

POSITIVE DEFINITE HYPERFUNCTIONS

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§0. Introduction

S. Bochner proved the following theorem in [B].

THEOREM 1 (Bochner). *If f is a continuous function in \mathbf{R}^n then the following conditions are equivalent:*

(i) *f is positive definite, that is, for any $x_1, \dots, x_m \in \mathbf{R}^n$ and for any complex numbers ζ_1, \dots, ζ_m*

$$(0.1) \quad \sum_{j,k=1}^m f(x_j - x_k) \zeta_j \bar{\zeta}_k \geq 0.$$

(ii) *f is the Fourier transform of a positive finite measure μ , i.e.,*

$$(0.2) \quad f(x) = \int e^{-i\lambda \cdot x} d\mu(\lambda).$$

(iii) *For any C^∞ function φ with compact support*

$$(0.3) \quad \iint \overline{f(x-y)} \varphi(x) \overline{\varphi(y)} dx dy = \langle f, \varphi * \varphi^* \rangle \geq 0$$

where $\varphi(x)^* = \overline{\varphi(-x)}$.

The definition (0.1) of the positive definiteness for the continuous functions cannot carry over to generalized functions. Instead, the equivalent definition (0.3) will be used to define the positive definiteness for the space of generalized functions, which can be represented as a dual space of test functions.

Thus the above Bochner theorem was generalized by L. Schwartz for the space of distributions and tempered distributions, which are the dual spaces of the spaces C_c^∞ and the Schwartz space \mathcal{S} as follows.

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THEOREM 2 (Bochner-Schwartz). (i) *Every positive definite distribution is the Fourier transform of a positive tempered measure, and vice versa.*

(ii) *Every positive definite tempered distribution is the Fourier transform of a positive tempered measure, and vice versa.*

Recall that a generalized function u is said to be *positive* if $u(\varphi) \geq 0$ for every nonnegative test function φ and is said to be *positive definite* (or *of positive type* in Schwartz [S]) if $u(\varphi * \varphi^*) \geq 0$ for any positive test function φ . Also, a positive measure μ is said to be *tempered* if for some $p \geq 0$

$$\int (1 + |x|^2)^{-p} d\mu < \infty.$$

Also, note that every positive definite distribution is a positive definite tempered distribution, that is, the class of positive definite distributions and the class of positive definite tempered distributions are the same.

Also, the above theorem are generalized to the more general case of hyperfunctions and Fourier hyperfunctions as follows:

THEOREM 3 [CK]. (i) *Every positive definite Fourier hyperfunction is the Fourier transform of a positive infra-exponentially tempered measure.*

(ii) *Every positive hyperfunction is a measure.*

(iii) *Every positive Fourier hyperfunction is an infra-exponentially tempered measure.*

(iv) *Every positive Aronszajn trace is a measure.*

Here, a positive measure μ is said to be *infra-exponentially tempered* if for every $k > 0$

$$\int e^{-k|x|} d\mu < \infty.$$

Note that Theorem 3 (i) is nothing but the Bochner-Schwartz theorem for the space \mathcal{F}' of Fourier hyperfunctions, which is the dual space of the space \mathcal{F} (see Definition 1.4).

In this paper we prove the Bochner-Schwartz theorem for the hyperfunctions. In other words, every positive definite hyperfunction is the Fourier transform of an infra-exponentially tempered measure, consequently the class of positive definite Fourier hyperfunctions and the class of positive definite hyperfunctions are

the same, which is the parallel result of Theorem 2 for the theory of hyperfunctions.

In order to prove this main result we note that a hyperfunction is defined locally as analytic functions, in other words, that the space \mathcal{B} of hyperfunctions is defined locally as the space of analytic functionals which is the dual space of analytic functions, but not globally. Hence it is difficult to define the *global* concept of positive definiteness for the hyperfunctions. To overcome this difficulty we apply the heat kernel method of T. Matsuzawa as in [M, KCK, CK]. We first make use of the representations of the generalized functions including distributions, hyperfunctions and Fourier hyperfunctions as the initial values of the solutions of the heat equation (see Theorems 1.5 and 1.6) and then we define the positive definite generalized functions in terms of the defining function.

As a consequence of these results we define the positive definite hyperfunctions (see Definition 2.2). As the natural definition of positive definiteness for the hyperfunctions is given we can easily prove the Bochner-Schwartz theorem for the hyperfunctions.

§1. Generalized functions as boundary values of solutions of the heat equation

We first briefly introduce analytic functionals, hyperfunctions and Fourier hyperfunctions. See [H, Ka, KCK] for more details.

DEFINITION 1.1 Let $K \subset \mathbf{R}^n$ be a compact set. Then $A(K)$ is the space of all real analytic functions in some neighborhood of K . In other words, $\varphi \in A(K)$ if φ is a C^∞ function in a neighborhood of K and there are positive constants C and h such that

$$\sup_{x \in K} \frac{|\partial^\alpha \varphi(x)|}{h^{|\alpha|} \alpha!} \leq C.$$

We denote by $A'(K)$ the strong dual space of $A(K)$ and call its element an *analytic functional carried by K* .

Here we use the multi-index notations: $|\alpha| = \alpha_1 + \dots + \alpha_n$ for $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n$ where \mathbf{N}_0 is the set of non-negative integers and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$, $\partial_j = \partial / \partial x_j$.

We set $A'(\mathbf{R}^n) = \cup_K A'(K)$ and the support of $u \in A'(\mathbf{R}^n)$ is the smallest compact set $K \subset \mathbf{R}^n$ such that $u \in A'(K)$.

We now define the space \mathcal{B} of hyperfunctions following A. Martineau as in [H].

DEFINITION 1.2. Let Ω be a bounded open set in \mathbf{R}^n . Then the space $\mathcal{B}(\Omega)$ of hyperfunctions is defined by

$$\mathcal{B}(\Omega) = A'(\bar{\Omega})/A'(\partial\Omega).$$

We now state the localization theorem to define hyperfunctions in every open set in \mathbf{R}^n .

THEOREM 1.3. Let $\Omega_j, j = 1, 2, \dots$, be bounded open subsets of \mathbf{R}^n such that $\Omega = \cup_{j=1}^{\infty} \Omega_j$. If $u_j \in \mathcal{B}(\Omega_j)$ and for all i, j we have $u_i = u_j$ in $\Omega_i \cap \Omega_j$, (that is, $\text{supp}(u_i - u_j) \cap \Omega_i \cap \Omega_j = \emptyset$) then there is a unique $u \in \mathcal{B}(\Omega)$ such that the restriction of u to Ω_j , is equal to u_j .

We introduce a real version of the Fourier hyperfunctions.

DEFINITION 1.4 [KCK]. (i) We denote by \mathcal{F} the set of all infinitely differentiable functions φ in \mathbf{R}^n such that for some $h, k > 0$

$$|\varphi|_{h,k} = \sup_{\substack{x \in \mathbf{R}^n \\ \alpha \in \mathbf{N}_0^n}} \frac{|\partial^\alpha \varphi(x)| \exp k|x|}{h^{|\alpha|} \alpha!} < \infty.$$

(ii) We say that $\varphi_j \rightarrow 0$ as $j \rightarrow \infty$ if $|\varphi_j|_{h,k} \rightarrow 0$ as $j \rightarrow \infty$ for some $h, k > 0$.

(iii) We denote by \mathcal{F}' the strong dual of \mathcal{F} and call its elements *Fourier hyperfunctions*.

We denote by $E(x, t)$ the n -dimensional heat kernel

$$E(x, t) = \begin{cases} (4\pi t)^{-n/2} \exp(-|x|^2/4t), & t > 0 \\ 0, & t \leq 0. \end{cases}$$

Note that $E(x, t)$ belongs to the space \mathcal{F} for each $t > 0$. Thus

$$(1.1) \quad U(x, t) = u_y(E(x - y, t))$$

is well defined in $\mathbf{R}_+^{n+1} = \{(x, t) \mid x \in \mathbf{R}^n, t > 0\}$ for all $u \in \mathcal{F}'$ and called the *defining function* of u . We now represent some generalized functions as the initial values of smooth solutions of heat equation.

THEOREM 1.5 [M]. (i) Let $u \in \mathcal{D}'(\mathbf{R}^n)$. Then there exists $U(x, t) \in C^\infty(\mathbf{R}_+^{n+1})$ and satisfies the following conditions:

$$(1.2) \quad (\partial/\partial t - \Delta)U(x, t) = 0 \text{ in } \mathbf{R}_+^{n+1}.$$

For any compact set $K \subset \mathbf{R}^n$ there exist positive integers $N = N(K)$ and C_K such that

$$(1.3) \quad |U(x, t)| \leq C_K t^{-N}, \quad t > 0, x \in K$$

and $U(x, t) \rightarrow u$ as $t \rightarrow 0^+$ in the sense that for every $\varphi \in C_c^\infty$

$$(1.4) \quad u(\varphi) = \lim_{t \rightarrow 0^+} \int U(x, t) \varphi(x) dx$$

Conversely, let $U(x, t) \in C^\infty(\mathbf{R}_+^{n+1})$ satisfy (1.2) and (1.3). Then there exists a unique $u \in \mathcal{D}'(\mathbf{R}^n)$ satisfying (1.4).

(ii) Let $u \in \mathcal{B}(\mathbf{R}^n)$. Then $U(x, t) \in C^\infty(\mathbf{R}_+^{n+1})$ satisfies the heat equation, and the following conditions: For every compact subset $K \subset \mathbf{R}^n$ and for every $\varepsilon > 0$ there exists a constant $C_{\varepsilon, K} > 0$ such that

$$(1.5) \quad |U(x, t)| \leq C_{\varepsilon, K} \exp(\varepsilon/t), \quad t > 0, x \in K$$

and

$$(1.6) \quad U(x, t) \rightarrow u \text{ as } t \rightarrow 0^+$$

in the sense that $U(x, t) - U_j(x, t) \rightarrow 0$ as $t \rightarrow 0^+$ in $\Omega_j, j = 1, 2, \dots$, where U_j is the defining function of u_j as in (1.1) and $u = (u_j) \in \mathcal{B}(\mathbf{R}^n)$, and $\mathbf{R}^n = \cup_j \Omega_j$.

The converse is also true as in (i)

(iii) Let $u \in \mathcal{S}'(\mathbf{R}^n)$. Then $U(x, t) = u_y(E(x - y, t))$ belongs to $C^\infty(\mathbf{R}_+^{n+1})$ and satisfies (1.2) and the following condition: There exist positive constants C, M and N such that

$$(1.7) \quad |U(x, t)| \leq Ct^{-M}(1 + |x|)^N \text{ in } \mathbf{R}_+^{n+1}.$$

and $U(x, t) \rightarrow u$ as $t \rightarrow 0^+$ in the following sense: for every $\varphi \in \mathcal{S}$

$$u(\varphi) = \lim_{t \rightarrow 0} \int U(x, t) \varphi(x) dx.$$

Conversely, every C^∞ -function defined in \mathbf{R}_+^{n+1} satisfying (1.2) and (1.7) can be expressed in the form $U(x, t) = u_y(E(x - y, t))$ for some $u \in \mathcal{S}'$.

THEOREM 1.6 [KCK]. *Let $u \in \mathcal{F}'(\mathbf{R}^n)$. Then the defining function $U(x, t)$ satisfies (1.2), (1.4) for every $\varphi \in \mathcal{F}$ and the following growth condition; for every $\varepsilon > 0$ there exists a constant $C_\varepsilon > 0$ such that*

$$(1.8) \quad |U(x, t)| \leq C_\varepsilon \exp[\varepsilon(|x| + 1/t)]$$

for $t > 0$, $x \in \mathbf{R}^n$.

Conversely, let $U(x, t) \in C^\infty(\mathbf{R}_+^{n+1})$ satisfy (1.2) and (1.8). Then there exists a unique $u \in \mathcal{F}'(\mathbf{R}^n)$ such that $U(x, t) = u_y(E(x - y, t))$.

§2. Bochner–Schwartz theorem for hyperfunctions

If a continuous function $f(x)$ is positive definite then it is easy to see that

$$f(-x) = \overline{f(x)}, \quad f(0) \geq 0$$

and

$$(2.1) \quad |f(x)| \leq f(0).$$

We first restate the Bochner–Schwartz theorem for the Fourier hyperfunctions.

Theorem 2.1 [CK]. *Every positive definite Fourier hyperfunction is the Fourier transform of a positive infra-exponentially tempered measure μ in the sense that*

$$(2.2) \quad u(\varphi) = \int \widehat{\varphi}(\lambda) d\mu, \quad \varphi \in \mathcal{F},$$

where $\widehat{\varphi}(\lambda)$ denotes the Fourier transform of $\varphi(x)$.

Conversely, the functional u defined by (2.2) is a positive definite Fourier hyperfunction.

We are now in a position to define the positive definite hyperfunction in terms of the defining function and the growth condition.

DEFINITION 2.2. A hyperfunction u is *positive definite* if there exists a defining function $U(x, t)$ of u is a positive definite function for each $t > 0$, that is,

$$\sum_{j,k=1}^n U(x_j - x_k, t) \zeta_j \bar{\zeta}_k \geq 0$$

for every $x_1, \dots, x_n \in \mathbf{R}^n$, $\zeta_1, \dots, \zeta_n \in \mathbf{C}$ and for each $t > 0$.

To justify the above definition for the positive definiteness of the hyperfunctions we prove the equivalence of our new definitions and the original definitions of the positive definiteness for the continuous functions, distributions, tempered

distributions and Fourier hyperfunctions respectively.

THEOREM 2.3. *Let $f(x)$ be a continuous positive definite function and $E_t(x) = E(x, t)$ be the heat kernel. Then $E_t * f$ is well defined for each $t > 0$ and is the Fourier transform of a measure $\exp(-t\lambda^2)d\mu(\lambda)$, where $d\mu$ is a finite positive measure, whence $E_t * f$ is a positive definite function, and vice versa.*

Proof. Let f be a positive definite continuous function. Then by the Bochner theorem we have

$$\begin{aligned}
 (2.2) \quad (E_t * f)(x) &= \int E(y, t) f(x - y) dy \\
 &= \iint E(y, t) e^{-i(x-y)\cdot\lambda} d\mu(\lambda) dy \\
 &= \int \left(\int E(y, t) e^{-iy\cdot\lambda} d\mu \right) e^{-ix\cdot\lambda} d\mu(\lambda) \\
 &= \int \tilde{E}(-\lambda, t) e^{-ix\cdot\lambda} d\mu(\lambda) \\
 &= \int \exp(-t\lambda^2) e^{-ix\cdot\lambda} d\mu(\lambda),
 \end{aligned}$$

where $\tilde{E}(\lambda, t)$ is the partial Fourier transform of $E(x, t)$ with respect to x . Thus $E_t * f$ is the Fourier transform of the measure $\exp(-t\lambda^2)d\mu(\lambda)$ where $d\mu$ is positive-finite measure. By Theorem 1, $E_t * f$ is a continuous positive definite function. The converse is also proved by (2.2).

The following theorem gives a new definition of the positive definite tempered distribution.

THEOREM 2.4. *Let $u \in \mathcal{S}'(\mathbf{R}^n)$. Then the following conditions are equivalent:*

- (i) *u is a positive definite tempered distribution.*
- (ii) *The defining function $U(\cdot, t)$ of u is a positive definite function for each $t > 0$.*

We first need the following lemma which is used to prove the Bochner theorem in [GS, pp.153–155].

LEMMA 2.5 [GS]. *A positive definite continuous function is a positive definite tempered distribution. Conversely, if a continuous function $f(x)$ is positive definite tem-*

pered distribution, that is,

$$f(\varphi * \varphi^*) = \int \overline{f(x)} (\varphi * \varphi^*)(x) dx \geq 0$$

for all $\varphi \in \mathcal{S}$ then $f(x)$ is a positive definite function.

Proof of Theorem 2.4. By Theorem 2.1 it suffices to prove the implication

(i) \Rightarrow (ii) that for all $\varphi \in \mathcal{S}$ and each $t > 0$

$$\langle U(\cdot, t), \varphi * \varphi^* \rangle \geq 0.$$

Let E_t denote the n -dimensional heat kernel $E(x, t)$. Then

$$\begin{aligned} \langle U(\cdot, t), \varphi * \varphi^* \rangle &= \langle u * E_t \varphi * \varphi^* \rangle \\ &= \langle u, E_t * \varphi * \varphi^* \rangle \\ &= \langle u, (E_{t/2} * \varphi) * (E_{t/2} * \varphi)^* \rangle \\ &\geq 0, \end{aligned}$$

since $E_{t/2} * \varphi \in \mathcal{S}$ and u is a positive definite tempered distribution.

Conversely, let $U(x, t)$ be a positive definite defining function. Then for each $t > 0$ we have $\langle U(\cdot, t), \varphi * \varphi^* \rangle \geq 0$ for all $\varphi \in \mathcal{S}$ by Theorem 1. Also it follows from Theorem 1.5 (iii) that

$$\begin{aligned} \langle u, \varphi * \varphi^* \rangle &= \lim_{t \rightarrow 0^+} \int U(x, t) (\varphi * \varphi^*)(x) dx \\ &= \lim_{t \rightarrow 0^+} \langle U(\cdot, t), \varphi * \varphi^* \rangle \\ &\geq 0. \end{aligned}$$

Thus u is a positive definite tempered distribution. This completes the proof.

The following theorem gives a new definition of the positive definite distributions.

THEOREM 2.6. *Let $u \in \mathcal{D}'(\mathbf{R}^n)$. Then the following conditions are equivalent:*

- (i) u is a positive definite distribution.
- (ii) The defining function $U(\cdot, t)$ of u is a positive definite continuous function for each $t > 0$.

Proof. If u is a positive definite distribution, then u is a positive definite tempered distribution by the Bochner-Schwartz theorem. The defining function $U(x, t)$ is given by $U(x, t) = (u * E_t)(x)$ which is a positive definite continuous function for each $t > 0$ by Theorem 2.4. Conversely, if the defining function $U(x, t)$ is a positive definite function for each $t > 0$ then $U(x, t)$ is a positive definite generalized function on \mathcal{S} for each $t > 0$. Thus $\langle U(\cdot, t), \varphi * \varphi^* \rangle \geq 0$ for all $\varphi \in C_c^\infty$ and for each $t > 0$. By Theorem 1.5 (i) we have

$$\begin{aligned} \langle u, \varphi * \varphi^* \rangle &= \lim_{t \rightarrow 0^+} \int U(x, t) (\varphi * \varphi^*)(x) dx \\ &= \lim_{t \rightarrow 0^+} \langle U(\cdot, t), \varphi * \varphi^* \rangle \geq 0. \end{aligned}$$

This completes the proof.

Applying the same method as in Theorem 2.4 we obtain the following theorem.

THEOREM 2.7. *Let $u \in \mathcal{F}'(\mathbf{R}^n)$. Then the following conditions are equivalent:*

- (i) u is a positive definite Fourier hyperfunction.
- (ii) The defining function $U(\cdot, t)$ of u is a positive definite function for each $t > 0$.

By virtue of the previous theorems we give new definitions for the positive definite distributions, positive definite tempered distributions and positive definite Fourier hyperfunctions as in Theorem 2.6 (ii), Theorem 2.5 (ii) and Theorem 2.7 (ii), respectively. In accordance with these new definitions we define the positive definite hyperfunctions as Definition 2.2.

We are now in a position to state and prove the main theorem.

THEOREM 2.8. *The following conditions are equivalent:*

- (i) u is a positive definite hyperfunction.
- (ii) u is a positive definite Fourier hyperfunction.

Proof. Let u be a positive definite Fourier hyperfunction. Then by Theorem 2.7, the defining function $U(x, t)$ is a positive definite function for each $t > 0$ and attains its maximum at the origin. Also by Theorem 1.6 the defining function $U(x, t)$ satisfies (1.8). Thus we have the following growth condition: for every $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that

$$|U(x, t)| \leq U(0, t) \leq C_\varepsilon \exp(\varepsilon/t).$$

Thus $U(x, t)$ defines a hyperfunction by Theorem 1.5 (ii), that is, u is a hyperfunction. The converse assertion is clear by putting $K = \{0\}$ in (1.5).

Combining the Theorem 1.5 and Theorem 1.6 with the Bochner-Schwartz theorem we have the following result:

COROLLARY 2.9. *The following conditions are equivalent:*

- (i) u is a positive definite distribution.
- (ii) u is a positive definite tempered distribution.
- (iii) u is the Fourier transform of a positive tempered measure.
- (iv) The defining function $U(\cdot, t)$ of $u \in \mathcal{D}'$ is a positive definite function for each $t > 0$.
- (v) The defining function $U(\cdot, t)$ of $u \in \mathcal{S}'$ is a positive definite function for each $t > 0$.

As a parallel result of the Bochner-Schwartz theorem we have the following:

THEOREM 2.10. *The following conditions are equivalent:*

- (i) u is a positive definite hyperfunction.
- (ii) u is a positive definite Fourier hyperfunction.
- (iii) u is the Fourier transform of a positive infra-exponentially tempered measure.
- (iv) The defining function $U(\cdot, t)$ of $u \in \mathcal{F}'$ is a positive definite function for each $t > 0$.

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