UNIFORM CONTINUITY OF CONTINUOUS FUNCTIONS ON UNIFORM SPACES

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1. Introduction. Recently several topologists have called attention to the uniform structures (in most cases, the coarsest ones) under which every continuous real function is uniformly continuous (let us call the structures the [coarsest] *uc-structures*), and some important results have been found which closely relate, explicitly or implicitly, to the uc-structures, such as in the υS of Hewitt **(3)** and in the e-complete space of Shirota **(7)**. Under these circumstances it will be natural to pose, as Hitotumatu did **(4)**, the problem: which are the uniform spaces with the uc-structures? In **(1; 2)**, we characterized the metric spaces with such structures, and in this paper we shall give a solution to the problem in uniform spaces (§ 1), together with some of its applications to normal uniform spaces and to the products of metric spaces (§ 2). It is evident that every continuous real function on a uniform space is uniformly continuous if and only if the uniform structure of the space is finer than the uniform structure defined by all continuous real functions on the space. Our main theorem (Theorem 1) shows an internal aspect of this necessary and sufficient condition, and thus it permits us to construct the coarsest uc-structures internally (Corollary to Theorem 1). Then it is shown that the coarsest uc-structures are equivalent to the e-structures of Shirota **(7)** on some kind of spaces (Theorem 2). The last section of this paper is devoted to characterizing, as it were, uniformly pseudo-compact spaces, a generalization of pseudo-compact spaces in which Theorem 1 is reduced to a known result, and the characterization is a generalization of Theorem 2 of **(2)**.

2. Main theorem. We shall first give some definitions used in this section. A *function* is a real-valued continuous mapping, a *space* S is, unless otherwise specified, a uniform space, and a space is said to be *uc* if every function on the space is uniformly continuous. A sequence of subsets is *discrete* if any point of the space has a neighbourhood intersecting at most one member of the sequence, and it is *uniformly discrete* if there is an entourage V of the uniform structure such that $V(x)$ meets at most one member of it for any point x of the space. A sequence of subsets $\{A_n\}$ is *discretely normally separated* by a sequence of subsets $\{B_n\}$ if $\{B_n\}$ is discrete and there is a function f, $0 \leqslant f \leqslant 1$, for each n with values 1 on A_n and 0 on the complement of B_n, let us call the function the *characteristic function for n*. $\{A_n\}$ is *uniformly separated* by $\{B_n\}$ if there is an entourage V with $V(A_n) \subset B_n$ for all n.

THEOREM 1. *A uniform space is uc if and only if any sequence of subsets*

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$\{A_n\}$, discretely normally separated by some sequence of subsets $\{B_n\}$, is uniformly separated by $\{B_n\}$.

Proof. Suppose the space is uc and f_n a characteristic function for n, then $f = \sup_n \inf(n, 2nf_n)$ and $g = \sup_n f_n$ are continuous, and so uniformly continuous, that is, there is an entourage V such that $(x, y) \in V$ implies $|f(x) - f(y)| < \frac{1}{2}$ and $|g(x) - g(y)| < \frac{1}{2}$. If x is in A_n, then $n - \frac{1}{2} < f(y) < n + \frac{1}{2}$ and $\frac{1}{2} < g(y)$. $g(y) \neq 0$ follows $y \in B_m$ for some m, and thus we have $g(y) = f_m(y)$, $\frac{1}{2} < f_m(y)$, $m < 2mf_m(y)$, $f(y) = m$, $n - \frac{1}{2} < m < n + \frac{1}{2}$, $m = n$, which means $V(A_n) \subset B_n$. For the converse, suppose f is a function on the space, consider the following closed intervals in the real line (n and k are integers and positive integers respectively):

$$A'_n = [(8n+1)/6k, \quad (8n+5)/6k],$$
$$B'_n = [4n/3k, (4n+3)/3k],$$

and put $A_n = f^{-1}(A_n')$, $B_n = f^{-1}(B_n')$. Since $\{A_n\}$ is discretely normally separated by $\{B_n\}$, there is an entourage V_1 with $V_1(A_n) \subset B_n$ for all n. Similarly, for the closed intervals

$$C'_n = [(8n+5)/6k, (8n+9)/6k],$$
$$D'_n = [(4n+2)/3k, (4n+5)/3k],$$

and the sets $C_n = f^{-1}(C_n')$, $D_n = f^{-1}(D_n')$, there is an entourage V_2 with $V_2(C_n) \subset D_n$. Any point of the space belongs to some $A_n \cup C_n$. Let $(x, y) \in V \subset V_1 \cap V_2$, and let x belong to A_n, then y is in B_n and $|f(x) - f(y)| \leqslant 1/k$; similarly for C_n.

COROLLARY. *For any topological space S, the coarsest uc-structure $\mathcal{U}$ is generated by the relations*

$$U = (S - \cup A_n) \times (S - \cup A_n) \cup \cup_n (B_n \times B_n),$$

where $\{A_n\}$ is any sequence of closed subsets, discretely normally separated by the sequence $\{B_n\}$ of open subsets.

*Proof.** If U is as stated and $V \in \mathcal{U}$ taken such that $V(A_n) \subset B_n$ for all n (by Theorem 1), then $V \subset U$; in fact, if $(x, y) \in V$ and x or y lies in $\cup A_n$, then $(x, y) \in B_k \times B_k$ for some k, and otherwise $(x, y) \in (S - \cup A_n) \times (S - \cup A_n)$, this shows $V \subset U$, and hence $U \in \mathcal{U}$. On the other hand, if $W = \{(x, y); |f(x) - f(y)| \leqslant e\}$ for some function f on S and some real $e > 0$, then there are, by the proof of Theorem 1, two relations U and U' of the type considered such that $U \cap U' \subset W$. Since these W generate $\mathcal{U}$, the same holds for the U.

It is easily seen that the structure defined in this corollary is compatible with the original topology when the space is completely regular.

*This proof, neater than the original, is due to the referee.

Let us now give some spaces on which the coarsest uc-structure and the e-structure **(7)** are equivalent.

DEFINITION 1. *A topological space is* normally disconnected *if, for any open set G containing any closed set A, there is an open and closed set contained in G and containing A.*

THEOREM 2. *A coarsest uc-structure is equivalent to an e-structure in a T_1-space S if the space is normally disconnected and at most one-dimensional.*

Proof. Obviously a normally disconnected T_1-space is normal. Suppose that $\{X_n\}$ is a countable normal open covering of a normally disconnected T_1-space with at most one dimension, then we may assume $\{X_n\}$ has at most order 2. There is, for every n, an open and closed set H_n contained in X_n and containing $X_n - \bigcup_{i\neq n} X_i$. $K_1 = H_1$, $K_n = H_n - \bigcup_{i<n} H_i$ are open and closed sets which are disjoint from each other. $L_{ij} = X_i \cap X_j - (K_i \cup K_j)$ is open and closed, because, since $\{X_n\}$ is of at most order 2 and a point included only in X_i belongs to K_i, $L_{ij} = S - \bigcup_{h\neq i,j} X_h - (K_i \cup K_j)$. Consequently, $\{K_i, L_{ij};\ i, j$ natural numbers$\}$ is an open and closed covering which refines $\{X_n\}$ and is discretely normally separated by itself.

3. Case of normal uniform spaces and others.

Using the result of Dowker **(5,** Lemma 3**)**, we get the following statement which is a slight modification of Theorem 1: *In order that a normal space is uc it is necessary and sufficient that if $\{B_n\}$ is a disjoint sequence of open subsets, if closed $A_n \subset B_n$ for every n, and if $\bigcup A_n$ is closed, then there is an entourage V such that $V(A_n) \subset B_n$ for all n.* Furthermore we have

THEOREM 3. *A normal uniform space is uc if and only if any discrete sequence of subsets is uniformly discrete.*

Proof. Let $\{A_n\}$ be discrete in a normal uc-space, then we have, by a simple induction, a disjoint sequence of open subsets B_n containing $\bar{A}_n$ for every n, and, by the above remark, an entourage V such that $V(A_n) \subset B_n$, that is, $V(A_m) \cap A_n = \emptyset$. To prove the converse, let $\{B_n\}$ discretely normally separate $\{A_n\}$ in a space S satisfying the property in the assertion, then there are entourages V_1 and V_2 with $V_1(B_m) \cap B_n = \emptyset$, $m \neq n$, $V_2(\bigcup A_n) \cap (S - \bigcup B_n) = \emptyset$, and we have $V(A_n) \subset B_n$ for $V \subset V_1 \cap V_2$ and for each n.

DEFINITION 2. *A sequence of subsets $\{A_n\}$ is said to be* shrinking (*or* uniformly shrinking) *if A_n^0 (interior of A_n) $\supset \bar{A}_{n+1}$ (or $A_n \supset V(A_{n+1})$ for some entourage V) for every n.*

THEOREM 4. *A normal uniform space is uc if and only if any shrinking sequence of subsets $\{A_n\}$ with vacuous intersection is uniformly shrinking.*

Proof. Suppose a normal space S is uc and put $F_n = \bar{A}_{2n-1} - A_{2n}^0$, then

$\{F_n\}$ is discrete and there is a discrete sequence of open subsets $\{G_n\}$ (**5**, Lemma 3) with $F_n \subset G_n \subset A_{2n-2}{}^0 - \bar{A}_{2n+1}$, $n \geqslant 1$, $A_0 = S$. Similarly, for the discrete $\{H_n\}$, $H_n = \bar{A}_{2n} - A_{2n+1}{}^0$, there is a discrete $\{K_n\}$ of open subsets with $H_n \subset K_n \subset A_{2n-1}{}^0 - \bar{A}_{2n+2}$. There is, by Theorem 1, an entourage V with $V(F_n) \subset G_n$ and $V(H_n) \subset K_n$ for all n. If a point x of A_n belongs to A_m and not to A_{m+1}, and when $m = 2k - 1$, then $x \in F_k$, $V(x) \subset G_k \subset A_{2k-2}{}^0 = A_{m-1}{}^0 \subset A_{n-1}$; when $m = 2k$, then $x \in H_k$, $V(x) \subset K_k \subset A_{2k-1}{}^0 = A_{m-1}{}^0 \subset A_{n-1}$. In any case, we have $V(A_n) \subset A_{n-1}$. Conversely, for a discrete sequence of closed subsets $\{A_n\}$, there is an open discrete $\{B_n\}$ with $B_n \supset A_n$. For every n we can make up a sequence of $2n - 2$ open sets $C_n{}^i$ such that $\{B_n, C_n{}^1, C_n{}^2, \ldots, C_n{}^{2n-2}, A_n\}$ is shrinking, and put

$$F_{2n-1} = A_n \cup \Big(\bigcup_{i>n} C_i^{2n-1} \Big),$$

$$F_{2n} = \bigcup_{i>n} C_i^{2n}.$$

Since $\{B_n\}$ is discrete, $\{F_n\}$ is shrinking, so that there is an entourage V with $V(F_n) \subset F_{n-1}$, that is, $V(A_m) \cap A_n = \emptyset$.

COROLLARY. *A normal space is uc if and only if for any decreasing sequence*

$$F_1 \supset G_1 \supset \ldots \supset F_n \supset G_n \supset \ldots$$

of closed F_n and open G_n with vacuous intersection, there is an entourage V with $G_{n-1} \supset V(F_n)$ for $n \geqslant 2$.

We shall now apply Theorem 1 to find the condition under which the product space of metric spaces is uc. We already know the various equivalent conditions a metric space to be uc (**1**; **2**), which are of course easily verified also by using the above theorems. Let us recall some of them for later use; for this, it is convenient to define the following. A subset A of a space S is said to be *uniformly isolated in* S if there is an entourage V in S such that for every point x in A, $V(x)$ contains no point of S except for x.

THEOREM 5 (**2**, Theorem 1). *A metric space S is uc if and only if it satisfies one of the following equivalent conditions.*

(i) *A set of all but finitely many members of a discrete sequence of points in S is uniformly isolated in S.*

(ii) *A set of all but finitely many points of the subset which has no accumulation point in S is uniformly isolated in S.*

LEMMA. *If a product space $S = \Pi_{a\in Z} S_a$ is uc, then so also is $S' = \Pi_{a\in Y} S_a$ for any $Y \subset Z$.*

Proof. We may suppose S' is a uniform subspace of S. We can make up, from a function f on S', an extension g over S by defining $g(x) = f(x')$ for every x of S, where x' is a projection of x on S'.

THEOREM 6. *A product space S of infinitely many metric spaces S_a is uc if and only if*

(i) *S is compact, or*

(ii) *all but finitely many factor spaces are one-point-spaces and* (1) *all are uniformly isolated or* (2) *all are finite except for one which is a non-compact and non-uniformly isolated uc-space.*

Proof. The verification of the "if" part is obvious, and is hence passed over. Let us suppose S is uc and not compact, and so there is a factor space, say S_0, including a discrete sequence $\{x_n\}$ of points; we may then, by the Lemma and Theorem 5, assume that every x_n is isolated. When there is a countable number of spaces $S_1, S_2, \ldots$, every one of which includes two points at least, we take a neighbourhood G_n of a point y_n in S_n for every $n \neq 0$ such that S_n includes a point which does not belong to G_n.

$$H_n = (x_n) \times G_1 \times \ldots \times G_n \times \prod_{a \in Z(n)} S_a,$$

where $Z(n)$ is the set of all indices excepting $0, 1, \ldots, n$, is a neighbourhood of a point p_n of S whose projection on S_0 (or S_i, $1 \leqslant i \leqslant n$) is x_n (or y_i). $\{p_n\}$ is discretely normally separated by $\{H_n\}$, but we cannot find an entourage V in S such that $V(p_n) \subset H_n$ for all n. Consequently all but finitely many factor spaces, for example, excepting $S_0, S_1, \ldots, S_n$, are one-point-spaces. If S_i, $i \neq 0$, has an accumulation point, then $S_0 \times S_i$ includes a discrete sequence of points which are not isolated, this, by Theorem 5, contradicts the Lemma. Therefore all S_i except S_0 have no accumulation point, that is, uniformly isolated by Theorem 5. If S_0 has no accumulation point, it is also uniformly isolated. If S_0 has an accumulation point, S_i, $i \neq 0$, cannot have infinitely many points just as above.

4. Remarks. The condition in order that every *bounded* function on a space is uniformly continuous is well known **(6)**, so Theorem 1 is essentially new for a non pseudo-compact space. In a uc-space, pseudo-compactness follows from precompactness, which is generalized in the following definition. The generalized concept closely relates to a property of a uniform space, which may be called uniform pseudo-compactness, as shown in Theorem 7, a generalization of Theorem 2 of **(2)**.

DEFINITION 3. *Let V be an entourage. The finite sequence of points $x_0, x_1, \ldots, x_m$ satisfying $(x_{i-1}, x_i) \in V$ is said to be a V-*chain with length m. *If for any entourage V there are finitely many points $p_1, \ldots, p_j$ and a positive integer m such that every point of the space can be bound with some p_i by a V-chain with length m, that is,*

$$\bigcup_{i=1}^{j} V^m(p_i) = S,$$

then the space is said to be finitely chainable.

THEOREM 7. *A uniform space is finitely chainable if and only if every uniformly continuous function is bounded.*

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Proof. The verification is essentially similar to that of Theorem 2 of **(2)**. We shall verify the "if" part only. Suppose a uniform space is not finitely chainable, and so there is an entourage V such that for any finite number of points and any positive integer n there is a point which cannot be bound with any one of the points selected above by a V-chain with length n. Let $A_0{}^0$ be a set consisting of a fixed point, and

$$A_0^n = V(A_0^{n-1}),\ V_0 = V,\ V_nV_n \subset V_{n-1},\ V_n^{-1} = V_n,$$

$$U_t = V_{nm}V_{nm-1}\ldots V_{n1},\ t = \sum_{i=1}^{m} 2^{-n_i},\ n_1 < n_2 < \ldots < n_m,$$

then the function

$$f_n(x) = \begin{cases} \sup\{t; x \notin U_t(A_0^{n-1})\}, \\ 0 \qquad \text{on } A_0^{n-1}, \end{cases}$$

is uniformly continuous (cf. (**8**, proof of Theorem 1, p. 13)).

(1) When $A_0{}^n \neq A_0{}^{n-1}$ for all n. Let us put $f(x) = n - 1 + f_n(x)$ for x belonging to $A_0{}^n$ and not to $A_0{}^{n-1}$, then $f(x)$ is uniformly continuous on $A_0 = \bigcup_n A_0{}^n$. To see this, let us suppose y is in $V_m(x)$. Then y belongs to $A_0{}^{n+1}$ and not to $A_0{}^{n-2}$. (i) If y is in $A_0{}^{n-1}$, then $f(y) = n - 2 + f_{n-1}(y) \leqslant f(x)$; $f_{n-1}(y) < 1 - 2^{-m} = t$ implies $y \in U_t(A_0{}^{n-2})$, $x \in V_mU_t(A_0{}^{n-2}) \subset A_0{}^{n-1}$, a contradiction; so we have $f_{n-1}(y) \geqslant 1 - 2^{-m}$. Therefore $|f(x) - f(y)| = f(x) - f(y) \leqslant 2^{-m+1}$. (ii) If y belongs to $A_0{}^n$ and not to $A_0{}^{n-1}$, then $f(y) = n - 1 + f_n(y)$, $|f(x) - f(y)| = |f_n(x) - f_n(y)| < 2^{-m+1}$. (iii) The remaining case for y is similar to (i).

(2) When $A_0{}^n = A_0{}^{n-1}$ for some n. If we can make up an unbounded function which is uniformly continuous on A_1 constructed from a point of the complement of A_0 in the similar way to (1), our proof will be complete.

(3) When we cannot get a desired function on A_m obtained in the same way with (2) for every natural number m, then we put $f(x) = m$ for x in A_m and $= 0$ otherwise. Since the space is not finitely chainable, the function is unbounded and uniformly continuous.

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