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RIEMANNIAN MANIFOLDS WHOSE CURVATURE OPERATOR $R(X,Y)$ HAS CONSTANT EIGENVALUES

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A Riemannian manifold M^n is called IP, if, at every point $x \in M^n$, the eigenvalues of its skew-symmetric curvature operator $R(X,Y)$ are the same, for every pair of orthonormal vectors $X, Y \in T_xM^n$. In [**5, 6, 12**] it was shown that for all $n \geqslant 4$, except $n = 7$, an IP manifold either has constant curvature, or is a warped product, with some specific function, of an interval and a space of constant curvature. We prove that the same result is still valid in the last remaining case $n = 7$, and also study 3-dimensional IP manifolds.

1. INTRODUCTION

An *algebraic curvature tensor* R in a Euclidean space $\mathbb{R}^n$ is a $(3,1)$ tensor having the same symmetries as the curvature tensor of a Riemannian manifold. Given an algebraic curvature tensor R, there is defined a quadrilinear functional on $\mathbb{R}^n$ by $R(X,Y,Z,W) = \langle R(X,Y)W, Z\rangle$. For any pair of vectors $X, Y \in \mathbb{R}^n$, $R(X,Y)$ is a skew-symmetric endomorphism of $\mathbb{R}^n$. One has $R(Y,X) = -R(X,Y)$, and, in particular, $R(X,Y) = 0$ when $X \parallel Y$. For any oriented two-plane $\pi \in G^+(2,n)$, there is a well-defined endomorphism $R(\pi)$ of $\mathbb{R}^n$, $R(\pi) = \|X \wedge Y\|^{-1}R(X,Y)$, where (X,Y) is any oriented pair of vectors spanning π, and $\|X \wedge Y\| = \left(\|X\|^2\|Y\|^2 - \langle X,Y\rangle^2\right)^{1/2}$.

DEFINITION. An algebraic curvature tensor R is called *IP*, if the eigenvalues of $R(\pi)$ are the same for all $\pi \in G^+(2,n)$. A Riemannian manifold M^n is called *IP*, if its curvature tensor at every point is IP (the eigenvalues may depend on a point).

For an IP algebraic curvature tensor R, its *rank* is the rank of any of the $R(\pi)$'s.

EXAMPLE 1. Any Riemannian manifold of constant curvature C is IP. Its curvature tensor R^C has rank 2 when $C \neq 0$.

EXAMPLE 2. ([**5**].) Let ϕ be a linear isometry of $\mathbb{R}^n$ with $\phi^2 = \mathrm{id}$ (all the eigenvalues of such a ϕ must be ± 1), and let $C \neq 0$. Then an algebraic curvature tensor R^C_ϕ defined by $R^C_\phi(X,Y) = R^C(\phi X, \phi Y)$ is IP, and $\mathrm{rk}\, R^C_\phi = 2$.

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EXAMPLE 3. ([**5, 12**].) A Riemannian manifold M^n with a metric of a warped product

$$ds^2 = dt^2 + f(t)\, ds_K^2, \tag{1}$$

where ds_K^2 is a metric of constant curvature K and $f(t) = Kt^2 + At + B > 0$, is IP. Its curvature tensor has the form $R_\phi^{C(t)}$, with $C(t) = (4KB - A^2)/(4f(t)^2)$. For every point $x \in M^n$, ϕ is a reflection of the tangent space T_xM^n in the hyperplane orthogonal to $\partial/\partial t$.

In Example 3, all but one eigenvalues of ϕ are $+1$. Clearly, if all the eigenvalues of ϕ are the same ($\phi = \pm\mathrm{id}$), the resulting algebraic curvature tensor (or manifold) has constant curvature. On the other hand, no IP curvature tensors R_ϕ^C of Example 2, with ϕ having more than one eigenvalue $+1$ and more than one eigenvalue -1, can locally be the curvature tensor of a Riemannian manifold [**5**].

Note that the metric (1) is not of constant curvature, unless $4KB - A^2 = 0$, but is conformally flat.

The IP manifolds were introduced and classified in dimension 4 by Ivanov, Petrova [**12**] (hence the name). Shortly after, in [**5**], Gilkey, Leahy and Sadofsky using powerful topological methods classified all the IP algebraic curvature tensors and manifolds of dimensions $n \geqslant 9$ and $n = 5, 6$. Later, in [**6**], Gilkey extended the result of [**5**] to $n = 8$, and gave a detailed description of all possible eigenvalue structures of $R(\pi)$ when $n = 7$. The case $n = 7$ was further studied in [**7**] using spinors.

In this paper, we complete the case $n = 7$:

THEOREM. *Any nonzero IP algebraic curvature tensor in $\mathbb{R}^7$ has rank* 2.

This, combined with the results of [**5, 6, 12**], gives the following classification:

COROLLARY.

1. *Any nonzero IP algebraic curvature tensor in $\mathbb{R}^n$, $n \neq 4$, has rank* 2 *and is of the form R_ϕ^C of Example 2.*
2. *Any Riemannian IP manifold M^n, $n \geqslant 4$, is either of constant curvature, or is locally isometric to the warped product* (1).

Note that the case $n = 2$ is of no interest: any algebraic curvature tensor (any Riemannian manifold) is IP. In dimension 3, IP algebraic curvature tensors can be easily classified (see [**12**, Remark 1]): they are either of constant curvature, or those whose Ricci tensor has rank 1 (this fits the construction of Example 2). However, the class of IP Riemannian manifolds of dimension 3 with such a Ricci tensor is much wider than in Example 3 (see Section 3 for discussion and some examples). In dimension 4, there exist IP algebraic curvature tensors of rank four (see [**12**] for classification, and [**7**] for construction using spinors), but only those of rank 2 can be realised as the curvature tensors of 4-dimensional manifolds.

The IP algebraic curvature tensors were also extensively studied in pseudoriemannian and in complex settings. We refer to [**8, 9, 10, 11**] for results in these directions.

The proof of the Theorem is given in Section 2. In Section 3, we study three-dimensional IP manifolds.

2. Proof of the Theorem

Let R be an IP algebraic curvature tensor in $\mathbb{R}^7$, whose rank is bigger than 2. For every two-plane $\pi \in G^+(2,7)$, the symmetric operator $R(\pi)^2$ has an odd-dimensional kernel and some negative eigenvalues, $-\lambda_j^2$, each of an even multiplicity $n_j, j = 1, \dots, p$. Let $E_j(\pi)$ be the corresponding eigenspaces, with $\dim E_j(\pi) = n_j$. Label the n_j's in a non-decreasing order and call the ordered set $(n_0 = \dim \operatorname{Ker} R(\pi), n_1, \dots, n_p)$ the *eigenvalue structure* for R. Then, according to [**6**, Theorem 0.4, 1a), (3)], one has only two possibilities:

(a) the kernel is one-dimensional and $n_1 = 2$, $n_2 = 4$: the eigenvalue structure $(1,2,4)$;

(b) the kernel is three-dimensional and $n_1 = 4$: the eigenvalue structure $(3,4)$.

We want to show that no IP algebraic curvature tensor with such eigenvalue structures can exist. We shall assume, in the both cases, that the eigenvalue of $R(\pi)^2$ of multiplicity 4 is -1. In an appropriate orthonormal basis for $\mathbb{R}^7$, the matrix of the operator $R(\pi)$ has the following normal form, respectively:

$$\text{(2)} \qquad \text{(a)} \begin{pmatrix} \mathcal{J} & 0 & 0 \\ 0 & \alpha J & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\text{(b)} \begin{pmatrix} \mathcal{J} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where } \mathcal{J} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}, \ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \alpha \neq 0, \pm 1.$$

For arbitrary X, Y, the normal form of the matrix of $R(X,Y)$ is the one above multiplied by $\|X \wedge Y\|$.

The proof goes as follows. We start with the case (a) (Section 2.1). The first step (Lemma 1) is to show that the kernel of $R(X,Y)$ is spanned by a vector depending linearly on X and Y. Next, in Lemma 2, we prove that for any nonzero vector X, the set $\mathfrak{K}_0(X) = \bigcup_{\pi \ni X} \operatorname{Ker} R(\pi)$ is a linear space of dimension six. What is more, there exists an orthogonal operator U in $\mathbb{R}^7$ such that UX is a normal vector to $\mathfrak{K}_0(X)$, for all $X \neq 0$. The key step in the proof is Lemma 3 saying that, for any nonzero X and any two-plane $\pi \ni X$, the two-dimensional eigenspace $E_1(\pi)$ of $R(\pi)$ contains the vector UX. It follows that $E_1(\pi) = U(\pi)$. We then show that U is symmetric, and that the tensor R splits on two: $R_U^{\pm\alpha}$ (as in Example 2), and the remaining part, which is an IP algebraic curvature tensor with the eigenvalue structure $(3,4)$, hence reducing (a) to (b).

The case (b) is done in Section 2.2 by a brute force of matrix algebra, using the fact that for all X, Y, the operator $R(X,Y)$ satisfies $R(X,Y)^3 + \|X \wedge Y\|^2 R(X,Y) = 0$, which follows from (2).

2.1. CASE (A), THE EIGENVALUE STRUCTURE $(1,2,4)$. We start with a brief introduction from commutative algebra. Let $\mathbf{D}$ be an integral domain (an associative commutative ring with a 1 and without zero divisors). A noninvertible element $p \in \mathbf{D}$ is prime, if it generates a prime ideal ($p \mid ab \implies p \mid a$ or $p \mid b$), and is irreducible, if $p = ab$ implies that either a or b is invertible. The domain $\mathbf{D}$ is a *unique factorisation domain*, if all irreducibles are primes and every element of $\mathbf{D}$ is a finite product of irreducibles. In a unique factorisation domain, every element a can be represented in the form $a = u \prod_i p_i^{m_i}$, with p_i primes, $p_i \nmid p_j$, and u invertible, and such a representation is unique up to invertible elements. In particular, in a unique factorisation domain, there defined (up to invertibles) the greatest common divisor of a finite set of elements. Also, for any four elements $a_{11}, a_{12}, a_{21}, a_{22}$ satisfying $a_{11}a_{22} = a_{12}a_{21}$, there exist b_1, b_2, c_1, c_2 such that $a_{ij} = b_i c_j$. Inductively, this implies the following fact ($\mathbf{D}^n$ is a free module of rank n over $\mathbf{D}$):

FACT 1. Let W be $n \times n$ matrix of rank 1 *(*all the 2×2 minors vanish*)* over a *unique factorisation domain* $\mathbf{D}$. Then there exist $a, b \in \mathbf{D}^n$ such that $W = ab^t$. If, in addition, W is symmetric, then there exist $a \in \mathbf{D}^n$, $r \in \mathbf{D}$ such that $W = raa^t$.

We shall use the fact that a polynomial ring over reals is a unique factorisation domain and the Nagata Theorem[**18**]:

FACT 2. The ring $\mathbf{R} = \mathbb{R}[x_1, \dots, x_n] \Big/ \left(\sum_{i=1}^{n} x_i^2 \right)$ is a *unique factorisation domain*, when $n \geqslant 5$.

Back to IP algebraic curvature tensors, we start by proving that the kernel of $R(X,Y)$ depends linearly on X and Y. More precisely:

LEMMA 1. *There exists a bilinear skew-symmetric map $B : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ such that*

$$\|B(X,Y)\|^2 = \|X \wedge Y\|^2, \qquad \text{for all } X, Y \in \mathbb{R}^7 \tag{3}$$

$$\operatorname{Ker} R(X,Y) = \operatorname{Span}\big(B(X,Y)\big), \qquad \text{for all } X \nparallel Y \in \mathbb{R}^7. \tag{4}$$

PROOF: For every pair of vectors $X, Y \in \mathbb{R}^7$, define a symmetric operator $W(X,Y) : \mathbb{R}^7 \to \mathbb{R}^7$ by

$$W(X,Y) = \big(R(X,Y)^2 + \|X \wedge Y\|^2\big)\big(R(X,Y)^2 + \alpha^2 \|X \wedge Y\|^2\big).$$

For arbitrary nonparallel X, Y, the operator $W(X,Y)$ has rank 1, with a nonzero eigenvalue $\alpha^2 \|X \wedge Y\|^4$ (this follows from (2)). The corresponding eigenvector spans $\operatorname{Ker} R(X,Y)$.

The matrix of $W(X,Y)$ can be viewed as a matrix over the ring $\mathbf{K} = \mathbb{R}[x_1, \dots, x_7, y_1, \dots, y_7,]$ of polynomials in 14 variables, the coordinates of X and Y (all its entries are homogeneous polynomials of degree 4). By Fact 1, there exist a polynomial $f(X,Y)$ and a 7-vector $P(X,Y)$ with polynomial components such that $W(X,Y) = f(X,Y)P(X,Y)P(X,Y)^t$. For any nonparallel X and Y, the vector $P(X,Y)$ spans Ker $R(X,Y)$. As Tr $W(X,Y) = \alpha^2\|X \wedge Y\|^4$ and the polynomial $\|X \wedge Y\|^2$ is irreducible in $\mathbf{K}$, we have two possibilities for f (up to multiplication by a positive constant): either $f = 1$, or $f = \|X \wedge Y\|^2$ (the case $f = \|X \wedge Y\|^4$ is not possible, as then the vector $P(X,Y)$, which spans Ker $R(X,Y)$, is constant. But if $Z \in \mathbb{R}^7$ is in the kernel of all the $R(X,Y)$'s, then $R(Z,\cdot)$ is zero).

We want to show that the case $f = 1$ leads to a contradiction. Assume $f = 1$, hence $W(X,Y) = P(X,Y)P(X,Y)^t$ for a polynomial vector $P \in \mathbf{K}^7$. As the entries of W are homogeneous in X, of degree 4, and homogeneous in Y, of degree 4, the P_i's, the components of P, must be polynomials homogeneous in X, of degree 2, and homogeneous in Y, of degree 2 (each component of P is a linear combination of terms $x_i x_j y_k y_l$). We also have

$$\sum_{i=1}^{7} P_i^2(X,Y) = \text{Tr } W(X,Y) = \alpha^2\|X \wedge Y\|^4.$$

For every nonzero $X \in \mathbb{R}^7$, define the the subset $\mathfrak{K}_0(X) \subset \mathbb{R}^7$ as follows:

$$\mathfrak{K}_0(X) = \bigcup_{Y \nparallel X} \text{Ker } R(X,Y) = \bigcup_{\pi \ni X} \text{Ker } R(\pi) = \Big\{\text{Span}\big(P(X,Y)\big) : Y \perp X,\ \|Y\| = 1\Big\}.$$

The set $\mathfrak{K}_0(X)$ is a cone over the image of the sphere $S^5 \subset \mathbb{R}^6 = X^\perp$ under the polynomial map, hence its complement $\mathbb{R}^7 \setminus \mathfrak{K}_0(X)$ is open and dense. It follows that the set of pairs $(X,Z) \in \mathbb{R}^7 \times \mathbb{R}^7$ such that $Z \notin \mathfrak{K}_0(X)$, $X \notin \mathfrak{K}_0(Z)$ is nonempty (even dense). Let (X,Z) be one such pair, and S^5 be the unit sphere in $X^\perp$. Consider a map $V : S^5 \to X^\perp$ defined by $V(Y) = R\big(Z, P(X,Y)\big)X$. We have $\langle Y, V(Y)\rangle = R\big(Z, P(X,Y), Y, X\big) = 0$, as $P(X,Y)$ is in the kernel of $R(X,Y)$. Furthermore, Range$(V) \not\ni 0$. Indeed, if $V(Y_0) = 0$ for some $Y_0 \in S^5$, then for all $T \in \mathbb{R}^7$, $0 = R(Z, P(X,Y_0), X, T)$. As $P(X,Y_0) \neq 0$ (since $\|P(X,Y_0)\|^2 = \alpha^2\|X \wedge Y_0\|^4 = \alpha^2\|X\|^4$), and $P(X,Y_0) \nparallel Z$ (since $Z \notin \mathfrak{K}_0(X)$), this implies that $X \in \mathfrak{K}_0(Z)$, which contradicts the choice of the pair (X,Z). Now the map $\widehat{V} : S^5 \to S^5$ defined by $\widehat{V}(Y) = V(Y)/\|V(Y)\|$ is even ($\widehat{V}(-Y) = \widehat{V}(Y)$), as such is $P(X,Y)$, and $\widehat{V}(Y) \perp Y$. This is not possible, since otherwise the homotopy $Y\cos t + \widehat{V}(Y)\sin t$ joins the identity map of S^5 with the one of an even degree.

It follows that $f = \|X \wedge Y\|^2$, and so

$$W(X,Y) = \|X \wedge Y\|^2 P(X,Y)P(X,Y)^t.$$

Comparing the degrees, we find that all the components of the polynomial vector $P(X,Y)$ are linear in X and in Y, so each $P_i(X,Y)$ is a bilinear form on $\mathbb{R}^7$. Also,

$$\sum_{i=1}^{7} P_i^2(X,Y) = \|X \wedge Y\|^{-2} \operatorname{Tr} W(X,Y) = \alpha^2 \|X \wedge Y\|^2,$$

which implies $P(X,X) = 0$, so P is skew-symmetric. Finally, for $X \nparallel Y$, $P(X,Y)$ is a nonzero eigenvector of $W(X,Y)$, hence it spans the kernel of $R(X,Y)$.

Now define the map B by setting $B(X,Y) = \alpha^{-1} P(X,Y)$. ∎

Lemma 2.

(1) *For $X \neq 0$, the set $\mathfrak{K}_0(X) = \bigcup_{\pi \ni X} \operatorname{Ker} R(\pi)$ is a six-dimensional subspace of $\mathbb{R}^7$.*

(2) *There exists an orthogonal operator U on $\mathbb{R}^7$ such that for all $X \neq 0$, the vector UX is orthogonal to $\mathfrak{K}_0(X)$, or equivalently,*

$$UX \perp \operatorname{Ker} R(X,Y) \quad \text{for all } Y \nparallel X. \tag{5}$$

Proof: Let $\mathbb{R}^8$ be an orthogonal sum of $\mathbb{R}e_0$ and $\mathbb{R}^7$, with $p : \mathbb{R}^8 \to \mathbb{R}^7$ the orthogonal projection. Define a bilinear map $\overline{B} : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^8$ as follows:

$$\overline{B}(X,Y) = \langle X,Y \rangle e_0 + B(X,Y),$$

where B is the map from Lemma 1. Then for all $X, Y \in \mathbb{R}^7$ and all $Y_1 \perp Y_2$,

$$\|\overline{B}(X,Y)\|^2 = \|X\|^2 \|Y\|^2, \quad \langle \overline{B}(X,Y_1), \overline{B}(X,Y_2) \rangle = 0, \tag{6}$$

(the first equation follows from (3), the second one follows from the first one), so $\overline{B}$ is a normed bilinear map. For every $X \in \mathbb{R}^7$, define an operator $A_X : \mathbb{R}^8 \to \mathbb{R}^7$ by

$$\langle A_X Z, Y \rangle = \langle \overline{B}(X,Y), Z \rangle,$$

where $Z \in \mathbb{R}^8$, $Y \in \mathbb{R}^7$. Then from (6), for all X and all $X_1 \perp X_2$,

$$A_X A_X^t = \|X\|^2 \operatorname{id}_{\mathbb{R}^7}, \quad A_{X_1} A_{X_2}^t \in \mathfrak{o}(7), \tag{7}$$

where $\mathfrak{o}(7)$ is the linear space of skew-symmetric operators in $\mathbb{R}^7$. In particular, from the first equation, $\operatorname{rk} A_X = 7$ when $X \neq 0$. This proves assertion 1 of the Lemma. Indeed, by Lemma 1, $\mathfrak{K}_0(X) = \bigcup_{Y \nparallel X} \operatorname{Ker} R(X,Y) = \operatorname{Range} B(X,\cdot)$, which is a linear subspace of $\mathbb{R}^7$, of dimension at most 6 (as B is skew-symmetric). If a nonzero vector $Z' \in \mathbb{R}^7$ is orthogonal to this subspace, then the vector $Z = 0e_0 + Z' \in \mathbb{R}^8$ is in the kernel of A_X, which is of dimension 1. Thus $\dim \mathfrak{K}_0(X) = 6$.

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Fix a unit vector $X_0 \in \mathbb{R}^7$. The kernel of the operator $A_0 = A_{X_0}$ is one-dimensional. Let Z_0 be a unit vector in this kernel. For a nonzero vector $X \perp X_0$, let $Y \in \mathbb{R}^7$ be a nonzero vector from the kernel of the skew-symmetric operator $A_0 A_X^t$, so that $A_0 A_X^t Y = 0$. As $\operatorname{rk} A_0 = 7$, with $\operatorname{Ker} A_0$ spanned by Z_0, it follows that $A_X^t Y$ is parallel to Z_0. Since $Y \neq 0$ and $\operatorname{rk} A_X^t = 7$, the vector $A_X^t Y$ is nonzero, and up to scaling, we can choose Y in such a way that $A_X^t Y = \|X\|^2 Z_0$. Acting on the both sides by A_X we find $Y = A_X Z_0$, so for all nonzero $X \perp X_0$,

$$A_X A_0^t A_X Z_0 = -A_0 A_X^t A_X Z_0 = 0, \quad A_X Z_0 \neq 0. \tag{8}$$

Define a linear operator $V : \mathbb{R}^7 \to \mathbb{R}^8$ by

$$V X_0 := Z_0, \quad VX := -A_0^t A_X Z_0 \quad \text{for } X \perp X_0. \tag{9}$$

We want to show that $U = p \circ V : \mathbb{R}^7 \to \mathbb{R}^7$ is the sought orthogonal operator. The operator V has the following properties:

(i) For all $X \in \mathbb{R}^7$, $VX \in \operatorname{Ker} A_X$.

(ii) V is an orthogonal embedding (note that it acts between Euclidean spaces of different dimension):
$\|VX_0\| = 1$, and for $X \perp X_0$: $VX \perp VX_0$, $\|VX\| = \|X\|$.

To check (i), take an arbitrary $X \perp X_0$ and $t \in \mathbb{R}$. We have:

$$\begin{aligned} A_{tX_0+X} V(tX_0 + X) &= (tA_0 + A_X)(tZ_0 - A_0^t A_X Z_0) \\ &= t^2 A_0 Z_0 + t(A_X Z_0 - A_0 A_0^t A_X Z_0) - A_X A_0^t A_X Z_0 = 0, \end{aligned}$$

by (8) and (7).

The first equation of (ii) immediately follows from (9). For the second one, we have: $\langle VX, VX_0 \rangle = \langle -A_0^t A_X Z_0, Z_0 \rangle = -\langle A_X Z_0, A_0 Z_0 \rangle = 0$. To check the third one, consider the vector $A_X^t A_X Z_0$. As $A_0(A_X^t A_X Z_0) = 0$ by (8), $A_X^t A_X Z_0 = f(X) Z_0$ for some function f. Acting on both sides by A_X we get $\left(f(X) - \|X\|^2\right) A_X Z_0 = 0$, so $A_X^t A_X Z_0 = \|X\|^2 Z_0$, since $A_X Z_0 \neq 0$ by (8). Then

$$\begin{aligned} \|VX\|^2 = \langle A_0^t A_X Z_0, A_0^t A_X Z_0 \rangle &= \langle A_X Z_0, A_0 A_0^t A_X Z_0 \rangle \\ &= \langle A_X Z_0, A_X Z_0 \rangle = \langle A_X^t A_X Z_0, Z_0 \rangle = \|X\|^2, \end{aligned}$$

as required.

From property (i) it follows that for all $Y \in \mathbb{R}^7$, $0 = \langle A_X VX, Y \rangle = \langle \overline{B}(X, Y), VX \rangle$. Taking $Y = X$ we get $VX \perp e_0$, for all X (as B is skew-symmetric). So Range $V = \mathbb{R}^7$, and the operator $U : \mathbb{R}^7 \to \mathbb{R}^7$ defined by $U = p \circ V$ is orthogonal (U acts exactly as V, but with a different codomain). Moreover, as $0 = \langle \overline{B}(X, Y), VX \rangle = \langle B(X, Y), UX \rangle$, we have $UX \perp B(X, Y) = \operatorname{Ker} R(X, Y)$, for all X, Y. ∎

REMARK. From the proof of Lemma 2, it is easy to see that the map $\Phi : \mathbb{R}^7 \to \mathrm{Hom}(\mathbb{R}^8, \mathbb{R}^8)$ defined by $\Phi(X)Z = A_X Z + \langle VX, Z\rangle e_0$ has the property $\Phi(X)\Phi(X)^t = \|X\|^2 \mathrm{id}_{\mathbb{R}^8}$, and so the map $\phi(X) = \Phi(X_0)^t\Phi(X)$ defined on the six-space $X_0^\perp$ satisfies $\phi(X)^2 = -\|X\|^2\mathrm{id}_{\mathbb{R}^8}$. Thus ϕ can be extended to a representation of the Clifford algebra Cl(6) in $\mathbb{R}^8$, which is a restriction of that for the Clifford algebra Cl(7), which, in turn, is equivalent to the right (or to the left) multiplication by imaginary octonions in the octonion algebra $\mathbb{O}$. One can then show that, identifying $\mathbb{R}^7$ with the space of imaginary octonions, $B(X,Y)$ is the imaginary part of XY, up to orthogonal transformations.

For every pair of nonparallel vectors X, Y, let $E_1(X,Y)$ be the two-dimensional eigenspace of $R(X,Y)^2$ with the eigenvalue $-\alpha^2\|X\wedge Y\|^2$ (that is, $E_1(X,Y) = E_1(\pi)$, where $\pi = \mathrm{Span}(X,Y)$).

LEMMA 3. *$E_1(X,Y) = \mathrm{Span}(UX, UY)$, where U is the orthogonal operator introduced in Lemma 2.*

PROOF: Fix a unit vector X. Introduce a new variable t, and define, for every $Y \in X^\perp$, the operators $G(Y), M(Y,t) : \mathbb{R}^7 \to \mathbb{R}^7$ by

$$G(Y) = R(X,Y)^2 + \|Y\|^2\mathrm{id} - B(X,Y)B(X,Y)^t,$$
$$M(Y,t) = \big(R(X,Y) + t\,\alpha\,\mathrm{id}\big)\,G(Y) = R(X,Y)\,G(Y) + t\,\alpha\,G(Y),$$

where B is the map from Lemma 1 spanning the kernel of $R(X,Y)$. Note that the operator $G(Y)$ is symmetric, while the operator $R(X,Y)\,G(Y)$ is skew-symmetric.

At this point, it will be more convenient to switch from operators to matrices fixing some orthonormal basis for $\mathbb{R}^7$. With a slight abuse of language, we shall use the same notation for an operator and its matrix. For $Y \in X^\perp$, let $y_1, \dots, y_6$ be its coordinates with respect to an orthonormal basis for $X^\perp$ (which is not related to the chosen orthonormal basis for $\mathbb{R}^7$). Denote $\mathbb{R}[Y] = \mathbb{R}[y_1, \dots, y_6]$ and $\mathbb{R}[Y,t] = \mathbb{R}[y_1, \dots, y_6, t]$ the corresponding polynomial rings.

From definition, it is clear that all the entries of $G(Y)$ and $R(X,Y)\,G(Y)$ are homogeneous polynomials of the y_i's, of degree 2 and 3, respectively.

From (2), the normal forms of the matrices $G(Y), R(X,Y)\,G(Y)$ and $M(Y,t)$ are, respectively,

$$(10)\quad \|Y\|^2(1-\alpha^2)\begin{pmatrix}0&0&0\\0&I_2&0\\0&0&0\end{pmatrix},\quad f(Y)\|Y\|\begin{pmatrix}0&0&0\\0&J&0\\0&0&0\end{pmatrix},\quad f(Y)\begin{pmatrix}0&0&0\\0&tI_2+\|Y\|J&0\\0&0&0\end{pmatrix},$$

in the same basis, where $f(Y) = \|Y\|^2\alpha(1-\alpha^2)$, $J = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$, and the nonzero blocks are in the 5th and the 6th rows and columns.

As it follows from (10), for every nonzero $Y \perp X$, both $G(Y)$ and $R(X,Y)\,G(Y)$ have rank 2. Moreover, the two-space $E_1(X,Y)$ is the image of the operator $R(X,Y)\,G(Y)$ and is the eigenspace of the operator $G(Y)$, with the eigenvalue $(1-\alpha^2)\|Y\|^2$.

It follows from (10) that

$$M(Y,t)^2 - 2\alpha(1-\alpha^2)\|Y\|^2 t M(Y,t) = -\alpha^3(1-\alpha^2)\|Y\|^2\big(\|Y\|^2+t^2\big)G(Y). \tag{11}$$

Moreover, as $M(Y,t)$ still has the normal form (10) for real Y and *complex* t, the rank of the complex matrix $M\big(Y, i\|Y\|\big)$ is 1 for all nonzero $Y \in \mathbb{R}^7$. So all the 2×2 minors of the polynomial matrix $M(Y,t)$ vanish for $t = i\|Y\|$. Any such minor has a form $q(Y,t) = f_1(Y) + t f_2(Y) + t^2 f_3(Y)$, with f_1, f_2, f_3 real polynomials. As $q\big(Y, i\|Y\|\big) = 0$, we get $f_2(Y) = 0$, $f_1(Y) = \|Y\|^2 f_3(Y)$, hence every 2×2 minor of $M(Y,t)$ is divisible by $t^2 + \|Y\|^2$ in the polynomial ring $\mathbb{R}[Y,t]$.

Let $\mathbf{I} \subset \mathbb{R}[Y,t]$ be the ideal generated by $t^2 + \|Y\|^2$, and let $\mathbf{R} = \mathbb{R}[Y,t]/\mathbf{I}$, with $\pi : \mathbb{R}[Y,t] \to \mathbf{R}$ the natural projection. Note that for every element $a \in \mathbf{R}$, there is a unique pair of polynomials $p, q \in \mathbb{R}[Y]$ such that $\pi(p + tq) = a$.

Consider the 7×7 matrix $\mathcal{M} = \pi(M)$, with entries from $\mathbf{R}$. As all the 2×2 minors of $M(Y,t)$ are in $\mathbf{I}$, the rank of the matrix $\mathcal{M}$ is 1 ($\mathcal{M}$ is nonzero, since nonzero entries of M are at most linear in t). Projecting the equation (11) to $\mathbf{R}$, we obtain

$$\mathcal{M}^2 = -2\alpha(1-\alpha^2)\,\bar{t}^{\,3}\,\mathcal{M}, \tag{12}$$

where $\bar{t} = \pi(t)$.

By Fact 2, the ring $\mathbf{R}$ is a unique factorisation domain. Let $d \in \mathbf{R}$ be the greatest common divisor of the entries of $\mathcal{M}$, and $\mathcal{M} = d\mathcal{L}$, with the greatest common divisor of the entries of $\mathcal{L}$ being 1. Let L_1, L_2 be matrices with entries from $\mathbb{R}[Y]$ such that $\pi(L_1 + tL_2) = \mathcal{L}$.

From (12), $d \mid \bar{t}^{\,3}$, and so (as $\bar{t}$ is prime in $\mathbf{R}$), $d = \bar{t}^{\,m}$, where $m = 0, 1, 2, 3$. Consider these cases separately.

First show that $m > 0$. As $\operatorname{rk}\mathcal{M} = 1$, by Fact 1, there exist $a, b \in \mathbf{R}^7$ such that $\mathcal{M} = ab^t$. Reducing $M(Y,t) + M(Y,t)^t = 2\alpha t G(Y)$ modulo $\mathbf{I}$ we get $ab^t + ba^t = 2\alpha\bar{t}\pi\big(G(Y)\big)$. So for all $i, j = 1, \dots, 7$,

$$a_i b_j + b_i a_j = 2\alpha\bar{t}\,\pi\big(G_{ij}(Y)\big). \tag{13}$$

Taking $j = i$ in (13) we find that $\bar{t} \mid a_i b_i$, so for every i, at least one of a_i, b_i is divisible by $\bar{t}$. If for some $i \neq j$, $\bar{t} \nmid a_i, b_j$, then $\bar{t} \mid b_i, a_j$, and we come to a contradiction with (13). It follows that either all the a_i's, or all the b_i's are divisible by $\bar{t}$. In both cases, all the entries of the matrix $\mathcal{M} = ab^t$ are divisible by $\bar{t}$, so $m > 0$.

Assume that $m = 3$. Lifting the equation $\mathcal{M} = \bar{t}^{\,3}\mathcal{L}$ to $\mathbb{R}[Y,t]$ we get, modulo

$$\mathbf{I},\quad M(Y,t) = t^3(L_1 + tL_2) = -t\|Y\|^2(L_1 + tL_2) = \|Y\|^2\big(\|Y\|^2 L_2 - tL_1\big),$$

hence

$$M(Y,t) = \|Y\|^2\big(\|Y\|^2 L_2 - tL_1\big) + \big(t^2 + \|Y\|^2\big)\widehat{M}$$

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for some matrix $\widehat{M}$ with entries in $\mathbb{R}[Y,t]$. As $M(Y,t) = R(X,Y)G(Y) + t\alpha G(Y)$, with all the entries of $R(X,Y)G(Y)$ and $G(Y)$ homogeneous polynomials of Y of degree 3 and 2, respectively, we get $R(X,Y)G(Y) = 0$, which contradicts the fact that $\operatorname{rk} R(X,Y)G(Y) = 2$, when Y is nonzero.

Let $m = 2$. We have $M(Y,t) = t^2(L_1 + tL_2) \mod \mathbf{I}$, so

$$M(Y,t) = -\|Y\|^2 L_1 - \|Y\|^2 tL_2 + \left(t^2 + \|Y\|^2\right)\widehat{M} = R(X,Y)G(Y) + t\alpha G(Y).$$

It follows that $G(Y) = \|Y\|^2 G_0$ for some constant symmetric matrix G_0 of rank 2. But then for all nonzero $Y \perp X$, the eigenspace $E_1(X,Y)$ is the same: it is the eigenspace of the fixed matrix G_0. This contradicts Lemma 2: the set

$$\mathfrak{K}_0(X) = \bigcup_{Y \perp X, Y \neq 0} \operatorname{Ker} R(X,Y)$$

is a six-dimensional subspace of $\mathbb{R}^7$, hence for some Y, the subspaces $\operatorname{Ker} R(X,Y)$ and $E_1(X,Y)$ have a nonzero intersection.

Finally, consider the case $m = 1$. Then $M(Y,t) = t(L_1 + tL_2) \mod \mathbf{I}$, so $M(Y,t) = -\|Y\|^2 L_2 + tL_1 + \left(t^2 + \|Y\|^2\right)\widehat{M}$ for some matrices L_1, L_2, with entries from $\mathbb{R}[Y]$, and a matrix $\widehat{M}$, with entries from $\mathbb{R}[Y,t]$. As $M(Y,t) = R(X,Y)G(Y) + t\alpha G(Y)$, it follows that $R(X,Y)G(Y) = -\|Y\|^2 L_2$, and $L_2 = L_2(Y)$ must be a skew-symmetric matrix, of rank 2 (when $Y \neq 0$), whose entries are linear in Y.

We get a linear map L_2 from $\mathbb{R}^6 = X^\perp$ to $\mathfrak{o}_2(7)$, the set of skew-symmetric 7×7 matrices of rank less than or equal to two. The map L_2 is injective (as $\operatorname{rk} L_2(Y) = 2$ for all $Y \neq 0$), so by [**5**, Lemma 2.2], there exists a unit vector $\xi \in \mathbb{R}^7$ such that $L_2(Y)Z = \langle \xi, Z\rangle L_2(Y)\xi - \langle L_2(Y)\xi, Z\rangle \xi$, for all $Z \in \mathbb{R}^7$. In particular, taking $Z = L_2(Y)\xi$ we get $L_2(Y)(L_2(Y)\xi) = -\left\|L_2(Y)\xi\right\|^2 \xi$. As $L_2(Y)\xi \neq 0$, unless $Y = 0$ (otherwise $L_2(Y) = 0$), we find that, for all $Y \neq 0$,

$$\xi \in \operatorname{Range} L_2(Y) = \operatorname{Range} R(X,Y)G(Y) = E_1(X,Y).$$

For every nonparallel X, Y, $\operatorname{Ker} R(X,Y) \perp E_1(X,Y)$ (they are the eigenspaces of the symmetric operator $R(X,Y)^2$, with different eigenvalues). It follows that $\xi \perp \bigcup_{Y \nparallel X} \operatorname{Ker} R(X,Y) = \mathfrak{K}_0(X)$. By Lemma 2, $\mathfrak{K}_0(X)$ is a six-dimensional subspace of $\mathbb{R}^7$, whose orthogonal complement is spanned by the vector UX.

It follows that $UX \in E_1(X,Y)$. Similarly, $UY \in E_1(X,Y)$. As the operator U is orthogonal (and, in particularly, nonsingular), the two-dimensional spaces $\operatorname{Span}(UX, UY)$ and $E_1(X,Y)$ must coincide. ∎

Let X, Y be any two orthonormal vectors. From Lemma 3 it follows that $E_1(X,Y) = \operatorname{Span}(UX, UY)$. Then $R(X,Y)UY = \varepsilon\alpha UX$, $\varepsilon = \pm 1$, and by continuity, ε is the same for all X, Y. Hence for any Z,

$$R(X,Y,UY,Z) = -\varepsilon\alpha\langle UX, Z\rangle. \tag{14}$$

We claim that the operator U is not only orthogonal, but also symmetric. Indeed, an orthogonal operator in an odd-dimensional space has at least one eigenvalue ± 1. Replacing U by $-U$, if necessary, we can assume that there exists a unit vector $Y \in \mathbb{R}^7$ such that $UY = Y$. Note that the space $Y^\perp$ is an invariant subspace of U. The equation (14), with $X, Z \in Y^\perp$ gives:

$$R(X, Y, Y, Z) = -\varepsilon\alpha\langle UX, Z\rangle.$$

The left-hand side is symmetric with respect to X, Z, and so such is the right-hand side. It follows that the operator U is symmetric on its invariant subspace $Y^\perp$, hence is symmetric on the whole $\mathbb{R}^7$.

As U is orthogonal and symmetric, $U^2 = \mathrm{id}$. Let now R_U^α be an algebraic curvature tensor constructed as in Example 2, with the operator U and the constant α:

$$R_U^\alpha(X, Y)Z = \alpha\big(\langle UY, Z\rangle UX - \langle UX, Z\rangle UY\big).$$

Define an algebraic curvature tensor $\overline{R} = R - \varepsilon R_U^\alpha$. We claim that $\overline{R}$ is IP, with the eigenvalue structure $(3, 4)$. Indeed, for any two orthonormal vectors X, Y, we have:

1. If $Z \in \operatorname{Ker} R(X, Y)$, then $Z \perp UX, UY$ (by assertion 2 of Lemma 2), and so $\overline{R}(X, Y)Z = -\varepsilon R_U^\alpha(X, Y)Z = -\varepsilon\alpha\big(\langle UY, Z\rangle UX - \langle UX, Z\rangle UY\big) = 0$, which implies $Z \in \operatorname{Ker} \overline{R}(X, Y)$.
2. Let $Z = UY$. By (14), $R(X, Y)UY = \varepsilon\alpha UX$. So $\overline{R}(X, Y)UY = \varepsilon\alpha UX - \varepsilon\alpha\big(\langle UY, UY\rangle UX - \langle UX, UY\rangle UY\big) = 0$, as U is orthogonal. The same is true for $Z = UX$. So $UX, UY \in \operatorname{Ker} \overline{R}(X, Y)$.
3. If $Z \in E_2(X, Y) = (\operatorname{Span}(\operatorname{Ker} R(X, Y), UX, UY))^\perp$, then $\overline{R}(X, Y)Z = R(X, Y)Z \in E_2(X, Y)$, So $E_2(X, Y)$ is an invariant subspace of $\overline{R}(X, Y)$, and the restriction of $\overline{R}(X, Y)$ to $E_2(X, Y)$ has the same eigenvalues as those of $R(X, Y)$, namely $\pm i$, both with multiplicity 2.

2.2. Case (b), the eigenvalue structure $(3, 4)$ Following [**6**], for a nonzero $X \in \mathbb{R}^7$, define a subset $\mathfrak{A}_0(X) \in \mathbb{R}^7$ by

$$\mathfrak{A}_0(X) = \bigcap_{Y \nparallel X} \operatorname{Ker} R(X, Y).$$

Lemma 4. *There exists an open, dense set $\mathcal{S} \subset \mathbb{R}^7$ such that $\mathfrak{A}_0(X) = 0$, when $X \in \mathcal{S}$.*

Proof: Let $Z \in \mathfrak{A}_0(X)$, for a given $X \neq 0$. Then for any $U, V \in \mathbb{R}^7$, $R(X, U, V, Z) = 0$ and $R(X, V, Z, U) = 0$. So, by the first Bianchi identity, $R(X, Z, U, V) = 0$, that is, the operator $R(X, Z)$ is zero. It follows that $Z \parallel X$. If $Z \neq 0$, then $R(X, V, X, U) = 0$ for all $U, V \in \mathbb{R}^7$, and, in particular, the curvature on any two-plane in $\mathbb{R}^7$ containing X must vanish. The set of X's with this property is closed. If it has a nonempty interior, then the sectional curvature vanishes identically, and so $R = 0$, which is a contradiction. ∎

As it follows from (2), for all $X, Y \in \mathbb{R}^7$,

$$(R(X,Y))^3 + \|X \wedge Y\|^2 R(X,Y) = 0. \tag{15}$$

Fix two orthonormal vectors $X, Y \in \mathbb{R}^7$, with $X \in \mathcal{S}$, and choose an orthonormal basis for $\mathbb{R}^7$ in such a way that the matrix of $R(X,Y)$ is

$$K = \begin{pmatrix} \mathcal{J} & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where } \mathcal{J} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}.$$

Let $\mathcal{W} = (\mathrm{Span}(X,Y))^\perp$. For a vector $Z \in \mathcal{W}$, let

$$L(Z) = \begin{pmatrix} A(Z) & B(Z) \\ -B(Z)^t & C(Z) \end{pmatrix}$$

be the matrix of $R(X,Z)$, with $A(Z), C(Z)$ skew-symmetric 4×4- and 3×3-matrices, respectively, and $B(Z)$ a 4×3-matrix, all depending linearly on $Z \in \mathcal{W}$. The equation (15), with Y replaced by $yY + Z$, gives

$$(yK + L(Z))^3 = -(y^2 + \|Z\|^2)(yK + L(Z)), \quad \text{for all } y \in \mathbb{R}. \tag{16}$$

Expanding (16) by the powers of y we find:

$$K^2 L(Z) + L(Z)K^2 + KL(Z)K = -L(Z), \tag{17}$$

$$L(Z)^2 K + KL(Z)^2 + L(Z)KL(Z) = -\|Z\|^2 K, \tag{18}$$

$$L(Z)^3 = -\|Z\|^2 L(Z). \tag{19}$$

From (17) it follows that $C(Z) = 0$ and $\mathcal{J}A(Z)\mathcal{J} = A(Z)$, hence

$$A(Z) = \begin{pmatrix} a(Z)J & b(Z)J \\ b(Z)J & -a(Z)J \end{pmatrix}, \quad \text{where} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and a, b are linear functionals on $\mathcal{W}$. Then from (18), (19),

$$B(Z)^t \mathcal{J} B(Z) = 0, \tag{20}$$

$$B(Z)B(Z)^t - \mathcal{J}B(Z)B(Z)^t\mathcal{J} = (\|Z\|^2 - a(Z)^2 - b(Z)^2) I_4 \tag{21}$$

$$B(Z)^t A(Z) B(Z) = 0. \tag{22}$$

Equation (20) implies that the column space of the matrix $B(Z)$ is an isotropic subspace of $\mathcal{J}$, hence $\mathrm{rk}\, B(Z) \leqslant 2$. If the set of vectors $Z \in \mathcal{W}$ with $\mathrm{rk}\, B(Z) < 2$ has a nonempty interior, then $\mathrm{rk}\, B(Z) < 2$ for all Z, hence the matrix on the right-hand side of (21) has rank at most two. It follows that $a(Z)^2 + b(Z)^2 = \|Z\|^2$, which is not possible for two linear functionals on a 5-space. So, for an open, dense set of vectors $Z \in \mathcal{W}$, $\mathrm{rk}\, B(Z) = 2$.

Multiplying the equation (21) by $B(Z)$ from the right and using (20), we get

$$B(Z)B(Z)^t B(Z) = (\|Z\|^2 - a(Z)^2 - b(Z)^2)B(Z).$$

This equation, together with the fact that $\operatorname{rk} B(Z) = 2$ for almost all $Z \in \mathcal{W}$, implies that the singular numbers of the 4×3 matrix $B(Z)$ are $c(Z), c(Z), 0$, where $c(Z) = \sqrt{\|Z\|^2 - a(Z)^2 - b(Z)^2}$ (where the *singular numbers* of a matrix M are the square roots of the eigenvalues of $M^t M$).

We need the following Lemma:

Lemma 5. *Let $\mathcal{V}$ be a linear space of 4×3-matrices B whose singular numbers are $c, c, 0$ (where $c = c(B) \geqslant 0$), and such that $\bigcap_{B \in \mathcal{V}} \operatorname{Ker} B = 0$. Then $\dim \mathcal{V} \leqslant 3$.*

In fact, up to orthogonal transformations, $\mathcal{V}$ is a subspace of the space of 3×3 skew-symmetric matrices, with a zero row added at the bottom.

Proof of Lemma 5 Let B_1, B_2, B_3, B_4 be linearly independent matrices in $\mathcal{V}$. Up to orthogonal transformation and scaling, we can assume that

$$B_1 = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix},$$

and $\operatorname{Tr} B_1^t B_i = 0$ for $i = 2, 3, 4$. The fact that $\operatorname{rk}(B_i + tB_1) \leqslant 2$ implies that $B_i = \begin{pmatrix} Q_i & u_i \\ T_i & 0 \end{pmatrix}$ for some 2×2-matrices Q_i, T_i and 2-vectors u_i satisfying $T_i u_i = 0$. At least one of the u_i's must be nonzero by assumption, and we can assume, up to orthogonal transformation and up to taking appropriate linear combinations, that $u_2 = (p, 0)$, $p \neq 0$ and $u_4 = 0$. The fact that the singular numbers of the matrix $(B_2 + sB_4) + tB_1$ are $c, c, 0$ (for some c depending on t and s), together with the condition $\operatorname{Tr} B_1^t B_i = 0$, gives

$$B_2 + sB_4 = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & 0 \\ 0 & q_1(s) & 0 \\ 0 & q_2(s) & 0 \end{pmatrix},$$

for some linear functions $q_1(s), q_2(s)$ satisfying $q_1(s)^2 + q_2(s)^2 = p^2$. It follows that q_1 and q_2 are constants, hence $B_4 = 0$. $\square$

Note that Lemma 5 applies in our situation, as $\bigcap_{Z \in \mathcal{W}} \operatorname{Ker} B(Z) = 0$. Otherwise, if u is a nonzero vector with $B(Z)u = 0$ for all $Z \in \mathcal{W}$, then the set $\mathfrak{A}_0(X)$ contains a nonzero vector $(0, 0, 0, 0, u)$, which contradicts the choice of $X \in \mathcal{S}$.

By Lemma 5, we can find two orthonormal vectors $Z_1, Z_2 \in \mathcal{W}$ such that $B(Z_1) = B(Z_2) = 0$. It then follows from (21) that $a(Z)^2 + b(Z)^2 = \|Z\|^2$, for all $Z \in \operatorname{Span}(Z_1, Z_2)$, so we can choose Z_1, Z_2 in such a way that $a(Z_1) = b(Z_2) = 1$, $a(Z_2) = b(Z_1) = 0$.

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Now for any $Z' \in \mathcal{W}$, the equation (22) with $Z = Z' + t_1 Z_1 + t_2 Z_2$ gives

$$B(Z')^t A(Z_1) B(Z') = B(Z')^t A(Z_2) B(Z') = 0.$$

As a common isotropic subspace of the matrices

$$A(Z_1) = \begin{pmatrix} J & 0 \\ 0 & -J \end{pmatrix}, \quad A(Z_2) = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad \mathcal{J} = \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix}$$

is at most one-dimensional, the latter equation, together with (20), implies that $\operatorname{rk} B(Z') \leqslant 1$ for all $Z' \in \mathcal{W}$. This is a contradiction with the fact that $\operatorname{rk} B(Z) = 2$ for a generic $Z \in \mathcal{W}$.

3. IP manifolds of dimension three

In the study of Riemannian IP manifolds, the three-dimensional case is exceptional. In dimension $n \geqslant 4$, the most difficult part is algebraic. Once all the IP algebraic curvature tensors are found, the corresponding Riemannian metrics can be produced in a closed form, and depend on a few constants. When $n = 3$, the situation is completely different: the IP algebraic curvature tensors can be easily classified [**12**, Remark 1]: they are either of constant curvature, or those whose Ricci tensor ρ has rank 1. However, the class of Riemannian manifolds satisfying the latter condition, $\operatorname{rk} \rho = 1$, is quite large: it depends on at least two arbitrary functions of one variable, and it seems doubtful that the description of these manifolds can be obtained in some nice form.

As in dimension 3 the Ricci tensor determines the curvature tensor, the question of finding IP manifolds can be viewed as the question of finding a Riemannian metric given its Ricci tensor. Even the existence of a solution g for the corresponding system of differential equations $\operatorname{Ric}(g) = \rho$ is a hard problem (see [**1**, Chapter 5] for examples of symmetric tensors which cannot be Ricci tensors of any Riemannian metric). For nondegenerate Ricci tensors, the existence problem is solved in affirmative by Deturck [**4**]. Recently, the existence of a Riemannian metric g with the given Ricci tensor ρ was also proved for degenerate ρ whose kernel distribution has constant rank and is integrable (under some additional assumptions on the first derivatives) [**3**].

Let M^3 be a Riemannian manifold whose Ricci tensor has constant rank one, with $2f$ the nonzero principal Ricci curvature.

If $f = \text{const}$ (and more generally, if the principal Ricci curvatures ρ_i are constant and $\rho_1 = \rho_2 \neq \rho_3$), the Riemannian manifold M^3, up to isometry, depends on two functions of one variable, as was shown by Kowalski [**14, 15**] and Bueken [**2**]. Despite of the fact that any such M^3 is curvature-homogeneous (the curvature tensor at every point is the same), the majority of them are not homogeneous. The only homogeneous 3-manifolds with $\operatorname{rk} \rho = 1$ are unimodular Lie groups with a specific left-invariant metric, whose explicit construction is given by Milnor [**17**, Chapter 4] (see also [**16**]). Let $\mathfrak{g}$ be a 3-dimensional Lie algebra with a basis e_1, e_2, e_3 and the Lie brackets defined by

$$[e_1, e_2] = \lambda_3 e_3, \quad [e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2.$$

Assuming e_1, e_2, e_3 orthonormal we get a left-invariant metric on the Lie group G of $\mathfrak{g}$. If $\lambda_1 + \lambda_2 = \lambda_3$, $\lambda_1\lambda_2 \neq 0$, the Ricci tensor of this metric has rank 1, with the nonzero principal Ricci curvature $2f = 2\lambda_1\lambda_2$. Depending on the signs of the λ_i's, the underlying Lie group G is $SU(2)$ (the sphere S^3, but not with a constant curvature metric), $SL(2,\mathbb{R})$, or $E(1,1)$, the group of motions of Minkowski plane. In the latter case, the metric has the form

$$ds^2 = dx^2 + e^{2ax}dy^2 + e^{-2ax}dz^2, \ a \neq 0, \tag{23}$$

and is the only 3-dimensional metric, which is generalised symmetric, but not symmetric [**13**, Chapter 6].

If $f \neq \text{const}$, only isolated examples are known. We shall consider here two particular cases: when the space M^3 is conformally flat, and when the principal Ricci direction corresponding to $2f$ is a geodesic vector field. This choice of the additional assumptions is motivated by the following facts. Firstly, all the IP manifolds of dimension $n \geqslant 4$ (Example 3) are conformally flat, which is no longer true when $n = 3$. Secondly, when $f = \text{const}$, the principal Ricci direction corresponding to $2f$ is a geodesic vector field, which follows from the second Bianchi identity (see (28) below).

Proposition. *Let M^3 be a Riemannian manifold whose Ricci tensor has rank one, with a nonconstant principal Ricci curvature $2f$, and the corresponding principal Ricci direction e_1.*

1. *If M^3 is conformally flat, then it is locally isometric to a manifold with metric* (1).
2. *If e_1 is a geodesic vector field, then M^3 is either conformally flat, or the metric form on M^3 is locally homothetic to*

$$ds^2 = dx^2 + x^{1+a}dy^2 + x^{1-a}dz^2, \qquad \text{with } a \neq \pm 1. \tag{24}$$

Before giving the proof, consider the general case. Let M^3 be a Riemannian manifold whose Ricci tensor has constant rank one. Introduce a local orthonormal frame e_1, e_2, e_3 in such a way that e_1 is the principal direction of the Ricci tensor corresponding to $2f$, and $\text{Span}(e_2, e_3) = \text{Ker}\,\rho$. The only nonzero components of the curvature tensor $R_{ijkl} = R(e_i, e_j, e_k, e_l)$, up to permutation of indices, are

$$R_{1212} = R_{1313} = f, \quad R_{2323} = -f.$$

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Let ω^i be the 1-forms dual to e_i, and let ψ_i^j, Ω_i^j be the connection and the curvature forms, respectively:

$$\psi_j^i = \Gamma_{jk}^i\omega^k,\ \Gamma_{jk}^i = \langle \nabla_k e_j, e_i\rangle,\ \Gamma_{ik}^j = -\Gamma_{jk}^i,\ \psi_j^i = -\psi_i^j,\ \Omega_j^i = -\Omega_i^j = \frac{1}{2}R_{ijkl}\omega^k \wedge \omega^l,$$

so that

$$\Omega_2^1 = f\omega^1 \wedge \omega^2, \qquad \Omega_3^1 = f\omega^1 \wedge \omega^3, \qquad \Omega_3^2 = -f\omega^2 \wedge \omega^3. \tag{25}$$

We have the structure equations

$$d\omega^i = -\psi_j^i \wedge \omega^j, \qquad d\psi_j^i = -\psi_k^i \wedge \psi_j^k + \Omega_j^i, \tag{26}$$

whose integrability condition is the second Bianchi identity

$$d\Omega_j^i = \Omega_k^i \wedge \psi_j^k - \Omega_j^k \wedge \psi_k^i. \tag{27}$$

Substituting (25) to (27) we find:

$$\frac{e_1(f)}{2f} = \Gamma_{22}^1 + \Gamma_{33}^1, \quad \frac{e_2(f)}{2f} = \Gamma_{11}^2, \quad \frac{e_3(f)}{2f} = \Gamma_{11}^3, \tag{28}$$

or, equivalently, $df/(2f) = (\Gamma_{22}^1 + \Gamma_{33}^1)\omega^1 + \Gamma_{11}^2\omega^2 + \Gamma_{11}^3\omega^3$. Another equivalent form of the second Bianchi identity is

$$d\big(\sqrt{|f|}\omega^1\big) = \sqrt{|f|}(\Gamma_{23}^1 - \Gamma_{32}^1)\omega^2 \wedge \omega^3, \qquad d\big(\sqrt{|f|}\,\omega^2 \wedge \omega^3\big) = 0. \tag{29}$$

Studying the system (25, 26, 29) further, it might be interesting to know, for example, whether the solution set (say, in the analytic case) depends on functions of two variables.

Proof of the Proposition: 1. A manifold M^3 is conformally flat, if its Schouten-Weyl tensor vanishes, that is, if the tensor $T(X,Y,Z) = (\nabla_X\rho)(Y,Z) - \langle Y,Z\rangle/4X(s)$ is symmetric with respect to X, Y, where s is the scalar curvature. In our case, $\rho = 2f\omega^1 \otimes \omega^1$, $s = 2f$. A direct calculation shows that M^3 is conformally flat, if and only if

$$e_1(f) = 4\Gamma_{22}^1 f = 4\Gamma_{33}^1 f, \quad e_2(f) = e_3(f) = 0, \quad \Gamma_{11}^2 = \Gamma_{11}^3 = 0, \quad \Gamma_{23}^1 = \Gamma_{32}^1 = 0, \tag{30}$$

and then the second Bianchi identity (28) is automatically satisfied.

From this point on, the proof goes word-by-word as in the four-dimensional case, starting from equation (3.21) on page 279 of [**12**], up to changing the notation.

2. As the field e_1 is geodesic, $d\omega^1 = 0$, and we can choose a (coordinate) function x on a neighbourhood $U \subset M^3$ in such a way that $\omega^1 = dx$.

The proof goes in seven steps:

Step 1. The distribution Ker $\rho = \mathrm{Span}(e_2, e_3)$ is integrable and $f = f(x)$.

As e_1 is geodesic, $\Gamma_{11}^2 = \Gamma_{11}^3 = 0$, so by (28), the fields e_2, e_3 are tangent to the level sets of f. Again, from (28), df is a scalar multiple of $\omega^1 = dx$, so f is a function of x. As $\mathrm{Span}(e_2, e_3)$ is integrable, $\Gamma_{23}^1 = \Gamma_{32}^1$.

STEP 2. The fields e_2, e_3 can be chosen in such a way that $\Gamma_{31}^2 = 0$.

Replacing e_2, e_3 by $\widetilde{e}_2 = \cos\alpha\, e_2 + \sin\alpha\, e_3$, $\widetilde{e}_3 = -\sin\alpha\, e_2 + \cos\alpha\, e_3$, respectively, with some function α, we find that $\widetilde{\Gamma}_{31}^2 = \langle \nabla_1 \widetilde{e}_3, \widetilde{e}_2 \rangle = \Gamma_{31}^2 - e_1(\alpha)$. Choosing α in such a way that $e_1(\alpha) = \Gamma_{31}^2$ we get what required.

Let H be a symmetric 2×2 matrix, with entries $h_{ij} = \Gamma_{ij}^1$, $i, j = 2, 3$ (the second fundamental form of the foliation $f = \mathrm{const}$). As it follows from (28),

$$\mathrm{Tr}\, H = f'/(2f). \tag{31}$$

STEP 3. The matrix H satisfies the differential equation

$$e_1(H) = H^2 + f I_2. \tag{32}$$

From the structure equations (26), the form of the curvature tensor (25), and the fact that $\psi_i^1 = h_{ij}\omega^j$, $i = 2, 3$, we obtain

$$\left(dH - \omega^1 (H^2 + f I_2) - \psi_3^2 [H, J]\right) \wedge \begin{pmatrix} \omega^2 \\ \omega^3 \end{pmatrix} = 0,$$

where $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Extracting the $\omega^1 \wedge \omega^i$ components from the both rows, and using the fact that $\Gamma_{31}^2 = 0$ (Step 2), we get (32).

STEP 4. Let $f(x) = \varepsilon \phi^{-4}(x)$, $\varepsilon = \pm 1$. Then

$$h_{22} = \phi^{-2} v - \phi^{-1}\phi', \quad h_{33} = -\phi^{-2} v - \phi^{-1}\phi', \quad h_{23} = \phi^{-2} u, \tag{33}$$

where the functions u, v satisfy $e_1(u) = e_1(v) = 0$, $u^2 + v^2 + \varepsilon = c_0 = \mathrm{const}$, and

$$\phi^2 = Ax^2 + Bx + C, \qquad \text{with constants } A, B, C \text{ satisfying } B^2 - 4AC = 4c_0. \tag{34}$$

This can be obtained directly by solving the system of ODE's (31, 32).

STEP 5. The fields e_2, e_3 can be chosen in such a way that $u = 0$, and both distributions $e_2^\perp$, $e_3^\perp$ are integrable.

Replacing e_2, e_3 by $\widetilde{e}_2 = \cos\beta\, e_2 + \sin\beta\, e_3$, $\widetilde{e}_3 = -\sin\beta\, e_2 + \cos\beta\, e_3$, respectively, with some function β such that $e_1(\beta) = 0$ (not to violate the condition of Step 2), we find: $\widetilde{\Gamma}_{23}^1 = \langle \nabla_{\widetilde{3}} \widetilde{e}_2, e_1 \rangle = \cos 2\beta\, \Gamma_{23}^1 + (\sin 2\beta)/2\, (\Gamma_{33}^1 - \Gamma_{22}^1) = \cos 2\beta\, h_{23} + (\sin 2\beta)/2\, (h_{33} - h_{22})$ $= \phi^{-2}(\cos 2\beta\, u - \sin 2\beta\, v)$ by (33). Choosing β accordingly, we obtain $\widetilde{\Gamma}_{23}^1 = 0$.

Omitting the tildes, we get $\Gamma_{23}^1 = \Gamma_{32}^1 = h_{23} = u = 0$. Then $d\omega^2 \wedge \omega^2 = (\Gamma_{31}^2 - \Gamma_{13}^2)\, \omega^1 \wedge \omega^2 \wedge \omega^3 = 0$ (from the above and Step 2), and similarly $d\omega^3 \wedge \omega^3 = 0$. Note also that $v^2 = c_0 - \varepsilon = \mathrm{const}$.

STEP 6. $v\,\psi_3^2 = 0$.

We already know that $\psi_2^1 = (\phi^{-2}v - \phi^{-1}\phi')\,\omega^2$, $\psi_3^1 = (-\phi^{-2}v - \phi^{-1}\phi')\,\omega^3$, $\psi_3^2 = \Gamma_{32}^2\omega^2 + \Gamma_{33}^2\omega^3$. Substituting this to the structure equation $d\psi_2^1 = -\psi_3^1 \wedge \psi_2^3 + f\omega^1 \wedge \omega^2$ and extracting the $\omega^2 \wedge \omega^3$ term we get $v\Gamma_{32}^2 = 0$. Similarly, $v\Gamma_{33}^2 = 0$.

STEP 7. Assuming $v = 0$, we go back to the conformally flat case. Indeed, equation (33) implies $h_{22} = h_{33} = -\phi^{-1}\phi' = e_1(f)/(4f)$, also from Step 1 we know that $e_i(f) = 0, \Gamma_{11}^i = 0$, $i = 2, 3$, and from Step 5, $\Gamma_{23}^1 = \Gamma_{32}^1 = 0$. Then (30) follows.

Let us take $v \neq 0$. By Step 6, $\psi_3^2 = 0$. It follows that $d\omega^2 = -(\phi^{-2}v - \phi^{-1}\phi')\,\omega^1 \wedge \omega^2 = -(\phi^{-2}v - \phi^{-1}\phi')\,dx \wedge \omega^2$, so we can find functions $\mu_2 = \mu_2(x)$ and y such that $\omega^2 = \mu_2(x)dy$. Similarly, for some functions $\mu_3(x)$ and z, $\omega^3 = \mu_3(x)dz$. Then

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 = dx^2 + \mu_2(x)^2dy^2 + \mu_3(x)^2dz^2.$$

Calculating the Ricci tensor (with MAPLE), and equating ρ_{22} and ρ_{33} to zero, we get $\mu_2''\mu_3 + \mu_2'\mu_3' = \mu_3''\mu_2 + \mu_2'\mu_3' = 0$, so, up to homothecy and translation,

$$\mu_2 = x^{(1+a)/2},\ \mu_3 = x^{(1-a)/2}, \qquad \text{or} \qquad \mu_2 = e^{ax},\ \mu_3 = e^{-ax}.$$

In the second case, we get the metric form (23), with $2f = \rho_{11} = -2a^2 = \text{const}$, which contradicts the assumption.

The first case gives the required metric (24). We have $2f = \rho_{11} = (1 - a^2)/2x^{-2}$, and the metric (24) is not isometric to any of (1), unless $a = 0$ (for instance, because the surfaces $f = \text{const}$ are not totally umbilical). ∎

We finish with yet another example of a metric whose Ricci tensor has rank one, and the scalar curvature is nonconstant:

$$ds^2 = e^y dx^2 + y^{-1}e^y dy^2 + ydz^2.$$

The nonzero principal Ricci curvature of this metric is $-e^{-y}/2$, with the corresponding principal Ricci direction $e^{-y/2}\partial/\partial x$ (which is not geodesic).

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