma_{\mathrm{s/a}}(\mathcal{Y}_i), \Omega_i\right) \to \Gamma_{\mathrm{s/a}}\left(\underset{i\in I}{\oplus} \mathcal{Y}_{\mathrm{i}}\right).$$

3.3.8 Dimension of Fock spaces

Let $\dim \mathcal{Y} = d$. Then it is easy to see that

$$\begin{aligned}\dim \Gamma_{\mathrm{s}}^n(\mathcal{Y}) &= \tfrac{(d+n-1)!}{(d-1)!n!}, \\ \dim \Gamma_{\mathrm{a}}^n(\mathcal{Y}) &= \tfrac{n!}{d!(n-d)!}.\end{aligned}$$

We have the following generating functions for the above quantities:

$$\begin{aligned}(1-t)^{-d} &= \sum_{n=0}^{\infty} t^n \tfrac{(d+n-1)!}{(d-1)!n!}, \\ (1+t)^d &= \sum_{n=0}^{d} t^n \tfrac{n!}{d!(n-d)!}.\end{aligned} \tag{3.18}$$

Recall that we have the identifications

$$\Gamma_{\mathrm{s/a}}^n(\mathcal{Y}_1 \oplus \mathcal{Y}_2) \simeq \overset{n}{\underset{m=0}{\oplus}} \Gamma_{\mathrm{s/a}}^m(\mathcal{Y}_1) \otimes \Gamma_{\mathrm{s/a}}^{n-m}(\mathcal{Y}_2). \tag{3.19}$$

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Assume that $\dim \mathcal{Y}_1 = d_1$, $\dim \mathcal{Y}_2 = d_2$. Then comparing the dimensions of both sides of (3.19) we obtain the following identities:

$$\frac{(d_1+d_2+n-1)!}{n!(d_1+d_2-1)!} = \sum_{m=0}^{n} \frac{(d_1+m-1)!}{m!(d_1-1)!} \frac{(d_2+n-m-1)!}{(n-m)!(d_2-1)!},$$
$$\frac{(d_1+d_2)!}{n!(d_1+d_2-n)!} = \sum_{m=0}^{n} \frac{d_1!}{m!(d_1-m)!} \frac{d_2!}{(n-m)!(d_2-n+m)!}.$$

These identities can be easily shown using the generating functions (3.18) and the identities

$$(1-t)^{-d_1}(1-t)^{-d_2} = (1-t)^{-(d_1+d_2)},$$
$$(1+t)^{d_1}(1+t)^{d_2} = (1+t)^{(d_1+d_2)}.$$

3.3.9 Super-Fock spaces

Let $(\mathcal{Y}, \epsilon)$ be a super-space (that is, a vector space equipped with an involution; see Subsect. 1.1.15). Then we introduce the action of the permutation group in $L(\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y})$ as follows:

Definition 3.44 *Let $\sigma \in \mathcal{S}_n$. Then $\Theta_\epsilon(\sigma)$ will denote the unique linear operator on $\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$ with the following property. Let $y_1, \dots, y_n \in \mathcal{Y}$ be homogeneous. Then*

$$\Theta_\epsilon(\sigma) y_1 \otimes \cdots \otimes y_n = \mathrm{sgn}_\epsilon(\sigma)\, y_{\sigma^{-1}(1)} \otimes \cdots \otimes y_{\sigma^{-1}(n)},$$

where $\mathrm{sgn}_\epsilon(\sigma)$ is the sign of the permutation σ restricted to the odd elements.

Definition 3.45 *We define*

$$\Theta_\epsilon^n := \frac{1}{n!} \sum_{\sigma \in S_n} \Theta_\epsilon(\sigma).$$

Clearly, Θ_ϵ^n is a projection on $\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$.

Definition 3.46 *We set*

$$\overset{\mathrm{al}}{\Gamma}{}^n_\epsilon(\mathcal{Y}) := \Theta_\epsilon^n \overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y},$$
$$\overset{\mathrm{al}}{\Gamma}_\epsilon(\mathcal{Y}) := \overset{\mathrm{al}}{\underset{0 \le n \le \infty}{\oplus}} \overset{\mathrm{al}}{\Gamma}{}^n_\epsilon(\mathcal{Y}).$$

If $\mathcal{Y}$ is a finite-dimensional space, we can write $\Gamma_\epsilon^n(\mathcal{Y})$ instead of $\overset{\mathrm{al}}{\Gamma}{}^n_\epsilon(\mathcal{Y})$.

If $\mathcal{Y}$ is a Hilbert space, then Θ_ϵ^n are orthogonal projections.

Definition 3.47 *We define the* super-Fock spaces

$$\Gamma_\epsilon^n(\mathcal{Y}) := \Theta_\epsilon^n \otimes^n \mathcal{Y},$$
$$\Gamma_\epsilon(\mathcal{Y}) := \overset{\infty}{\underset{n=0}{\oplus}} \Gamma_\epsilon^n(\mathcal{Y}).$$

We extend various definitions from the context of bosonic, resp. fermionic Fock spaces to super-spaces in an obvious way. In particular, we define the operation $\otimes_\epsilon$*, creation, resp. annihilation operators (generalizing the definitions of Sect. 3.4 below) and the operators* $\Gamma(\cdot)$ *and* $\mathrm{d}\Gamma(\cdot)$*.*

$\Gamma_\epsilon^n(\mathcal{Y})$ is naturally a super-space with the involution $\Gamma(\epsilon)$.

Super-Fock spaces enjoy the exponential property analogous to that described in Thm. 3.43 for bosonic and fermionic Fock spaces. Thus if $(\mathcal{Y},\epsilon)$, $(\mathcal{W},\varepsilon)$ are two super-Hilbert spaces, then

$$\Gamma_{\epsilon\oplus\varepsilon}(\mathcal{Y}\oplus\mathcal{W})\simeq\Gamma_\epsilon(\mathcal{Y})\otimes\Gamma_\varepsilon(\mathcal{W}). \tag{3.20}$$

In particular, if $\mathcal{Y}=\mathcal{Y}_0\oplus\mathcal{Y}_1$ is the decomposition into the even and odd subspace, we then have

$$\Gamma_\epsilon(\mathcal{Y})\simeq\Gamma_{\mathrm{s}}(\mathcal{Y}_0)\otimes\Gamma_{\mathrm{a}}(\mathcal{Y}_1), \tag{3.21}$$

which can be treated as an alternative definition of a super-Fock space.

We will often drop the index ϵ in (3.21)

Note that if $c\in L(\mathcal{Y})$ is odd, then $\mathrm{d}\Gamma(c)^2=\mathrm{d}\Gamma(c^2)$. In the matrix notation:

$$\mathrm{d}\Gamma\left(\begin{bmatrix}0 & c_{01}\\ c_{10} & 0\end{bmatrix}\right)^2=\mathrm{d}\Gamma\left(\begin{bmatrix}c_{01}c_{10} & 0\\ 0 & c_{10}c_{01}\end{bmatrix}\right).$$

This identity plays an important role in super-symmetric quantum physics.

3.4 Creation and annihilation operators

Creation and *annihilation operators* belong to the most useful constructions of quantum physics. This section is devoted to their basic properties, in both the bosonic and the fermionic case.

Throughout this section we will use the standard convention for the scalar product in the Fock spaces. Some of the properties of creation and annihilation operators actually look simpler on *modified Fock spaces*, which will be discussed in Subsect. 3.5.7.

Throughout the section, $\mathcal{Z}$, $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are Hilbert spaces.

3.4.1 Creation and annihilation operators: abstract approach

We prepare for the definitions of the creation and annihilation operators with two lemmas in an attract setting. We start with the bosonic case.

Lemma 3.48 *Let* $\mathcal{H}$ *be a Hilbert space,* $\mathcal{D}\subset\mathcal{H}$ *a dense subspace, and c, a two linear operators on* $\mathcal{D}$ *such that*

(1) $c,a:\mathcal{D}\to\mathcal{D}$*;*
(2) $c\subset a^*$, $a\subset c^*$*;*

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(3) *$ac - ca = \mathbb{1}$, as an operator identity on $\mathcal{D}$;*
(4) *ca is essentially self-adjoint on $\mathcal{D}$.*

Then c, a are closable with $a^{\mathrm{cl}} = c^*$, $c^{\mathrm{cl}} = a^*$. If we write a for a^{cl}, one has

$$aa^* - a^*a = \mathbb{1}, \text{ as a quadratic form identity on } \mathrm{Dom}(a) \cap \mathrm{Dom}(a^*).$$

Proof Since $c \subset a^*$ and $a \subset c^*$, c^* and a^* are densely defined, and hence c and a are closable. Moreover, since $c \subset a^*$ we have $c^{\mathrm{cl}} \subset a^{\mathrm{cl}*}$. From now on we will denote a^{cl}, c^{cl} simply by a, c.

Set $N := (ca)^{\mathrm{cl}}$. Using (4), we see that N is a positive self-adjoint operator and $\mathcal{D}$ is a core for N. Since

$$\|a\Phi\|^2 = (\Phi|ca\Phi), \ \|c\Phi\|^2 = (\Phi|ca\Phi) + (\Phi|\Phi) \text{ for } \Phi \in \mathcal{D},$$

we see that $\mathrm{Dom}\, a = \mathrm{Dom}\, c = \mathrm{Dom}\, N^{\frac{1}{2}}$. This implies that

$$a(N + \mathbb{1})^{-\frac{1}{2}}, \ c(N + \mathbb{1})^{-\frac{1}{2}}, \ (N + \mathbb{1})^{-\frac{1}{2}}c, \ (N + \mathbb{1})^{-\frac{1}{2}}a \in B(\mathcal{H}). \tag{3.22}$$

Next, for $\Phi, \Psi \in \mathcal{D}$, we have

$$|(c\Phi|\Psi)| = |(\Phi|a\Psi)| = |((N + \mathbb{1})^{\frac{1}{2}}\Phi|(N + \mathbb{1})^{-\frac{1}{2}}a\Psi)| \leq C\|(N + \mathbb{1})^{\frac{1}{2}}\Phi\|\|\Psi\|.$$

Since $\mathcal{D}$ is dense in $\mathrm{Dom}\, N^{\frac{1}{2}}$ and in $\mathrm{Dom}\, a$, we obtain that $\mathrm{Dom}\, N^{\frac{1}{2}} \subset \mathrm{Dom}\, a^*$, and $a^*\big|_{\mathrm{Dom}\, N^{\frac{1}{2}}} = c$.

To prove that $a^* = c$, it remains to prove that $\mathrm{Dom}\, a^* = \mathrm{Dom}\, N^{\frac{1}{2}}$. Note that $\Phi \in \mathrm{Dom}\, N^{\frac{1}{2}}$ iff

$$\|N(\epsilon N + 1)^{-1}\Phi\| \leq C\epsilon^{-\frac{1}{2}}, \ \epsilon > 0. \tag{3.23}$$

From the identity $a(N + \mathbb{1}) = Na$ valid on $\mathcal{D}$, we deduce first that $(\epsilon N + \mathbb{1})^{-1}a(\epsilon N + \mathbb{1} - \epsilon) = a$ on $\mathcal{D}$ and then on $\mathrm{Dom}\, N$, and then that

$$(\epsilon N + \mathbb{1})^{-1}a = a(\epsilon N + \mathbb{1} - \epsilon)^{-1} \text{ on } \mathcal{H}, \tag{3.24}$$

since both operators are bounded by (3.22). For $\Phi \in \mathrm{Dom}\, a^*$ and $\Psi \in \mathcal{D}$, we have

$$\begin{aligned}
|(\Phi|N(\epsilon N + \mathbb{1})^{-1}\Psi)| &= |(\Phi|(\epsilon N + \mathbb{1})^{-1}ca\Psi)| \\
&= |(\Phi|(\epsilon N + \mathbb{1})^{-1}ac\Psi) - (\Phi|(\epsilon N + \mathbb{1})^{-1}\Psi)| \\
&= (\Phi|a(\epsilon N + \mathbb{1} - \epsilon)^{-1}c\Psi) - (\Phi|(\epsilon N + \mathbb{1})^{-1}\Psi)| \\
&\leq C\|(\epsilon N + \mathbb{1} - \epsilon)^{-1}c\Psi\| + C\|\Psi\| \\
&\leq C\|(N + \mathbb{1})^{\frac{1}{2}}(\epsilon N + \mathbb{1} - \epsilon)^{-1}\|\|(N + \mathbb{1})^{-\frac{1}{2}}c\Psi\| + C\|\Psi\| \\
&\leq C\epsilon^{-\frac{1}{2}}\|\Psi\|,
\end{aligned}$$

where we have used (3.24) and the fact that $\Phi \in \mathrm{Dom}\, a^*$. Using (3.23), we obtain that $\Phi \in \mathrm{Dom}\, N^{\frac{1}{2}}$, which completes the proof that $a^* = c$, and hence that $c^* = a$. The quadratic form identity on $\mathrm{Dom}\, a \cap \mathrm{Dom}\, a^*$ follows then by density from the operator identity on $\mathcal{D}$. □

The following lemma describes properties of fermionic creation and annihilation operators in the abstract setting:

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Lemma 3.49 *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{D} \subset \mathcal{H}$ a dense subspace, and c, a two linear operators on $\mathcal{D}$ such that*

(1) $c, a : \mathcal{D} \to \mathcal{D}$;
(2) $c \subset a^*$, $a \subset c^*$;
(3) $a^2 = c^2 = 0$, $ac + ca = \mathbb{1}$ *as operator identities on* $\mathcal{D}$.

Then c, a extend as bounded operators on $\mathcal{H}$, $c = a^*$ and $\|a\| = \|c\| = 1$.

Proof We obtain from (2) and (3) that

$$\|c\Phi\|^2 + \|a\Phi\|^2 = \|\Phi\|^2, \quad \Phi \in \mathcal{D},$$

and hence c and a extend as bounded operators on $\mathcal{H}$ with $a = c^*$, $c = a^*$. Next we use

$$a^*aa^*a = a^*a - (a^*)^2a^2 = a^*a,$$

and hence $\|a\|^4 = \|a^*aa^*a\| = \|a^*a\| = \|a\|^2$. By $[a, a^*]_+ = \mathbb{1}$, $\|a\|$ cannot be 0. Therefore, $\|a\| = \|a^*\| = \mathbb{1}$. □

3.4.2 Creation and annihilation operators on Fock spaces

We consider the bosonic or fermionic Fock space $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$.

Definition 3.50 *Let $w \in \mathcal{Z}$. The* creation operator of w, *resp. the* annihilation operator of w, *are defined as operators on $\Gamma^{\mathrm{fin}}_{\mathrm{s/a}}(\mathcal{Z})$ by*

$$c(w)\Psi := \sqrt{n+1}w \otimes_{\mathrm{s/a}} \Psi,$$
$$a(w)\Psi := \sqrt{n}(w|\otimes \mathbb{1}_{\mathcal{Z}}^{\otimes(n-1)}\ \Psi, \quad \Psi \in \Gamma^n_{\mathrm{s/a}}(\mathcal{Z}).$$

Theorem 3.51 (Bosonic case) *In the bosonic case, the operators $c(w)$ and $a(w)$ are densely defined and closable. We denote their closures by the same symbols. They satisfy $a(w)^* = c(w)$. Therefore, we will write $a^*(w)$ instead of $c(w)$.*

(1) *The following quadratic form identities are valid:*

$$[a^*(w_1), a^*(w_2)] = [a(w_1), a(w_2)] = 0,$$
$$[a(w_1), a^*(w_2)] = (w_1|w_2)\mathbb{1}.$$

(2) *For $\Psi \in \Gamma_{\mathrm{s}}(\mathcal{Z})$, $w \in \mathcal{Z}$,*

$$\|a(w)\Psi\| \le \|w\|\|N^{\frac{1}{2}}\Psi\|, \quad \|a^*(w)\Psi\| \le \|w\|\|(N+\mathbb{1})^{\frac{1}{2}}\Psi\|.$$

Proof We apply Lemma 3.48 to $c(w)$, $a(w)$ with $\mathcal{D} = \Gamma^{\mathrm{fin}}_{\mathrm{s}}(\mathcal{Z})$ (without loss of generality we can assume that $\|w\| = 1$). Then $c(w)a(w) = \mathrm{d}\Gamma\big(|w)(w|\big)$, which is essentially self-adjoint on $\mathcal{D}$.

We have

$$a^*(w)a(w) = \mathrm{d}\Gamma\big(|w)(w|\big), \quad a(w)a^*(w) = \mathrm{d}\Gamma\left(|w)(w|\right) + \|w\|^2\mathbb{1}.$$

Using that $|w)(w| \leq \|w\|^2 \mathbb{1}$ on $\mathcal{Z}$, we get

$$\mathrm{d}\Gamma\big(|w)(w|\big) \leq \|w\|^2 N,$$

which implies (2). □

Theorem 3.52 (Fermionic case) *In the fermionic case, the operators $c(w)$ and $a(w)$ are densely defined and bounded. We denote their closures by the same symbols. They satisfy $a(w)^* = c(w)$. Therefore, we will write $a^*(w)$ instead of $c(w)$.*

(1) *The following operator identities are valid:*

$$[a^*(w_1), a^*(w_2)]_+ = [a(w_1), a(w_2)]_+ = 0,$$
$$[a(w_1), a^*(w_2)]_+ = (w_1|w_2)\mathbb{1}.$$

(2) $\|a(w)\| = \|a^*(w)\| = \|w\|$.

Proof We apply Lemma 3.49 to $c(w)$, $a(w)$ with $\mathcal{D} = \Gamma_{\mathrm{a}}^{\mathrm{fin}}(\mathcal{Z})$ (without loss of generality we can assume that $\|w\| = 1$). □

Proposition 3.53 *If $p \in B(\mathcal{Z}_1, \mathcal{Z}_2)$ and $h \in Cl(\mathcal{Z})$, one has*

(1) $a(w_2)\Gamma(p) = \Gamma(p)a(p^*w_2)$, $\Gamma(p)a^*(w_1) = a^*(pw_1)\Gamma(p)$,
(2) $[\mathrm{d}\Gamma(h), a(w)] = -a(h^*w)$, $[\mathrm{d}\Gamma(h), a^*(w)] = a^*(hw)$,

the last two identities being quadratic form identities on $\overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathrm{Dom}\, h)$.

For further reference we note the following obvious facts:

$$\big\{\Psi \in \Gamma_{\mathrm{s/a}}(\mathcal{Z}) \;:\; a(w)\Psi = 0, \quad w \in \mathcal{Z}\big\} = \mathbb{C}\Omega, \tag{3.25}$$

$$\mathrm{Span}^{\mathrm{cl}} \left\{ \prod_{i=0}^{n} a^*(w_i)\Omega, \quad w_1, \dots, w_n \in \mathcal{Z}, \; n = 0, 1, \dots \right\} = \Gamma_{\mathrm{s/a}}(\mathcal{Z}). \tag{3.26}$$

Remark 3.54 *The notation for creation and annihilation operators introduced in this section is typical for the mathematically oriented literature. In the physical literature it is common to assume that the one-particle space has a distinguished o.n. basis $\{e_j\}_{j \in J}$. One writes a_j^* and a_j instead of $a^*(e_j)$ and $a(e_j)$, $j \in J$.*

Clearly, every vector $w \in \mathcal{Z}$ can then be written as $\sum_{i \in J} w_j e_j$, and we have the following dictionary between "mathematician's" and "physicist's" notations:

$$a^*(w) = \sum_{j \in J} w_j a_j^*,$$
$$a(w) = \sum_{j \in J} \overline{w}_j a_j. \tag{3.27}$$

Note that the latter notation is heavier and depends on the choice of a basis, but has a useful advantage: it does not hide the anti-linearity of the annihilation operator.

Sometimes, instead of choosing an o.n. basis of $\mathcal{Z}$ it is more natural to assume that $\mathcal{Z} = L^2(\mathcal{Q}, \mathrm{d}q)$ for some measure space $(\mathcal{Q}, \mathrm{d}q)$. Clearly, $w \in \mathcal{Z}$ can be represented as a function $\mathcal{Q} \ni q \mapsto w(q)$. One introduces "operator-valued distributions" $\mathcal{Q} \ni q \mapsto a_q^*, a_q$, which are then "smeared out" with test functions to obtain creation and annihilation operators:

$$a^*(w) = \int w(q) a_q^* \mathrm{d}q,$$
$$a(w) = \int \overline{w(q)} a_q \mathrm{d}q. \tag{3.28}$$

(3.28) can be viewed as a generalization of (3.27).

The following operator seems to have no name, but is useful, especially on fermionic Fock spaces:

Definition 3.55 *Set*

$$\Lambda := (-1)^{N(N-\mathbb{1})/2}. \tag{3.29}$$

The following property is valid in both the bosonic and the fermionic case:

$$\begin{aligned} \Lambda a^*(z)\Lambda &= -I a^*(z) = a^*(z) I, \\ \Lambda a(z)\Lambda &= I a(z) = -a(z) I, \end{aligned} \tag{3.30}$$

where I denotes the parity operator. In the fermionic case, (3.30) allows the conversion of the anti-commutation relations into commutation relations:

$$[\Lambda a^*(z_1)\Lambda, a^*(z_2)] = [\Lambda a(z_1)\Lambda, a(z_2)] = 0,$$
$$[\Lambda a^*(z_1)\Lambda, a(z_2)] = I(z_2|z_1).$$

3.4.3 Exponential law for creation and annihilation operators

Let N_i, I_i, Λ_i be the operators on $\Gamma_{\mathrm{s/a}}(\mathcal{Z}_i)$ defined as in (3.9), (3.10) and (3.29). Recall that the unitary operator $U : \Gamma_{\mathrm{s/a}}(\mathcal{Z}_1) \otimes \Gamma_{\mathrm{s/a}}(\mathcal{Z}_2) \to \Gamma_{\mathrm{s/a}}(\mathcal{Z}_1 \oplus \mathcal{Z}_2)$ was defined in Thm. 3.43.

The exponential law for creation and annihilation operators is slightly different in the bosonic and fermionic cases:

Proposition 3.56 *Let $(w_1, w_2) \in \mathcal{Z}_1 \oplus \mathcal{Z}_2$.*

(1) *In the bosonic case we have*

$$a^*(w_1, w_2)U = U(a^*(w_1) \otimes \mathbb{1} + \mathbb{1} \otimes a^*(w_2)),$$
$$a(w_1, w_2)U = U(a(w_1) \otimes \mathbb{1} + \mathbb{1} \otimes a(w_2)).$$

(2) *In the fermionic case, we have*

$$a^*(w_1, w_2)U = U(a^*(w_1) \otimes \mathbb{1} + I_1 \otimes a^*(w_2)),$$
$$a(w_1, w_2)U = U(a(w_1) \otimes \mathbb{1} + I_1 \otimes a(w_2)).$$

Proposition 3.57 (1) $IU = UI_1 \otimes I_2$,
(2) $\Lambda U = U(\Lambda_1 \otimes \Lambda_2)(-1)^{N_1 \otimes N_2}$.
(3) *In the fermionic case,*

$$\Lambda a^*(w_1, w_2)\Lambda\ U = U(a^*(w_1)I_1 \otimes I_2 + \mathbb{1} \otimes a^*(w_2)I_2),$$

$$\Lambda a(w_1, w_2)\Lambda\ U = U(-a(w_1)I_1 \otimes I_2 - \mathbb{1} \otimes a(w_2)I_2).$$

Proof We use

$$NU = U(N_1 \otimes \mathbb{1} + \mathbb{1} \otimes N_2),$$

$$\frac{1}{2}N(N - \mathbb{1}) = \frac{1}{2}N_1(N_1 - \mathbb{1}) + \frac{1}{2}N_2(N_2 - \mathbb{1}) + N_1 N_2,$$

$$(-1)^{N_1 \otimes N_2}(a(w) \otimes \mathbb{1})(-1)^{N_1 \otimes N_2} = a(w) \otimes I_2.$$ □

3.4.4 Multiple creation and annihilation operators

Let $\Phi \in \Gamma^m_{\mathrm{s/a}}(\mathcal{Z})$.

Definition 3.58 *We define the* operator of creation of Φ *with the domain* $\Gamma^{\mathrm{fin}}_{\mathrm{s/a}}(\mathcal{Z})$ *as*

$$a^*(\Phi)\Psi := \sqrt{(n+1)\cdots(n+m)}\Phi \otimes_{\mathrm{s/a}} \Psi, \quad \Psi \in \Gamma^n_{\mathrm{s/a}}(\mathcal{Z}).$$

$a^*(\Phi)$ is a densely defined closable operator. We denote its closure by the same symbol.

Definition 3.59 *We set*

$$a(\Phi) := (a^*(\Phi))^*.$$

$a(\Phi)$ *is called the* operator of annihilation of Φ*.*

For $w_1, \dots, w_m \in \mathcal{Z}$ we have

$$a^*(w_1 \otimes_{\mathrm{s/a}} \cdots \otimes_{\mathrm{s/a}} w_m) = a^*(w_1)\cdots a^*(w_m),$$

$$a(w_1 \otimes_{\mathrm{s/a}} \cdots \otimes_{\mathrm{s/a}} w_m) = a(w_m)\cdots a(w_1).$$

Note that in the fermionic case we have

$$a(\Lambda\, w_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} w_m) = a(w_1)\cdots a(w_m),$$

where Λ was defined in (3.29).

3.5 Multi-linear symmetric and anti-symmetric forms

We continue to discuss symmetric and anti-symmetric tensors. In this section we will look at them mostly as multi-linear functions. This leads to somewhat different notational conventions.

Let $\mathcal{Y}$ be a real or complex vector space.

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3.5.1 Polynomials

Let $\Psi \in \overset{\mathrm{al}}{\Gamma}{}^n_{\mathrm{s/a}}(\mathcal{Y})$. Then Ψ determines the function

$$\begin{aligned} \mathcal{Y}^{\#} \times \cdots \times \mathcal{Y}^{\#} \ni (v_1, \dots, v_n) \mapsto \Psi(v_1, \dots, v_n) & \\ := \langle v_1 \otimes_{\mathrm{s/a}} \cdots \otimes_{\mathrm{s/a}} v_n | \Psi \rangle \in \mathbb{K}. & \end{aligned} \tag{3.31}$$

Definition 3.60 *The space* $\overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathcal{Y})$ *will often be denoted by* $\mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Y}^{\#})$*, if we want to stress the interpretation of its elements given by (3.31). (*Pol *stands for "poly-linear" or "a polynomial".) It will be called the symmetric, resp. anti-symmetric tensor algebra written in the* polynomial notation*.*

More generally, if $\mathcal{Y} = \mathcal{Y}_0 \oplus \mathcal{Y}_1$ *is a super-space, the super-tensor algebra* $\overset{\mathrm{al}}{\Gamma}_{\epsilon}(\mathcal{Y})$ *will be also sometimes denoted by* $\mathrm{Pol}_{\epsilon}(\mathcal{Y}^{\#})$*. Clearly,*

$$\mathrm{Pol}_{\epsilon}(\mathcal{Y}^{\#}) \simeq \mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}_0^{\#}) \overset{\mathrm{al}}{\otimes} \mathrm{Pol}_{\mathrm{a}}(\mathcal{Y}_1^{\#}). \tag{3.32}$$

Thus an element of $\mathrm{Pol}_{\epsilon}(\mathcal{Y}^{\#})$ *is a polynomial in commuting variables from* $\mathcal{Y}_0$ *and in anti-commuting variables from* $\mathcal{Y}_1$*.*

We will often drop the subscript ϵ *in (3.32).*

In the symmetric case we can make yet another identification. Let $\Psi = \sum_{n=0}^{\infty} \Psi_n$ with $\Psi_n \in \mathrm{Pol}_{\mathrm{s}}^n(\mathcal{Y}^{\#})$.

Definition 3.61 *We introduce the function called the* polynomial function associated with Ψ*:*

$$\mathcal{Y}^{\#} \ni v \mapsto \Psi(v) := \sum_{n=0}^{\infty} \langle v^{\otimes n} | \Psi_n \rangle. \tag{3.33}$$

Note that if we know the function (3.33), we have full knowledge of $\Psi \in \mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}^{\#})$.

In the following proposition $\Psi, \Phi \in \mathrm{Pol}_{\mathrm{s}}(\mathcal{Y}^{\#})$ are interpreted as polynomial functions and $v \in \mathcal{Y}^{\#}$:

Proposition 3.62 (1) $\Gamma(p)\Psi(v) = \Psi(p^{\#} v)$.
(2) $\Psi \otimes_{\mathrm{s}} \Phi(v) = \Psi(v)\Phi(v)$.

Motivated by Prop. 3.62, we will often replace $\Psi \otimes_{\mathrm{s}} \Phi$ with $\Psi \cdot \Phi$. We will often do the same in the anti-symmetric case as well.

In (3.31), $v_1, \dots, v_n$ are elements of $\mathcal{Y}^{\#}$. In (3.33) and in Prop. 3.62, v has the same meaning. Sometimes, however, we will write $\Psi(v)$ without having in mind a concrete $v \in \mathcal{Y}^{\#}$. We will treat the symbol v as the "generic variable in $\mathcal{Y}^{\#}$"; see Subsect. 2.1.2.

In the anti-symmetric case we do not have an analog of (3.33). Nevertheless, following the common usage of theoretical physics, one often calls elements of $\mathrm{Pol}_{\mathrm{a}}^n(\mathcal{Y}^{\#})$ "polynomials in non-commuting variables from $\mathcal{Y}^{\#}$". This suggests

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the notation $\Psi(v)$ instead of $\Psi(v_1, \dots, v_n)$. In this context v is just the name of the generic variable in $\mathcal{Y}^{\#}$. Similarly, motivated by Prop. 3.62 (1), we will write $\Psi(p^{\#} v)$ instead of $\Gamma(p)\Psi(v)$. This point of view will be further developed in Chap. 7.

3.5.2 Multiplication and differentiation operators

As mentioned above, we will use the letter v as the name of the generic variable in $\mathcal{Y}^{\#}$. This symbol will appear in the multiplication and derivative operators that we define below.

Definition 3.63 *For $y \in \mathcal{Y}$, the* operator of multiplication by y *is defined by*

$$y(v)\Psi := y \otimes_{\mathrm{s/a}} \Psi, \quad \Psi \in \mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Y}^{\#}).$$

We will often write $y{\cdot}v$ instead of $y(v)$.

More generally, if $\Phi \in \mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Y}^{\#})$, $\Phi(v)$ will denote the operator of multiplication by Φ*:*

$$\Phi(v)\Psi := \Phi \otimes_{\mathrm{s/a}} \Psi.$$

Definition 3.64 *For $w \in \mathcal{Y}^{\#}$, the* derivative in the direction of w *is defined by*

$$w(\nabla_v)\Psi := n\langle w| \otimes \mathbb{1}^{\otimes(n-1)}\Psi, \quad \Psi \in \mathrm{Pol}^n_{\mathrm{s/a}}(\mathcal{Y}^{\#}). \tag{3.34}$$

We will often write $w{\cdot}\nabla_v$ instead of $w(\nabla_v)$.

More generally, if $\Phi \in \mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Y})$, we define the derivative $\Phi(\nabla_v)$. *For $\Phi \in \mathrm{Pol}^m_{\mathrm{s/a}}(\mathcal{Y})$, it acts on $\Psi \in \mathrm{Pol}^n_{\mathrm{s/a}}(\mathcal{Y}^{\#})$ as*

$$\Phi(\nabla_v)\Psi = n(n-1)\cdots(n-m+1)\langle\Phi| \otimes \mathbb{1}^{\otimes(n-m)}\Psi. \tag{3.35}$$

Then we extend this definition by linearity.

Note that in the symmetric case the differentiation operator defined above is the usual differentiation of polynomials. In particular, $w(\nabla_v)$ in (3.34) coincides with the directional derivative Def. 2.50.

The operators of multiplication and differentiation are essentially equivalent to the creation and annihilation operators. We will discuss this equivalence in Subsect. 3.5.7.

In the following propositions $y, y_1, y_2 \in \mathcal{Y}$, $w, w_1, w_2 \in \mathcal{Y}^{\#}$ and $\Phi, \Psi \in \mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Y}^{\#})$.

Proposition 3.65 (Symmetric case)

(1) $[y_1(v), y_2(v)] = 0$, $[w_1(\nabla_v), w_2(\nabla_v)] = 0$,
(2) $[w(\nabla_v), y(v)] = \langle w|y\rangle \mathbb{1}$,
(3) $w(\nabla_v)\Psi \otimes_{\mathrm{s}} \Phi = (w(\nabla_v)\Psi) \otimes_{\mathrm{s}} \Phi + \Psi \otimes_{\mathrm{s}} (w(\nabla_v)\Phi)$,
(4) $w(\nabla_v) = \sum_{i=1}^{n} \mathbb{1}^{\otimes(i-1)} \otimes \langle w| \otimes \mathbb{1}^{\otimes(n-j)}$ *on* $\mathrm{Pol}^n_{\mathrm{s}}(\mathcal{Y}^{\#})$.

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Proposition 3.66 (Anti-symmetric case)

(1) $[y_1(v), y_2(v)]_+ = 0$, $[w_1(\nabla_v), w_2(\nabla_v)]_+ = 0$,
(2) $[w(\nabla_v), y(v)]_+ = \langle w|y\rangle \mathbb{1}$,
(3) $w(\nabla_v)\Psi \otimes_{\mathrm{a}} \Phi = (w(\nabla_v)\Psi) \otimes_{\mathrm{a}} \Phi + (I\Psi) \otimes_{\mathrm{a}} (w\nabla_v)\Phi)$,
(4) $w(\nabla_v) = \sum_{i=1}^{n} (-1)^{i-1} \mathbb{1}^{(i-1)\otimes} \otimes \langle w| \otimes \mathbb{1}^{(n-j)\otimes}$ *on* $\mathrm{Pol}_{\mathrm{a}}^{n}(\mathcal{Y}^{\#})$.

Proposition 3.67 *Let $p, h \in L(\mathcal{Y})$. In both the symmetric and the anti-symmetric case we have*

(1) $\Gamma(p)y(v) = py(v)\Gamma(p)$, $\quad w(\nabla_v)\Gamma(p) = \Gamma(p)p^{\#} w(\nabla_v)$,
(2) $[\mathrm{d}\Gamma(h), y(v)] = hy(v)$, $\quad [\mathrm{d}\Gamma(h), w(\nabla_v)] = -h^{\#} w(\nabla_v)$,
(3) $\Phi(\nabla_v)\Psi(\nabla_v) = (\Phi \otimes_{\mathrm{s/a}} \Psi)(\nabla_v)$.

$\Psi \in \mathrm{Pol}_{\mathrm{s/a}}^{n}(\mathcal{Y}^{\#})$ can be treated as an n-linear function on $(\mathcal{Y}^{\#})^n$. Let us denote the generic variable of the j-th $\mathcal{Y}^{\#}$ by v_j. We can write an identity

$$\nabla_v \Psi = (\nabla_{v_1} + \cdots + \nabla_{v_n})\,\Psi, \tag{3.36}$$

where on the left we use the functional notation, and on the right we treat Ψ as a function depending on n separate variables. (3.36) should be compared with (4) of Props. 3.65 and 3.66. Note that in the anti-symmetric case one has to remember that ∇_{v_i} anti-commutes with the operator of multiplication by v_j, hence the alternating sign.

3.5.3 Right derivative

Definition 3.68 *In the anti-symmetric case the derivative defined in Def. 3.64 should actually be called the* left derivative*. One can also introduce another operator with the name of the* right derivative*. For $w \in \mathcal{Y}^{\#}$, the* right derivative in the direction of w *acts on $\Psi \in \mathrm{Pol}_{\mathrm{a}}^{n}(\mathcal{Y}^{\#})$ as*

$$w(\overleftarrow{\nabla}_v)\Psi := n\mathbb{1}^{(n-1)\otimes}\otimes\langle w|\Psi.$$

More generally, if $\Phi \in \mathrm{Pol}_{\mathrm{a}}(\mathcal{Y})$, we can define the right derivative $\Phi(\overleftarrow{\nabla}_v)$. *For $\Phi \in \mathrm{Pol}_{\mathrm{a}}^{m}(\mathcal{Y})$ and $\Psi \in \mathrm{Pol}_{\mathrm{a}}^{n}(\mathcal{Y}^{\#})$, it is given by*

$$\Phi(\overleftarrow{\nabla}_v)\Psi = n(n-1)\cdots(n-m+1)\mathbb{1}^{(n-m)\otimes}\otimes\langle\Phi|\Psi. \tag{3.37}$$

Note that we need to invert the order (compare with Prop. 3.67 (3)):

$$\Phi_1(\overleftarrow{\nabla}_v)\Phi_2(\overleftarrow{\nabla}_v)\Psi = (\Phi_2 \otimes_{\mathrm{a}} \Phi_1)(\overleftarrow{\nabla}_v)\Psi. \tag{3.38}$$

Here is the relation between the left and right derivative:

$$\Phi(\nabla_v)\Psi = (-1)^{n(m-n)}\Phi(\overleftarrow{\nabla}_v)\Psi, \quad \Phi \in \mathrm{Pol}_{\mathrm{a}}^{m}(\mathcal{Y}), \Psi \in \mathrm{Pol}_{\mathrm{a}}^{n}(\mathcal{Y}^{\#}).$$

3.5.4 Exponential law in the polynomial notation

The exponential law described in Subsect. 3.3.7 is not the only convention used in the context of the tensor product of symmetric and anti-symmetric tensor algebras. In fact, there exists another convention that avoids the complicated multiplier involving the square roots of factorials. This convention is commonly used in the "algebraic case" (when we are not interested in the Hilbert space structure).

Let $\mathcal{Y}_1, \mathcal{Y}_2$ be two vector spaces. Let $j_i : \mathcal{Y}_i \to \mathcal{Y}_1 \oplus \mathcal{Y}_2$, $i = 1, 2$, be the canonical embeddings.

Definition 3.69

$$U^{\mathrm{mod}} : \mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Y}_1^{\#}) \overset{\mathrm{al}}{\otimes} \mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Y}_2^{\#}) \to \mathrm{Pol}_{\mathrm{s/a}}((\mathcal{Y}_1 \oplus \mathcal{Y}_2)^{\#})$$

is defined as the unique linear map such that if $\Psi_1 \in \mathrm{Pol}_{\mathrm{s/a}}^{n_1}(\mathcal{Y}_1^{\#})$, $\Psi_2 \in \mathrm{Pol}_{\mathrm{s/a}}^{n_2}(\mathcal{Y}_2^{\#})$, *then (in the tensor notation)*

$$U^{\mathrm{mod}}\Psi_1 \otimes \Psi_2 := (\Gamma(j_1)\Psi_1) \otimes_{\mathrm{s/a}} (\Gamma(j_2)\Psi_2). \tag{3.39}$$

We will use v_i as the generic variables in $\mathcal{Y}_i^{\#}$, $i = 1, 2$. In the "polynomial notation", $\Psi := U^{\mathrm{mod}}\Psi_1 \otimes \Psi_2$ will be simply written as

$$\Psi(v) = \Psi_1(v_1) \otimes_{\mathrm{s/a}} \Psi_2(v_2), \quad v = (v_1, v_2). \tag{3.40}$$

Often, we will even omit $\otimes_{\mathrm{s/a}}$ between the factors. Note that in the symmetric case, if we use the "polynomial interpretation", the exponential law is just the usual multiplication of polynomials in two separate variables, which is consistent with the notation (3.40).

Clearly, the identities (3.16) and (3.17) hold with U replaced with U^{mod}.

Proposition 3.70 (1) (Symmetric case)

$$\begin{aligned}(y_1, y_2)(v)U^{\mathrm{mod}} &= U^{\mathrm{mod}}(y_1(v_1) \otimes \mathbb{1} + \mathbb{1} \otimes y_2(v_2)),\\ (w_1, w_2)(\nabla_v)U^{\mathrm{mod}} &= U^{\mathrm{mod}}(w_1(\nabla_{v_1}) \otimes \mathbb{1} + \mathbb{1} \otimes w_2(\nabla_{v_2})).\end{aligned}$$

(2) (Anti-symmetric case)

$$\begin{aligned}(y_1, y_2)(v)U^{\mathrm{mod}} &= U^{\mathrm{mod}}(y_1(v_1) \otimes \mathbb{1} + I_1 \otimes y_2(v_2)),\\ (w_1, w_2)(\nabla_v)U^{\mathrm{mod}} &= U^{\mathrm{mod}}(w_1(\nabla_{v_1}) \otimes \mathbb{1} + I_1 \otimes w_2(\nabla_{v_2})).\end{aligned}$$

3.5.5 Holomorphic continuation of polynomials

Let $\mathcal{Y}$ be a real vector space. The identification (3.3) leads to the following isomorphism:

$$\mathbb{C}\Gamma_{\mathrm{s/a}}(\mathcal{Y}) \simeq \Gamma_{\mathrm{s/a}}(\mathbb{C}\mathcal{Y}). \tag{3.41}$$

In the polynomial notation this isomorphism is written as

$$\mathbb{C}\mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Y}^{\#}) \simeq \mathrm{Pol}_{\mathrm{s/a}}(\mathbb{C}\mathcal{Y}^{\#}).$$

Note that in the polynomial interpretation $\Psi \in \mathbb{C}\mathrm{Pol}_{\mathrm{s/a}}^{n}(\mathcal{Y}^{\#})$ is a complex multi-linear function on $\mathcal{Y}^{\#}$, whereas the corresponding $\Psi_{\mathbb{C}} \in \mathrm{Pol}_{\mathrm{s/a}}(\mathbb{C}\mathcal{Y}^{\#})$ is a multi-linear function on $\mathbb{C}\mathcal{Y}^{\#}$, which restricted to $\mathcal{Y}^{\#}$ equals Ψ.

Definition 3.71 *The polynomial* $\Psi_{\mathbb{C}}$ *will be called the* holomorphic extension of Ψ.

(Of course, instead of polynomials one can consider more general holomorphic functions.)

3.5.6 Polynomials on complex spaces

Let $\mathcal{Z}$ be a complex vector space. Recall that $\mathcal{Z}_{\mathbb{R}}$ denotes its realification. We can distinguish four basic families of polynomials related to $\mathcal{Z}$:

Definition 3.72 (1) *Elements of* $\mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Z}_{\mathbb{R}})$ *are called* real-valued polynomials.
(2) *Elements of* $\mathbb{C}\mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Z}_{\mathbb{R}})$ *are called* complex-valued polynomials.
(3) *Elements of* $\mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Z})$ *are called* holomorphic polynomials.
(4) *Elements of* $\mathrm{Pol}_{\mathrm{s/a}}(\overline{\mathcal{Z}})$ *are called* anti-holomorphic polynomials.

As sets, $\mathcal{Z}_{\mathbb{R}}$, $\mathcal{Z}$ and $\overline{\mathcal{Z}}$ can be identified. With these identifications, $\mathbb{C}\mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Z}_{\mathbb{R}})$ is the largest family – it contains the other three.

Let us use the notation and results from Subsect. 1.3.6. In particular, we recall the space $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) = \{(z, \overline{z}) \ : \ z \in \mathcal{Z}\}$, whose complexification can be identified with $\mathcal{Z} \oplus \overline{\mathcal{Z}}$. We have the obvious map (which according to Def. 1.84 is called T_1^{-1})

$$\mathcal{Z}_{\mathbb{R}} \ni z \mapsto (z, \overline{z}) \in \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}). \tag{3.42}$$

Its complexification is

$$\mathbb{C}\mathcal{Z}_{\mathbb{R}} \ni z_1 + \mathrm{i}z_2 \mapsto (z_1 + \mathrm{i}z_2, \overline{z}_1 + \mathrm{i}\overline{z}_2) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}. \tag{3.43}$$

With these identifications, we have

$$\begin{aligned} \mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Z}_{\mathbb{R}}) &\simeq \mathrm{Pol}_{\mathrm{s/a}}\left(\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})\right), \\ \mathbb{C}\mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Z}_{\mathbb{R}}) \simeq \mathrm{Pol}_{\mathrm{s/a}}(\mathbb{C}\mathcal{Z}_{\mathbb{R}}) &\simeq \mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}) \simeq \mathrm{Pol}_{\mathrm{s/a}}(\mathcal{Z}) \otimes \mathrm{Pol}_{\mathrm{s/a}}(\overline{\mathcal{Z}}). \end{aligned}$$

In the last line, we first used (3.41), then (3.43), and finally the exponential law.

In the symmetric case, the polynomial functions corresponding to all four cases of Def. 3.72 can be viewed as functions on the same space $\mathcal{Z}_{\mathbb{R}}$. By allowing convergent series, we can also consider more general functions, in particular *holomorphic* and *anti-holomorphic functions on* $\mathcal{Z}$, with obvious definitions.

3.5.7 Modified Fock spaces

Let $\mathcal{Z}$ be a Hilbert space. Recall that the scalar product in the Fock space $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$ is inherited from the scalar product in the tensor algebra $\otimes\mathcal{Z}$. This choice has some disadvantages. Instead, many authors adopt a different convention, which we will describe in this subsection.

Recall that N denotes the number operator on $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$.

Definition 3.73 *Let us set* $\Gamma^{\mathrm{mod}}_{\mathrm{s/a}}(\mathcal{Z}) := \mathrm{Dom}\sqrt{N!}$ *equipped with the scalar product* $(\Psi|\Phi)_{\mathrm{mod}} := (\Psi|N!\Phi)$. *We introduce also the unitary operator*

$$\Gamma_{\mathrm{s/a}}(\mathcal{Z}) \ni \Psi \mapsto T^{\mathrm{mod}}\Psi := \frac{1}{\sqrt{N!}}\Psi \in \Gamma^{\mathrm{mod}}_{\mathrm{s/a}}(\mathcal{Z}). \tag{3.44}$$

Sometimes we will write Ψ^{mod} *for* $T^{\mathrm{mod}}\Psi$.

The operators $\mathrm{d}\Gamma(h)$ and $\Gamma(p)$ keep the same form after conjugation by T^{mod}. If $\{e_i\}_{i\in I}$ is an o.n. basis of $\mathcal{Z}$, then

$$\left\{ \frac{1}{\sqrt{\vec{k}!}}e_{\vec{k}} \,:\, \vec{k} \in (\mathbb{N}^I)_{\mathrm{fin}} \right\}$$

is an o.n. basis of $\Gamma^{\mathrm{mod}}_{\mathrm{s}}(\mathcal{Z})$, and

$$\left\{ e_J \,:\, J \in 2^I_{\mathrm{fin}} \right\}$$

is an o.n. basis of $\Gamma^{\mathrm{mod}}_{\mathrm{a}}(\mathcal{Z})$, where $e_{\vec{k}}$ and e_J are defined in Subsects. 3.3.5 and 3.3.6.

Often we will consider the "polynomial notation" for $\overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathcal{Z})$, where $\mathcal{Z}$ is a Hilbert space. In this case, it is convenient to use elements of the topological dual of $\mathcal{Z}$, instead of the algebraic dual, as arguments of the polynomial. The topological dual of $\mathcal{Z}$ is identified with $\overline{\mathcal{Z}}$. Thus the polynomial notation for $\overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathcal{Z})$ will be $\mathrm{Pol}_{\mathrm{s/a}}(\overline{\mathcal{Z}})$. Clearly, $\mathrm{Pol}_{\mathrm{s/a}}(\overline{\mathcal{Z}})$ is dense in $\Gamma^{\mathrm{mod}}_{\mathrm{s/a}}(\mathcal{Z})$.

The generic variable of $\overline{\mathcal{Z}}$ will be often denoted $\overline{z}$. Thus an element of $\mathrm{Pol}_{\mathrm{s/a}}(\overline{\mathcal{Z}})$ in the polynomial notation will be written as $\Psi(\overline{z})$. If $w \in \mathcal{Z}$, then the corresponding multiplication and differentiation operators are $w(\overline{z})$ and $\overline{w}(\nabla_{\overline{z}})$. They are related to the creation and annihilation operators as

$$T^{\mathrm{mod}}a(w)(T^{\mathrm{mod}})^{-1} = \overline{w}(\nabla_{\overline{z}}), \quad T^{\mathrm{mod}}a^*(w)(T^{\mathrm{mod}})^{-1} = w(\overline{z}). \tag{3.45}$$

If $\mathcal{Z}_1, \mathcal{Z}_2$ are Hilbert spaces, then the map U^{mod} defined in Def. 3.69 extends to a unitary map from $\Gamma^{\mathrm{mod}}_{\mathrm{s/a}}(\mathcal{Z}_1) \otimes \Gamma^{\mathrm{mod}}_{\mathrm{s/a}}(\mathcal{Z}_2)$ to $\Gamma^{\mathrm{mod}}_{\mathrm{s/a}}(\mathcal{Z}_1 \oplus \mathcal{Z}_2)$. It is related to the map $U : \Gamma_{\mathrm{s/a}}(\mathcal{Z}_1) \otimes \Gamma_{\mathrm{s/a}}(\mathcal{Z}_2) \to \Gamma_{\mathrm{s/a}}(\mathcal{Z}_1 \oplus \mathcal{Z}_2)$ defined in Subsect. 3.3.7 by

$$U^{\mathrm{mod}} = T^{\mathrm{mod}}U(T^{\mathrm{mod}}_1 \otimes T^{\mathrm{mod}}_2)^{-1},$$

where $T^{\mathrm{mod}}, T^{\mathrm{mod}}_1, T^{\mathrm{mod}}_2$ are the unitary identifications of the corresponding Fock and modified Fock spaces; see (3.44).

3.6 Volume forms, determinant and Pfaffian

In this section we recall some well-known concepts related to anti-symmetric tensors, such as *volume forms*, the *determinant* of a matrix and the *Pfaffian* of an anti-symmetric matrix. They are usually introduced in a coordinate-dependent fashion. In our presentation, we try to stress the coordinate-independent approach based on the anti-symmetric tensor algebra.

3.6.1 Volume forms

Let $\mathcal{X}$ be a (real or complex) d-dimensional space. A special role is played by the space $\wedge^d \mathcal{X}^\#$ of anti-symmetric d-forms on $\mathcal{X}$, which is one-dimensional.

Definition 3.74 *A non-zero element of $\wedge^d \mathcal{X}^\#$ will be called a* volume form *on $\mathcal{X}$. If the name of the generic variable in $\mathcal{X}$ is x, then a volume form on $\mathcal{X}$ will be often denoted by $\mathrm{d}x$.*

Suppose that we choose a basis $(e_1, \dots, e_d)$ in $\mathcal{X}$. Let $(e^1, \dots, e^d)$ be the dual basis in $\mathcal{X}^\#$. Then we have a distinguished volume form on $\mathcal{X}$ defined by

$$\Xi = e^d \wedge \cdots \wedge e^1. \tag{3.46}$$

(Note the reverse order and the use of $\wedge$ and not of $\otimes_a$.) We have

$$\langle \Xi | e_1 \otimes_\mathrm{a} \cdots \otimes_\mathrm{a} e_d \rangle = 1.$$

Definition 3.75 *If $\mathcal{X}$ is a Euclidean, resp. unitary space of dimension d, then we say that a volume form Ξ is* compatible with its Euclidean, *resp.* unitary structure *if there exists an o.n. basis of $\mathcal{X}$ $(e_1, \dots, e_d)$ such that*

$$\Xi = e^d \wedge \cdots \wedge e^1.$$

If $\mathcal{X}_i$, $i = 1, 2$, are vector spaces with volume forms Ξ_i, then on $\mathcal{X}_1 \oplus \mathcal{X}_2$ we take the volume form $\Xi_2 \wedge \Xi_1$. (Note again the reverse order and the use of $\wedge$ and not of $\otimes_\mathrm{a}$.)

Definition 3.76 *If we use the notation $\mathrm{d}x_i$ for Ξ_i, we will often write $\mathrm{d}x_2 \mathrm{d}x_1$ for $\mathrm{d}x_2 \wedge \mathrm{d}x_1$.*

Definition 3.77 *If Ξ is a distinguished volume form on $\mathcal{X}$, then we have a distinguished volume form Ξ^{dual} on $\mathcal{X}^\#$ determined by*

$$\langle \Xi | \Xi^{\mathrm{dual}} \rangle = d!.$$

If in coordinates Ξ is given by (3.46), then

$$\Xi^{\mathrm{dual}} = e_1 \wedge \cdots \wedge e_d.$$

Note that $(\Xi^{\mathrm{dual}})^{\mathrm{dual}} = \Xi$. We will often use ξ as the generic variable of $\mathcal{X}^\#$, and then the dual volume form on $\mathcal{X}^\#$ will be denoted by $\mathrm{d}\xi$.

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3.6.2 Hodge star operator

Let $d = \dim \mathcal{X}$. Let us fix a volume form $\Xi \in \Gamma_{\mathrm{a}}^{d}(\mathcal{X}^{\#}) = \mathrm{Pol}_{\mathrm{a}}^{d}(\mathcal{X})$ on $\mathcal{X}$.

Definition 3.78 *The* Hodge star operator *is defined as the map*

$$\theta : \mathrm{Pol}_{\mathrm{a}}(\mathcal{X}^{\#}) \to \mathrm{Pol}_{\mathrm{a}}(\mathcal{X})$$

by

$$\langle \Phi | \theta \Psi \rangle := \frac{1}{(d-n)!} \langle \Psi \otimes_{\mathrm{a}} \Phi | \Xi \rangle, \quad \Psi \in \mathrm{Pol}_{\mathrm{a}}^{n}(\mathcal{X}^{\#}), \quad \Phi \in \mathrm{Pol}_{\mathrm{a}}^{d-n}(\mathcal{X}^{\#}).$$

Note that θ maps $\mathrm{Pol}_{a}^{n}(\mathcal{X}^{\#})$ onto $\mathrm{Pol}_{a}^{d-n}(\mathcal{X})$. We will see in Subsect. 7.1.7 that the Hodge star operator can be viewed as an analog of the *Fourier transformation.*

Let us fix a basis $(e_1, \dots, e_d)$ of $\mathcal{X}$ such that $\Xi = e^d \wedge \cdots \wedge e^1$. Let $\sigma \in S_d$ be a permutation and $0 \le n \le d$. Then

$$\theta\ e_{\sigma(1)} \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} e_{\sigma(n)} = \mathrm{sgn}(\sigma)\ e^{\sigma(d)} \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} e^{\sigma(n+1)}.$$

3.6.3 Liouville volume forms

Let $(\mathcal{Y}, \omega)$ be a symplectic space of dimension $2d$. Note that $\omega \in L_{\mathrm{a}}(\mathcal{Y}, \mathcal{Y}^{\#}) \simeq \Gamma_{\mathrm{a}}^{2}(\mathcal{Y}^{\#})$.

Definition 3.79 $\mathcal{Y}$ *possesses a distinguished volume form called the* Liouville form*,*

$$\Xi^{\mathrm{Liouv}} := \frac{1}{d!} \wedge^{d} \omega. \tag{3.47}$$

Recall that $\mathcal{Y}^{\#}$ is equipped with the symplectic form ω^{-1}. Thus it possesses its own Liouville form $\frac{1}{d!} \wedge^{d} \omega^{-1}$. It is easy to see that it equals the volume form dual to Ξ^{Liouv}.

3.6.4 Liouville volume forms on $\mathcal{X}^{\#} \oplus \mathcal{X}$

Assume that $\mathcal{X}$ is a vector space of dimension d. Consider $\mathcal{Y} = \mathcal{X}^{\#} \oplus \mathcal{X}$ with its canonical symplectic form (1.9). If we choose an arbitrary basis $e_1, \dots, e_d$ of $\mathcal{X}$ and $e^1, \dots, e^d$ is the dual basis, then one can use the wedge product to write the canonical symplectic form as

$$\omega = \sum_{i=1}^{d} e_i \wedge e^i. \tag{3.48}$$

Hence the Liouville form on $\mathcal{X}^{\#} \oplus \mathcal{X}$ is

$$e_1 \wedge e^1 \wedge \cdots e_d \wedge e^d = e_1 \wedge \cdots \wedge e_d \wedge e^d \wedge \cdots \wedge e^1. \tag{3.49}$$

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Proposition 3.80 *Choose an arbitrary volume form Ξ on $\mathcal{X}$. Then the Liouville volume form on $\mathcal{Y}$ is equal to $\Xi^{\mathrm{dual}} \wedge \Xi$.*

Proof We choose a basis of $\mathcal{X}$ and the dual basis of $\mathcal{X}^{\#}$ as above. Now for any volume form Ξ on $\mathcal{X}$ there exists $\lambda \neq 0$ such that

$$\Xi = \lambda e^d \wedge \cdots \wedge e^1, \qquad \Xi^{\mathrm{dual}} = \lambda^{-1} e_1 \wedge \cdots \wedge e_d. \qquad \square$$

The symplectic form ω^{-1} on $\mathcal{Y}^{\#} = \mathcal{X} \oplus \mathcal{X}^{\#}$ can be also written as (3.48). Hence the Liouville form on $\mathcal{Y}^{\#}$ can be written as (3.49).

Recall that often the symbols $\mathrm{d}x$ is used for a fixed volume form on $\mathcal{X}$, and its dual form on $\mathcal{X}^{\#}$ is denoted by $\mathrm{d}\xi$. Then the symbol $\mathrm{d}x\mathrm{d}\xi$ denotes the Liouville volume form on $\mathcal{X}^{\#} \oplus \mathcal{X}$ and on $\mathcal{X} \oplus \mathcal{X}^{\#}$.

3.6.5 Densities and Lebesgue measures

Let $\mathcal{X}$ be a real d-dimensional vector space.

Definition 3.81 *An element of $\wedge^d \mathcal{X}^{\#}/\{1, -1\}$ will be called a* density *on $\mathcal{X}$. The density associated with a volume form Ξ will be denoted by $|\Xi|$. Thus $|\Xi| = \{\Xi, -\Xi\}$. If $|\Xi|$ is a density on $\mathcal{X}$, we define the corresponding* dual density *on $\mathcal{X}^{\#}$ by $|\Xi|^{\mathrm{dual}} := |\Xi^{\mathrm{dual}}|$.*

Clearly, the set of volume forms compatible with a Euclidean structure is a density.

Recall from Def. 3.74 that if the generic variable of $\mathcal{X}$ is denoted x, then $\mathrm{d}x$ denotes a fixed volume form on $\mathcal{X}$. Thus, according to Def. 3.81, the corresponding density should be denoted by $|\mathrm{d}x|$.

Definition 3.82 *By a* Lebesgue measure on $\mathcal{X}$ *we mean a non-zero translation invariant Borel measure on $\mathcal{X}$ finite on compact sets.*

If $|\Xi|$ is a density on $\mathcal{X}$, then $|\Xi|$ induces a Lebesgue measure μ on $\mathcal{X}$ by setting

$$\mu\left(V(e_1, \dots, e_d)\right) := |\langle \Xi | e_1 \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} e_d \rangle|,$$

where $V(e_1, \dots, e_d) := \left\{\sum_{i=1}^d t_i e_i \ : \ t_i \in [0,1]\right\}$ is the parallelepiped with edges $e_1, \dots, e_d$. Conversely a Lebesgue measure on $\mathcal{X}$ yields a unique density on $\mathcal{X}$. Therefore, we will often identify the concepts of a Lebesgue measure and a density.

The integral w.r.t. a Lebesgue measure is called a *Lebesgue integral.* If F is a function on $\mathcal{X}$, its Lebesgue integral is denoted $\int F(x)\mathrm{d}x$ (although, according to Def. 3.81, the notation $\int F(x)|\mathrm{d}x|$ would be more appropriate, since a Lebesgue integral depends only on the density $|\mathrm{d}x|$, and not on the volume form $\mathrm{d}x$). For further reference let us list elementary properties of a Lebesgue integral:

$$\int \Phi(\nabla_x) f(x) \mathrm{d}x = 0, \qquad \Phi \in \mathbb{C}\mathrm{Pol}_{\mathrm{s}}^{\geq 1}(\mathcal{X}^{\#});$$
$$\int f(x+y)\mathrm{d}x = \int f(x)\mathrm{d}x, \quad y \in \mathcal{X};$$
$$\int f(mx)\mathrm{d}x = (\det m)^{-1} \int f(x)\mathrm{d}x, \quad m \in L(\mathcal{X}). \tag{3.50}$$

3.6.6 Determinants

Definition 3.83 *If $a = [a_{ij}]$ is a $d \times d$ matrix, one defines its* determinant *as*

$$\det(a) := \sum_{\sigma \in S_d} \mathrm{sgn}(\sigma) \prod_{i=1}^{d} a_{i\sigma(i)}.$$

It is possible to give a manifestly coordinate-independent definition of the determinant. Let $\mathcal{X}$ be a d-dimensional vector space over $\mathbb{K}$.

Definition 3.84 *For $a \in L(\mathcal{X})$, its* determinant *is defined as the unique number* $\det a$ *satisfying*

$$\Gamma(a)\big|_{\wedge^d \mathcal{X}} =: \det a \mathbb{1}.$$

Clearly, this definition is possible, because $\Gamma(a)$ sends $\wedge^d \mathcal{X}$ into itself. If $(e_1, \dots, e_d)$ is a basis of $\mathcal{X}$ and $(e^1, \dots, e^d)$ its dual basis, then $\det a$ coincides with the determinant of the matrix $[\langle e^i | a e_j \rangle]$.

Proposition 3.85 (1) *If $\mathcal{X}$ is real and $a \in L(\mathcal{X})$, then* $\det a = \det a_{\mathbb{C}}$.
(2) *If $a, b \in L(\mathcal{X})$, then* $\det ab = \det a \det b$.
(3) *If $a_i \in L(\mathcal{X}_i)$, $i = 1, 2$, then* $\det(a_1 \oplus a_2) = \det a_1 \det a_2$.
(4) *If $a \in L(\mathcal{X})$, then* $\det a = \det a^{\#}$.

3.6.7 Determinant of a bilinear form

Now let $\mathcal{X}$ be a finite-dimensional vector space equipped with a density $|\Xi|$.

Definition 3.86 *If $\zeta \in L(\mathcal{X}, \mathcal{X}^{\#})$, we define the* determinant of ζ w.r.t. the density $|\Xi|$ *as the unique number* $\det \zeta$ *satisfying*

$$\Gamma(\zeta^{\#})\Xi = \det \zeta \ \Xi^{\mathrm{dual}}. \tag{3.51}$$

(Note that the above definition does not depend on the sign of Ξ.)

If $(e_1, \dots, e_d)$ is a basis of $\mathcal{X}$ such that $\Xi = e^d \wedge \cdots \wedge e^1$, then $\det \zeta$ is equal to the determinant of the matrix $[\langle e_i, \zeta e_j \rangle]$.

If $|\Xi|$ is compatible with a Euclidean scalar product ν, then the determinant of ζ w.r.t. $|\Xi|$ is equal to the determinant of the operator $\nu^{-1}\zeta \in L(\mathcal{X})$.

3.6.8 Orientations of vector spaces

Let $\mathcal{X}$ be a finite-dimensional real vector space.

Definition 3.87 *Two bases of* $\mathcal{X}$ *are said to be* equivalent *if the determinant of the matrix of the change of basis is positive. An* orientation of $\mathcal{X}$ *is the choice of one of two equivalence classes of bases. Bases in this class are called* compatible with the orientation*. A space equipped with an orientation is called* oriented.

Sometimes it is useful to have the concept of an orientation also on a complex vector space. Its definition is identical to that on the real vector space. The only difference is that on a complex vector space the set of orientations is not made of two elements but is homeomorphic to a circle.

3.6.9 Volume forms on complex spaces

Let $\mathcal{Z}$ be a complex space of dimension d equipped with a complex volume form denoted by Ξ. On $\overline{\mathcal{Z}}$ we have the volume form $\overline{\Xi}$ defined by $\langle\overline{\Xi}|\overline{\Psi}\rangle = \overline{\langle\Xi|\Psi\rangle}$, $\Psi \in \Gamma_{\mathrm{a}}^{d}(\mathcal{Z})$. We will also need $\overline{\Xi}^{\mathrm{rev}} = (-1)^{\frac{1}{2}d(d-1)}\overline{\Xi}$. We will usually denote Ξ by $\mathrm{d}z$ and $\overline{\Xi}^{\mathrm{rev}}$ by $\mathrm{d}\overline{z}$. If $e^1, \dots, e^d$ is a basis of $\mathcal{Z}^{\#}$ and $\mathrm{d}z = e^d \wedge \cdots \wedge e^1$ then $\mathrm{d}\overline{z} = \overline{e}^1 \wedge \cdots \wedge \overline{e}^d$.

On $\overline{\mathcal{Z}} \oplus \mathcal{Z}$ we have a distinguished volume form given by

$$\mathrm{i}^{-d}\overline{\Xi}^{\mathrm{rev}} \wedge \Xi = \mathrm{i}^{-d}\mathrm{d}\overline{z} \wedge \mathrm{d}z. \tag{3.52}$$

We claim that the restriction of $\mathrm{i}^{-d}\mathrm{d}\overline{z} \wedge \mathrm{d}z$ to $\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$ is a real volume form. This can be seen by noting that the canonical conjugation

$$\overline{\mathcal{Z}} \oplus \mathcal{Z} \ni (\overline{z}_1, z_2) \mapsto \epsilon(\overline{z}_1, z_2) := (\overline{z}_2, z_1) \in \overline{\mathcal{Z}} \oplus \mathcal{Z}$$

fixes $\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$ and transforms $\mathrm{i}^{-d}\mathrm{d}\overline{z} \wedge \mathrm{d}z$ into its complex conjugate.

Recall that the realification of $\mathcal{Z}$ is denoted $\mathcal{Z}_{\mathbb{R}}$. It is a real $2d$-dimensional space. $\mathcal{Z}_{\mathbb{R}}$ has a distinguished real volume form obtained by pulling back $\mathrm{i}^{-d}\mathrm{d}\overline{z} \wedge \mathrm{d}z$ from $\mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$ to $\mathcal{Z}_{\mathbb{R}}$ by the transformation

$$\mathcal{Z}_{\mathbb{R}} \ni z \mapsto (z, \overline{z}) \in \mathrm{Re}(\mathcal{Z} \oplus \overline{\mathcal{Z}}), \tag{3.53}$$

(which we encountered before, e.g. in (3.42)). The Lebesgue measure obtained from this volume form will be adopted as the standard measure on $\mathcal{Z}$. Thus, a typical notation for the Lebesgue integral of a function $\mathcal{Z} \in z \mapsto F(z)$ will be

$$\int F(z)\mathrm{i}^{-d}\mathrm{d}\overline{z} \wedge \mathrm{d}z. \tag{3.54}$$

Let us give an argument for why $\mathrm{i}^{-d}\mathrm{d}\overline{z} \wedge \mathrm{d}z$ is a natural choice for the distinguished volume form on a complex vector space. Assume that $\mathcal{Z}$ is a unitary space and that $\mathrm{d}z$ is compatible with its structure (see Subsect. 3.6.5). We have seen in Subsect. 1.3.9 that $\mathrm{Re}(\overline{\mathcal{Z}} \oplus \mathcal{Z})$ is equipped with a Euclidean scalar product

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and with a symplectic form. We claim that $\mathrm{i}^{-d}\mathrm{d}\overline{z}\wedge\mathrm{d}z$ is compatible with these two structures. To see this, note that if $(e^1,\dots,e^d)$ is an o.n. basis in $\mathcal{Z}^\# \simeq \overline{\mathcal{Z}}$, then

$$\begin{aligned}\mathrm{i}^{-d}\mathrm{d}\overline{z}\mathrm{d}z &= \mathrm{i}^{-d}\overline{e}^1\wedge\cdots\wedge\overline{e}^d\wedge e^d\wedge\cdots\wedge e^1\\ &= \tfrac{e^1+\overline{e}^1}{\sqrt{2}}\wedge\tfrac{-\mathrm{i}e^1+\mathrm{i}\overline{e}^1}{\sqrt{2}}\wedge\cdots\wedge\tfrac{e^d+\overline{e}^d}{\sqrt{2}}\wedge\tfrac{-\mathrm{i}e^d+\mathrm{i}\overline{e}^d}{\sqrt{2}},\end{aligned}$$

and $(\frac{e^1+\overline{e}^1}{\sqrt{2}},\frac{-\mathrm{i}e^1+\mathrm{i}\overline{e}^1}{\sqrt{2}},\cdots,\frac{e^d+\overline{e}^d}{\sqrt{2}},\frac{-\mathrm{i}e^d+\mathrm{i}\overline{e}^d}{\sqrt{2}})$ is both an o.n. and a symplectic basis in $\mathrm{Re}(\overline{\mathcal{Z}}\oplus\mathcal{Z})$.

Remark 3.88 *The following remark may sound academic, but actually it is related to a true computational nuisance – factors of* $\sqrt{2}$ *in various formulas, which are often difficult to keep track of.*

We saw that the volume form $\mathrm{i}^{-d}\mathrm{d}\overline{z}\wedge\mathrm{d}z$ *is compatible with the natural Euclidean structure on* $\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$*. Its pullback to* $\mathcal{Z}_{\mathbb{R}}$*, however, is not compatible with the usual Euclidean structure on* $\mathcal{Z}_{\mathbb{R}}$*, that is, with the real scalar product* $\mathrm{Re}\,\overline{z}_1\cdot z_2$*. To see this, note that the map (3.53) is not orthogonal. (3.53) becomes orthogonal only after we multiply it by* $\frac{1}{\sqrt{2}}$*. A volume form compatible with the Euclidean structure of* $\mathcal{Z}_{\mathbb{R}}$ *is* $(2\mathrm{i})^{-d}\mathrm{d}\overline{z}\wedge\mathrm{d}z$*.*

One can say that when we consider integrals on $\mathcal{Z}$*, we actually view them as integrals on* $\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$*, where the integrand has been pulled back from* $\mathcal{Z}$ *onto* $\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ *by (3.53). Therefore, when normalizing the Lebesgue measure in (3.54), we prefer the convention adapted to* $\mathrm{Re}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$ *rather than to* $\mathcal{Z}$*.*

3.6.10 Pfaffians

Definition 3.89 *Let* $d\in\mathbb{N}$*. We denote by* Pair_{2d} *the set of* pairings *of* $\{1,\dots,2d\}$*, i.e. the set of partitions of* $\{1,\dots,2d\}$ *into pairs.*

A pairing can be uniquely written as

$$\big((i_1,j_1),(i_2,j_2),\dots,(i_d,j_d)\big),$$

where $i_k<j_k$ and $i_1<i_2<\cdots<i_d$, and we can identify Pair_{2d} with the subset of permutations

$$\mathrm{Pair}_{2d}=\big\{\sigma\in S_{2d}\ :\ \sigma(2i-1)<\sigma(2i),\quad \sigma(2i-1)<\sigma(2i+1),\quad 1\le i\le d\big\}.$$

It is easy to see that Pair_{2d} has $\frac{(2d)!}{d!2^d}$ elements.

Definition 3.90 *If* $\zeta=[\zeta_{ij}]$ *is a* $2d\times 2d$ *anti-symmetric matrix, one defines its* Pfaffian *by*

$$\mathrm{Pf}(\zeta):=\sum_{\sigma\in\mathrm{Pair}_{2d}}\mathrm{sgn}(\sigma)\prod_{i=1}^{d}\zeta_{\sigma(2i-1),\sigma(2i)}.$$

It is possible to give a manifestly coordinate-independent definition of the Pfaffian. Now let $\mathcal{Y}$ be a (real or complex) vector space of dimension $2d$, equipped with the volume form Ξ.

Definition 3.91 *For* $\zeta \in L_{\mathrm{a}}(\mathcal{Y}^{\#}, \mathcal{Y}) \simeq \Gamma_{\mathrm{a}}^{2}(\mathcal{Y})$*, its* Pfaffian w.r.t. Ξ *is defined by*

$$\mathrm{Pf}(\zeta) := \frac{1}{2^d d!}\langle \otimes_{\mathrm{a}}^{d} \zeta | \Xi \rangle. \tag{3.55}$$

An alternative definition is

$$\wedge^d \zeta =: d! \mathrm{Pf}(\zeta) \Xi^{\mathrm{dual}}. \tag{3.56}$$

If $(e^1, \dots, e^{2d})$ is a basis of $\mathcal{Y}^{\#}$ such that $\Xi = e^{2d} \wedge \cdots \wedge e^1$, then $\mathrm{Pf}(\zeta)$ coincides with the Pfaffian of the matrix $[\langle e^i | \zeta e^j \rangle]$.

Proposition 3.92 (1) *If* $\zeta \in L_{\mathrm{a}}(\mathcal{Y}^{\#}, \mathcal{Y})$*,* $r \in L(\mathcal{Y})$*, then*

$$\mathrm{Pf}(r \zeta r^{\#}) = \mathrm{Pf}(\zeta) \det r.$$

(2) *Let* $\zeta_i \in L_{\mathrm{a}}(\mathcal{Y}_i^{\#}, \mathcal{Y}_i)$*,* $i = 1, 2$*. Then*

$$\mathrm{Pf}\left(\begin{bmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{bmatrix}\right) = \mathrm{Pf}(\zeta_1)\mathrm{Pf}(\zeta_2),$$

where the Pfaffian on the l.h.s. is computed w.r.t. $\Xi_1 \wedge \Xi_2$*.*

(3) *For* $\zeta \in L_{\mathrm{a}}(\mathcal{Y}^{\#}, \mathcal{Y})$*, one has*

$$\mathrm{Pf}(\zeta)^2 = \det \zeta,$$

where $\det a$ *is computed w.r.t. the density* $|\Xi^{\mathrm{dual}}|$*.*

(4) *Let* $\mathcal{X}$ *be a finite-dimensional vector space equipped with a volume form* Ξ *and let us equip* $\mathcal{Y} = \mathcal{X}^{\#} \oplus \mathcal{X}$ *with the volume form* $\Xi^{\mathrm{dual}} \wedge \Xi$*. Let* $a \in L(\mathcal{X})$*, so that* $\begin{bmatrix} 0 & a^{\#} \\ -a & 0 \end{bmatrix} \in L_{\mathrm{a}}(\mathcal{Y}^{\#}, \mathcal{Y})$*. Then*

$$\mathrm{Pf}\left(\begin{bmatrix} 0 & a^{\#} \\ -a & 0 \end{bmatrix}\right) = \det a.$$

3.7 Notes

The tensor product of Hilbert spaces is studied e.g. in the monograph by Reed–Simon (1980). The notions of Fock spaces and second quantization were originally introduced by Fock (1932). Mathematical expressions can be found e.g. in Reed–Simon (1980), Simon (1974), Bratteli–Robinson (1996) and Glimm–Jaffe (1987).