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EPICOMPLETE ARCHIMEDEAN LATTICE-ORDERED GROUPS

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In this paper we give the structure of an $\aleph_i$-complete ℓ-group and the epicomplete objects in the category $\mathcal{A}^\ell$.

1. INTRODUCTION

We determined the structure of a complete ℓ-group in [**8**] and the structure of an Archimedean ℓ-group in [**9**]. In this paper we will determine the structue of an $\aleph_i$-complete ℓ-group, in particular, of a σ-complete ℓ-group.

Let $\mathcal{L}$ be the category of all ℓ-homomorphisms between abelian ℓ-groups. In [**1**], Anderson and Conrad determined the epicomplete objects in $\mathcal{L}$. An object G in $\mathcal{L}$ is epicomplete if and only if it is divisible. Let $\mathcal{A}^\ell$ be the category of all ℓ-homomorphisms between Archimedean ℓ-groups. In [**2, 3**], Ball and Hager determined the epicomplete objects in $\mathcal{A}^\ell$. An object G^ℓ in $\mathcal{A}$ is epicomplete if and only if G is divisible and σ-complete and σ-laterally complete (meaning each countable subset of positive elements of G which is either bounded or pairwise disjoint has a supremum). In this paper we will give the structure of the epicomplete objects in $\mathcal{A}^\ell$.

Our general terminology and notation are standard, as in [**5**]; for the special notations to be discussed here the reader may refer to [**8, 9**].

2. THE STRUCTURE OF AN $\aleph_i$-COMPLETE ℓ-GROUP

Let G be an ℓ-group. We denote the least cardinal α such that $|A| \leqslant \alpha$ for each bounded disjoint subset A of G by vG, where $|A|$ denotes the cardinal of A. G is said to be v-homogeneous if $vH = vG$ for any convex ℓ-subgroup $H \neq \{0\}$ of the ℓ-group G. Let G be a v-homogeneous ℓ-group and $\aleph_i$ a cardinal number. If $vG = \aleph_i$, we call G an ℓ-group of $\aleph_i$ *type*. For example, an ℓ-group of countable type is an ℓ-group of $\aleph_0$. (For the definition of an ℓ-group of countable type the reader may consult [**10**].) The free abelian ℓ-group A_η of rank η ($\eta > 1$) is an ℓ-group of $\aleph_0$ type (see Proposition 8.1 in [**9**]). A Riesz space (vector lattice) V is said to be of $\aleph_i$ type if it is an ℓ-group of $\aleph_i$ type.

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Lemma 2.1. *Any Archimedean ℓ-group G of $\aleph_i$ type with a weak unit can be embedded into a complete Riesz space of $\aleph_i$ type.*

Proof: Let $G^\frown$ be the Dedekind-MacNeille completion of G. It follows from Proposition 2.12 in [8] that $G^\frown$ is a complete ℓ-group of $\aleph_i$ type. From 1.16 in [6] we see that $G^\frown$ can be embedded into a complete Riesz space $U(G^\frown)$; that is

$$G \to G^\frown \to Z(G^\frown) \to U(G^\frown), \tag{1}$$

where $Z(G^\frown) = \{\frac{x}{n} \mid x \in G^\frown \& \; n \in \mathbb{N}\}$ and $U(G^\frown)$ is the Dedekind-MacNeille completion of $Z(G^\frown)$. Now we prove that $Z(G^\frown)$ and $U(G^\frown)$ are both of $\aleph_i$ type. Without loss of generality, from Proposition 2.2 in [8] we may assume

$$\sum_{\delta\in\Delta} T_\delta \subseteq G^\frown \subseteq^* \prod_{\delta\in\Delta} T_\delta \tag{2}$$

where each T_δ $(\delta \in \Delta)$ is a real group, or an integer group, or a complete v-homogeneous ℓ-group of $\aleph_i$ type. Put $\overline{T}_\delta = \{g \in G^\frown \mid \delta' \neq \delta \implies g_{\delta'} = 0\}$ for $\delta \in \Delta$. Then $\overline{T}_\delta$ is a convex ℓ-subgroup of $G^\frown$ for $\delta \in \Delta$. So $v\overline{T}_\delta = vG^\frown = \aleph_i$. Hence each T_δ $(\delta \in \Delta)$ is a continuous complete ℓ-group of $\aleph_i$ type. From (1) we have

$$\sum_{\delta\in\Delta} T_\delta \subseteq G^\frown \subseteq Z(G^\frown) \subseteq' \prod_{\delta\in\Delta} Z(T_\delta), \tag{3}$$

where $Z(T_\delta) = \{\frac{x_\delta}{n} \mid x_\delta \in T_\delta \; \& \; n \in \mathbb{N}\}$. Let $\{x^\alpha \mid \alpha \in A\}$ be a disjoint subset with an upper bound in $Z(G^\frown)$. Then there exists a division of Δ from formula (3)

$$\Delta = \left(\bigcup_{\alpha\in A} \Delta_\alpha\right) \bigcup \Delta',$$

where $\Delta_\alpha = \{\delta \in \Delta \mid x_\delta^\alpha \neq 0\}$, and $\Delta' = \{\delta \in \Delta \mid (\forall \alpha \in A)(x_\delta^\alpha = 0)\}$. It is clear that, if $\alpha \neq \alpha'$ then $\Delta_\alpha \cap \Delta_{\alpha'} = \emptyset$. Let x be a weak unit in G. Then x is also a weak unit in $G^\frown$. In fact, for any $y \in G^\frown$, $y = \bigvee_{\alpha\in A} \{y_\alpha \in G \mid \alpha \in A\}$. Then $0 = x \wedge y = x \wedge \left(\bigvee_{\alpha\in A} y_\alpha\right) = \bigvee_{\alpha\in A} (x \wedge y_\alpha)$ implies $x \wedge y_\alpha = 0$ for all $\alpha \in A$. So $y = 0$. Let $\overline{x}_\alpha$ be the element whose δ component is x_δ and all other components are zero. Put

$$\overline{x}_\delta^\alpha = \begin{cases} x_\delta & \delta \in \Delta_\alpha \\ 0 & \delta \overline{\in} \Delta_\alpha, \end{cases}$$

then $\overline{x}^\alpha = (\ldots \overline{x}_\delta^\alpha \ldots) = \bigvee_{\delta\in\Delta_\alpha} (G^\frown)_{\overline{x}_\alpha}$, hence $\{\overline{x}^\alpha \mid \alpha \in A\}$ is a disjoint subset with an upper bound x in $G^\frown$. So $|A| \leqslant \aleph_i$ and $vZ(G^\frown) \leqslant \aleph_i$. On the other hand, since

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$G^{\frown} \subseteq Z(G^{\frown})$, we have $vZ(G^{\frown}) \geqslant vG^{\frown} = \aleph_i$. Therefore, $Z(G^{\frown})$ is of $\aleph_i$ type. From Proposition 2.12 in [8] we see that $v\overline{\mathrm{U}}(G^{\frown}) = \aleph_i$. ∎

A lattice L is called $\aleph_i$*-complete* if each of its non-empty subsets with cardinality number $\alpha \leqslant \aleph_i$ has a supremum and an infimum. A Boolean algebra X is said to be $\aleph_i$-complete if X is an $\aleph_i$-complete lattice. A Boolean algebra $\mathcal{P}$ is called *an algebra of* $\aleph_i$ *type* if the cardinal number of each disjoint subset is at most $\aleph_i$.

LEMMA 2.2. *Let $\mathcal{P}$ be an $\aleph_i$-complete Boolean algebra of $\aleph_i$ type. Then in an arbitrary infinite subset E of $\mathcal{P}$ there exists a subset $E' \subseteq E$ with $|E'| \leqslant \aleph_i$ such that*

$$\vee E' = \vee E, \qquad \wedge E' = \wedge E.$$

PROOF: The proof is similar to the proof of Theorem VI.1.1 in [**10**]. Let E be an arbitrary infinite subset of $\mathcal{P}$. We denote by $\mathcal{N}$ the set of all subsets $N \subseteq \mathcal{P}$ with $|N| \leqslant \aleph_i$ possessing the following properties:

a) N is a disjoint set;

b) if $e \notin N$, then there exists an $e_1 \in E$ such that $e \leqslant e_1$.

The set $\mathcal{N}$ is non-empty. We assume that $\mathcal{N}$ is ordered by inclusion. We will show that $\mathcal{N}$ satisfies the condition of Zorn's Lemma. In fact, if $\mathcal{N}' \subset \mathcal{N}$ $\mathcal{N}' = \{N_\alpha \mid \alpha \in A\}$ and $\mathcal{N}'$ is a chain, then we put $N' = \bigcup_{\alpha \in A} N_\alpha$. Since for arbitrary e_1, $e_2 \in N'$ an index α can be found for which e_1, $e_2 \in N_\alpha$, then N' consists of pairwise disjoint elements and hence $|N'|$ is at most $\aleph_i$ and $N' \in \mathcal{N}$. By Zorn's Lemma, there exists a maximal set N_0 in $\mathcal{N}$. Let $e_0 = \vee N_0$. We can show that $e_0 = \vee E$. Suppose that there exists $e_1 \in E$ such that e_1 is not $\leqslant e_0$. Then $e_1 \wedge e_0' = e > 0$ (see Theorem II.5.2 a), b) in [**10**]). Now, adjoining the element e to the set N_0, we obtain a set which also occurs in $\mathcal{N}$ which contradicts the maximality of N_0. On the other hand, since N_0 satisfies the condition b), then there exists a subset E' in E with $|E'| \leqslant |N_0| \leqslant \aleph_i$ such that $e_0 \leqslant \vee E'$. Consequently, $e_0 \vee E' = \vee E$. ∎

A Riesz space X is called a space of $\aleph_i$ type if the cardinal number of each bounded disjoint subset is at most $\aleph_i$.

LEMMA 2.3. *Let X be a Dedekind complete Riesz space of $\aleph_i$ type. Then in every infinite subset $E \subseteq X$ which is bounded above (below), there exists a subset $E' \subseteq E$ with $|E'| \leqslant \aleph_i$ such that $\vee E' = \vee E$ ($\wedge E' = \wedge E$).*

PROOF: This is similar to the proof of Theorem VI.2.2 in [**10**], using Lemma 2.2 to replace Theorem VI.1.1. ∎

LEMMA 2.4. *In any Archimedean ℓ-group G of $\aleph_i$ type with a weak unit, if $z = \bigvee_{\alpha \in A} (G)_{z_\alpha}$, then there exists a subset $\{z_{\alpha'} \mid \alpha' \in A\}$ with $|A'| \leqslant \aleph_i$ of $\{z_\alpha \mid \alpha \in A\}$*

such that $z = \bigvee_{\alpha' \in A'}{}^{(G)} z_{\alpha'}$.

PROOF: Let G be an Archimedean ℓ-group of $\aleph_i$ type with a weak unit. From Lemma 2.1 we see G can be embedded into a complete Riesz space $U(G)$ of $\aleph_i$ type according to the process of (1). Since $U(G)$ is a regular extension of $Z(G)$, we can assume

$$z' = \bigvee_{\alpha \in A}{}^{(Z(G))} z_\alpha = \bigvee_{\alpha \in A}{}^{(U(G))} z_\alpha. \tag{4}$$

By Lemma 2.3 there exists a subset $\{z_{\alpha'} \mid \alpha' \in A'\}$ with $|A'| \leqslant \aleph_i$ of $\{z_\alpha \mid \alpha \in A\}$ such that

$$z' = \bigvee_{\alpha' \in A'}{}^{(Z(G))} z_{\alpha'} = \bigvee_{\alpha' \in A'}{}^{(U(G))} z_{\alpha'}. \tag{5}$$

We denote the set of all upper bounds of the subset M of G in $Z(G)$ by $M^\star_{Z(G)}$ and the set of all upper bounds of M in G by $M^\star_G$. From (4) and (5) we have

$$\{z_\alpha \mid \alpha \in A\}^\star_{Z(G)} = \{z_{\alpha'} \mid \alpha' \in A'\}^\star_{Z(G)},$$

then

$$\{z_\alpha \mid \alpha \in A\}^\star_{Z(G)} \cap G = \{z_{\alpha'} \mid \alpha' \in A'\}^\star_{Z(G)} \cap G,$$
$$\{z_\alpha \mid \alpha \in A\}^\star_G = \{z_{\alpha'} \mid \alpha' \in A\}^\star_G.$$

Therefore

$$\bigvee_{\alpha' \in A'}{}^{(G)} z_{\alpha'} = \bigvee_{\alpha \in A}{}^{(G)} z_\alpha = z.$$

∎

An ℓ-group G is said to be $\aleph_i$-*complete* if each upper bounded subset E with $|E| \leqslant \aleph_i$ in G has a least upper bound. For example, a σ-complete ℓ-group G is $\aleph_0$-complete. Since a σ-complete ℓ-group is Archimedean, if $\aleph_i \geqslant \aleph_0$, an $\aleph_i$-complete ℓ-group is Archimedean.

LEMMA 2.5. *Any $\aleph_i$-complete ℓ-group of $\aleph_i$ type with a weak unit is complete.*

PROOF: Let G be an $\aleph_i$-complete ℓ-group of $\aleph_i$ type with a weak unit ε. Then G is an Archimedean ℓ-group of $\aleph_i$ type with a weak unit ε and G has a Dedekind completion $G^\wedge$. Let $\{x^\alpha \mid \alpha \in A\}$ be an arbitrary upper bounded subset in G. Assume

$$x = \bigvee_{\alpha \in A}{}^{(G^\wedge)} x^\alpha.$$

By Proposition 2.12 in [8] we see that $G^\frown$ is also an ℓ-group of $\aleph_i$ type. And ε is also a weak unit in $G^\frown$. So, by Lemma 2.4, there exists a subset $\{x^{\alpha'} \mid \alpha' \in A'\}$ with $|A'| \leqslant \aleph_i$ of $\{x^\alpha \mid \alpha \in A\}$ such that

$$x = \bigvee_{\alpha' \in A'} {}^{(G^\frown)} x^{\alpha'}.$$

Therefore

$$x = \bigvee_{\alpha' \in A'} {}^{(G)} x^{\alpha'}.$$

∎

LEMMA 2.6. *Let G be an Archimedean ℓ-group. If the completion $G^\frown$ has a weak unit, then G also has a weak unit.*

PROOF: Let ε be a weak unit in $G^\frown$. From Theorem 2.4 in [4] there exists ε' in G such that $\varepsilon' \geqslant \varepsilon$. Then ε' is a weak unit in G. Because, if $\varepsilon' \wedge x = 0$ for $x \in G$, then $\varepsilon \wedge x = 0$. Hence $x = 0$. ∎

Let ℓ-group G be a subdirect sum of $\{G_\delta \mid \delta \in \Delta\}$. If there exists a subset $\Delta_1 \subseteq \Delta$ such that $\sum_{\delta \in \Delta_1} G_\delta \subseteq G$, then we call G a *semicomplete subdirect sum* of $\{G_\delta \mid \delta \in \Delta\}$.

An ℓ-group G is said to be *projectable* or a *P-group*, if $G = g^{\perp\perp} \boxplus g^\perp$ for each $g \in G$, where $g^\perp = \{g\}^\perp = \{x \in G \mid |g| \wedge |x| = 0\}$ and $g^{\perp\perp} = \left(g^\perp\right)^\perp$. It is well-known that any σ-complete ℓ-group G is projectable. So any $\aleph_i$-complete ℓ-group G is projectable for $\aleph_i \geqslant \aleph_0$. An Archimedean ℓ-group G is said to be *continuous*, if for any strictly positive element x we have $x = x_1 + x_2$ and $x_1 \wedge x_2 = 0$, where $x_1 \neq 0$ and $x_2 \neq 0$.

LEMMA 2.7. *Any $\aleph_i$-complete ℓ-group G of $\aleph_j$ type with $\aleph_i$ and $\aleph_j \geqslant \aleph_0$ is continuous.*

PROOF: Let $0 < x \in G$. Since $v[x] = vG = \aleph_j$, by 4.3 in [7], $[0, x]$ is not a chain, and there exists $0 < x_1 < x$ and $0 < x_2 < x$ such that $x_2 \wedge x_1 = 0$. It is clear that $x^{\perp\perp}$ is also $\aleph_i$-complete. In fact, if $\{x^\alpha \mid \alpha \in A\}$ is a subset in $x^{\perp\perp}$ with $|A| \leqslant \aleph_i$ and $x^\alpha \leqslant \overline{x} \in x^{\perp\perp}$ for $\alpha \in A$, then there exists $\bigvee_{\alpha \in A} {}^{(G)} x^\alpha = x_0 \in G$. Since $\overline{x} \wedge y = 0$ for each $y \in x^\perp$, $x_0 \wedge y = 0$ for each $y \in x^\perp$ and so $x_0 \in x^{\perp\perp}$. Because G is projectable, we have

$$x^{\perp\perp} = x_1^{\perp\perp} \boxplus x_1^\perp. \tag{6}$$

It is easy to see that $x \overline{\in} x_1^{\perp\perp}$. In fact, if $x \in x_1^{\perp\perp}$, then $x^{\perp\perp} \subseteq x_1^{\perp\perp}$, hence $x^{\perp\perp} = x_1^{\perp\perp}$. But $x_2 \in x_1^\perp \subseteq x^{\perp\perp}$, giving a contradiction. On the other hand, since $x_1 \overline{\in} x_1^\perp$, we have

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$x \overline{\in} x_1^{\perp}$. From (6) we see that there exist $0 < x^1 \in x_1^{\perp\perp}$ and $0 < x^2 \in x_1^{\perp}$ such that $x = x^1 + x^2$. Therefore G is continuous. ∎

THEOREM 2.8. *Any $\aleph_i$-complete ℓ-group G is ℓ-isomorphic to an semicomplete subdirect sum of real groups, integer groups, continuous complete ℓ-groups of $\aleph_i$ type and continuous $\aleph_i$-complete ℓ-groups of $\aleph_j$ type with $\aleph_j > \aleph_i$.*

PROOF: We will proceed in three steps.

(1) Let G be an $\aleph_i$-complete ℓ-group. Then G has a Dedekind completion $G^\frown$. From Theorem III.4.4 and Theorem III.4.6 in **[11]** we see that

(7) $$G^\frown \subseteq^* \prod_{\lambda \in \Lambda} G_\lambda,$$

where G_λ is a complete ℓ-group with a weak unit x_λ for $\lambda \in \Lambda$. From Proposition 2.2 in [**8**], without loss of generality, we have

(8) $$G_\lambda \subseteq^* \prod_{\lambda_\delta \in \Delta_\lambda} G_{\lambda_\delta}$$

for each $\lambda \in \Lambda$, where G_{λ_δ} ($\lambda_\delta \in \Delta_\lambda$) is a real group, or an integer group or a continuous v-homogeneous complete ℓ-group. We can show that x_{λ_δ} is a weak unit in G_{λ_δ} for each $\lambda_\delta \in \Delta_\lambda$. In fact, let $y_{\lambda_\delta} \in G_{\lambda_\delta}$ and $x_{\lambda_\delta} \wedge y_{\lambda_\delta} = 0$. Let $\overline{y}_{\lambda_\delta}$ be the element in G_{λ_δ} whose λ_δ component is y_{λ_δ} and all other components are zero. Then $x_\lambda \wedge \overline{y}_{\lambda_\delta} = 0$ and so $\overline{y}_{\lambda_\delta} = 0$, therefore $y_{\lambda_\delta} = 0$.

From (7) and (8) we have

$$G^\frown \subseteq^* \prod_{\lambda \in \Lambda} G_\lambda \subseteq^* \prod_{\lambda \in \Lambda} \left(\prod_{\lambda_\delta \in \Delta_\lambda} G_{\lambda_\delta} \right) \subseteq^* \prod_{\substack{\lambda \in \Lambda \\ \lambda_\delta \in \Delta_\lambda}} G_{\lambda_\delta}.$$

Putting $\Delta = \bigcup_{\lambda \in \Lambda} \Delta_\lambda$ and $T_\delta = G_{\lambda_\delta}$ for each $\delta \in \Delta$, we get

$$G^\frown \subseteq^* \prod_{\delta \in \Delta} T_\delta.$$

Let ρ_δ be the projection from $\prod_{\delta \in \Delta} T_\delta$ to T_δ and $T'_\delta = G\rho_\delta$ for each $\delta \in \Delta$. Then

(9) $$G \subseteq' \prod_{\delta \in \Delta} T'_\delta,$$

where each T_δ is a real group, or an integer group or a continuous v-homogenous complete ℓ-group with a weak unit and T'_δ is a subgroup of reals or a v-homogenous Archimedean ℓ-group with a weak unit for $\delta \in \Delta$ (see Lemma 2.6).

(2) It is clear that the projection ρ_δ from G onto T'_δ is complete. Let $\{x^\alpha_\delta \mid \alpha \in A\}$ be a subset in T'_δ with $|A| \leqslant \aleph_i$ and $x^\alpha_\delta \leqslant x_\delta \in T'_\delta$ for $\alpha \in A$. Let x^α be the element in G whose δ component is x^α_δ and x the element in G whose δ component is x_δ. Put $y^\alpha = x^\alpha \wedge x$. Then $\{y^\alpha \mid \alpha \in A\}$ is a subset of G with an upper bound x and $|A| \leqslant \aleph_i$. Thus there exists $y = \bigvee_{\alpha\in A}{}^{(G)} y^\alpha$. Clearly, $y^\alpha \rho_\delta = x^\alpha_\delta$. Put $y\rho_\delta = y_\delta \in T'_\delta$. Then $y_\delta = \bigvee_{\alpha\in A}{}^{(T'_\delta)} x^\alpha_\delta$. So each T'_δ is $\aleph_i$-complete for $\delta \in \Delta$. If T'_δ is a subgroup of the reals, then T'_δ is $\mathbf{R}$ or $\mathbf{Z}$. If T'_δ is not a subgroup of the reals, then T'_δ is v-homogeneous. Suppose $vT'_\delta = \aleph_j$. If $\aleph_j = \aleph_i$, T'_δ is an $\aleph_i$-complete ℓ-group of $\aleph_i$ type. It follows from Lemma 5 that T'_δ is complete for those $\delta \in \Delta$ for which T'_δ is of $\aleph_i$ type. It then follows from Lemma 2.7 that each T'_δ is continuous for $\delta \in \Delta$.

(3) Finally we prove G is a semicomplete subdirect sum. Put $\Delta_1 = \{\delta \in \Delta \mid T'_\delta$ is $\mathbf{R}$ or $\mathbf{Z}$ or a continuous complete ℓ-group of $\aleph_i$ type$\}$. For each $\delta \in \Delta_1$, set

$$\overline{T}'_\delta = \{g \in G \mid \delta' \neq \delta \implies g_{\delta'} = 0\}.$$

Let $\overline{z}_\delta$ be the strictly positive element in $\overline{T}'_\delta$ whose δ component is z_δ. Then $\overline{z}_\delta \in G^\frown$ because $G^\frown \supseteq \sum_{\delta\in\Delta} T_\delta \supseteq \sum_{\delta\in\Delta} T'_\delta$. From Theorem 1.1 in [4] we have

$$\overline{z}_\delta = \bigvee_{\alpha\in A}{}^{G^\frown}\{z^\alpha \in G \mid 0 \leqslant z^\alpha \leqslant \overline{z}_\delta\}. \tag{10}$$

It is clear that $z^\alpha \in= \overline{T}'_\delta$. It follows from (10) that

$$\overline{z}_\delta = \bigvee_{\alpha\in A}{}^{(T'_\delta)} z^\alpha_\delta.$$

By Lemma 2.4 there exists a subset $\{z^{\alpha'} \mid \alpha' \in A'\}$ with $|A'| \leqslant \aleph_i$ of $\{z^\alpha \mid \alpha \in A\}$ such that

$$z_\delta = \bigvee_{\delta'\in A'}{}^{(T'_\delta)} z^{\alpha'}.$$

Therefore

$$\overline{z}_\delta = \bigvee_{\alpha'\in A'}{}^{(G^\frown)} z^{\alpha'}_\delta.$$

From Theorem 2.4 in [4] there exists $z' \in G$ such that $\overline{z}_\delta \leqslant z'$. Since G is $\aleph_i$-complete, there exists $\bigvee_{\alpha'\in A'}{}^{(G)} z^{\alpha'}$. From Lemma 2.2 and Theorem 2.4 in [4] and (11) above, we see that

$$\bigvee_{\alpha'\in A'}{}^{(G)} z^{\alpha'} = \bigvee_{\alpha'\in A'}{}^{(G^\frown)} z^{\alpha'} = \overline{z}_\alpha.$$

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Thus $\overline{z}_\delta \in G$. This proves that $\overline{T}'_\delta \subseteq G$ for each $\delta \in \Delta_1$. Therefore

$$\sum_{\delta\in\Delta_1} T'_\delta \subseteq G \subseteq' \prod_{\delta\in\Delta} T'_\delta.$$

∎

3. Epicomplete objects in the category $\mathcal{A}^\ell$

LEMMA 3.1. *The following are equivalent for $G \in \mathcal{A}$:*

(a) *G is epicomplete in $\mathcal{A}^\ell$;*

(b) *G is conditionally and laterally σ-complete, and G is divisible (see Theorem 4.9 in* [**3**]*).*

From Theorem 2.8 we have the following result.

COROLLARY 3.2. *Any σ-complete ℓ-group is ℓ-isomorphic to a semicomplete subdirect sum of real groups, integer groups, continuous complete ℓ-groups of countable type and continuous σ-complete ℓ-groups of $\aleph_j$ type with $\aleph_j > \aleph_0$. That is, there exists an ℓ-isomorphism f such that*

$$\sum_{\delta\in\Delta_1\subseteq\Delta} T_\delta \subseteq f(G) \subseteq' \prod_{\delta\in\Delta} T_\delta, \tag{12}$$

where $\Delta_1 = \{\delta \in \Delta \mid T_\delta$ is a real group, or an integer group or a continuous complete ℓ-group of countable type$\}$ and $\Delta\backslash\Delta_1 = \{\delta \in \Delta \mid T_\delta$ is a continuous σ-complete ℓ-group of $\aleph_j$ type with $\aleph_j > \aleph_0\}$.

Now let G be a divisible σ-complete ℓ-group. Without loss of generality, from (12) we have

$$\sum_{\delta\in\Delta_1\subseteq\Delta} T_\delta \subseteq G \subseteq' \prod_{\delta\in\Delta} T_\delta.$$

Each T_δ is a homomorphic image of G, hence T_δ is divisible for each $\delta \in \Delta$. Thus we get:

COROLLARY 3.3. *Any divisible σ-complete ℓ-group is ℓ-isomorphic to a semicomplete subdirect sum of real groups, continuous divisible complete ℓ-groups of ocuntable type and continuous divisible σ-complete ℓ-groups of $\aleph_j$ type with $\aleph_j > \aleph_0$.*

Let $\{T_\delta \mid \delta \in \Delta\}$ be a set of ℓ-groups. Put

$$\prod_{\delta\in\Delta} {}^\sigma T_\delta = \{x \in \prod_{\delta\in\Delta} T_\delta \mid \exists \text{ a countable subset } \Delta_\sigma \text{ in } \Delta \text{ such that } x_\delta = 0 \text{ if } \delta \overline{\in} \Delta_\sigma\}.$$

Let ℓ-group G be a semicomplete subdirect sum of $\{T_\delta \mid \delta \in \Delta\}$. That is.

$$\sum_{\delta\in\Delta_1\subseteq\Delta} T_\delta \subset G \subseteq' \prod_{\delta\in\Delta} T_\delta.$$

If $\prod_{\delta\in\Delta_1} {}^\sigma T_\delta \subseteq G$, then we call G a *σ-semicomplete subdirect sum* of $\{T_\delta \mid \delta \in \Delta\}$.

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THEOREM 3.4. *Let G be an epicomplete object in the category $\mathcal{A}^\ell$. Then G is ℓ-isomorphic to a σ-semicomplete subdirect sum of real groups, continuous complete epicomplete ℓ-groups of countable typs and continuous ℓ-groups of $\aleph_j$ type with $\aleph_j > \aleph_0$.*

PROOF: By Lemma 3.1 and Corollary 3.3 we have

$$\sum_{\delta \in \Delta_1 \subseteq \Delta} T_\delta \subseteq G \subseteq' \prod_{\delta \in \Delta} T_\delta, \tag{13}$$

where T_δ is a real group or a continuous divisible complete ℓ-group of countable type for each $\delta \in \Delta_1$ and T_δ is a continuous divisible σ-complete ℓ-group of $\aleph_j$ type with $\aleph_j > \aleph_0$. By 6.1 in [2] the real group $\mathbf{R}$ is epicomplete in $\mathcal{A}^\ell$. For each $\delta \in \Delta_1$ put

$$\overline{T}_\delta = \{g \in G \mid \delta' \neq \delta \implies g_\delta = 0\}.$$

From (13) we have

$$G = \overline{T}_\delta \boxplus G_\delta$$

for each $\delta \in \Delta_1$, where $G_\delta = \{g \in G \mid g_\delta = 0\}$. If $\overline{T}_\delta \leqslant T'_\delta$ is epic in $\mathcal{A}^\ell$, put $G' = T'_\delta \boxplus G_\delta$. Then $G \leqslant G'$. Suppose α_1 and α_2 are two ℓ-homomorphisms from G' to an Archimedean ℓ-group P such that $\alpha_1|_G = \alpha_2|_G$. Then $\alpha_1|_{\overline{T}_\delta} = \alpha_2|_{\overline{T}_\delta}$. So $\alpha_1|_{\overline{T'_\delta}} = \alpha_2|_{T'_\delta}$ and $\alpha_1 = \alpha_2$. This means $G \leqslant G'$ is epic in $\mathcal{A}^\ell$. Since G is epicomplete, $G = G'$ and $\overline{T}_\delta = T'_\delta$. Therefore each $\overline{T}_\delta$ or T_δ is epicomplete in $\mathcal{A}^\ell$ for $\delta \in \Delta_1$.

On the other hand, G is σ-laterally complete. Hence

$$\prod_{\delta \in \Delta_1} {}^\sigma T_\delta \subseteq G \subseteq' \prod_{\delta \in \Delta} T_\delta.$$

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