Can. J. Math. Vol. 49 (2), 1997 pp. 283–300

THE 2-RANK OF THE CLASS GROUP OF IMAGINARY BICYCLIC BIQUADRATIC FIELDS

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ABSTRACT. A formula is obtained for the rank of the 2-Sylow subgroup of the ideal class group of imaginary bicyclic biquadratic fields. This formula involves the number of primes that ramify in the field, the ranks of the 2-Sylow subgroups of the ideal class groups of the quadratic subfields and the rank of a Z_2-matrix determined by Legendre symbols involving pairs of ramified primes. As applications, all subfields with both 2-class and class group $Z_2 \times Z_2$ are determined. The final results assume the completeness of D. A. Buell's list of imaginary fields with small class numbers.

1. **Introduction.** In our recent work [8], we determined all imaginary bicyclic biquadratic fields with cyclic 2-class group. Here we present a general method for determining the 2-rank of the ideal class group of any imaginary bicyclic biquadratic field. This rank depends only on the ramification of primes in the field K and the rank of a Z_2-matrix associated with K. As an application of our techniques, we determine all imaginary bicyclic biquadratic fields with 2-class group having rank 2. We also determine all such fields with 2-class group $Z_2 \times Z_2$ and assuming completeness of Buell's lists [6], we determine all such fields K with ideal class groups $Z_2 \times Z_2$ and Z_4. The techniques developed here can be modified to apply to more general types of fields. This will be done in future articles. The structure of the 2-class group of a bicyclic biquadratic field is closely related to that of its quadratic subfields. The structure of the 2-class groups of quadratic fields has been studied by L. Rédei [11, 12], L. Rédei and H. Reichardt [10] and others.

2. **Notation and Terminology.** The following notation will be used for the remainder of this article.

K: An imaginary bicyclic biquadratic number field.

k_1, k_2, k_3: The quadratic subfields of K with k_2 real.

d_1, d_2, d_3: Square free integers with $k_i = Q(\sqrt{d_i})$ for $i = 1, 2, 3$.

f: The conductor of K.

H, H_1, H_2, H_3: The ideal class groups of K, k_1, k_2 and k_3 respectively.

h, h_1, h_2, h_3: The class numbers of K, k_1, k_2 and k_3 respectively.

$h^{(2)}, h_1^{(2)}, h_2^{(2)}, h_3^{(2)}$: The maximum power of 2 dividing h, h_1, h_2, h_3 respectively.

$G^{(2)}$: The 2-Sylow subgroup of a group G.

Received by the editors June 22, 1995.

AMS subject classification: Primary: 11R16, 11R29; Secondary: 11R20.

https://doi.org/10.4153/CJM-1997-014-2 Published online by Cambridge University Press

$\hat{H}_i$: The group of quadratic characters on the group H_i. Each element of $\hat{H}_i$ corresponds to a genus in H_i for $i = 1, 2, 3$.

(l, q, r): An element of $H_1 \times H_2 \times H_3$ determined by the ideal classes of prime divisors of l, q and r in k_1, k_2 and k_3 respectively.

$\tilde{A}$: The ideal class determined by the ideal A.

$\hat{S}$: The subgroup of $\hat{H}_1 \times \hat{H}_2 \times \hat{H}_3$ consisting of those character values which are consistent on each pair of H_1, H_2 and H_3.

S: The subgroup of $H_1 \times H_2 \times H_3$ with character group $\hat{S}$.

θ: The homomorphisms $\theta: H_1 \times H_2 \times H_3 \longrightarrow H$ defined by $\theta(C_1, C_2, C_3) = C_1C_2C_3$.

ker: The kernel of θ.

H_0: The image of θ.

t: The positive integer determined so that 2^t is the product of the ramification indices of all rational primes for the extension K/Q.

t_1, t_2, t_3: The number of rational primes ramified in k_i for $i = 1, 2, 3$.

r_a: The rank of the 2-Sylow subgroup of $H_1 \times H_2 \times H_3$.

r_2: The rank of the 2-Sylow subgroup of H.

ψ: The isomorphism from the multiplicative group $\{\pm 1\}$ to the additive group Z_2.

M: A Z_2-matrix determined by $\hat{H}_1 \times \hat{H}_2 \times \hat{H}_3$.

l, q, r, s: Distinct prime numbers.

$(\frac{a}{b})$: Kronecker Symbol using the convention $(\frac{b}{2}) = (\frac{2}{b})$ for all odd positive integers.

x, y, z, u, v, w: $\psi(\frac{l}{q}), \psi(\frac{l}{r}), \psi(\frac{l}{s}), \psi(\frac{q}{s}), \psi(\frac{r}{s}), \psi(\frac{q}{r})$ respectively.

$\bar{x} = 1 - x$.

3. **A general formula for the 2-rank of H.** In this section we describe the theoretical basis of an algorithm for determining the 2-rank of H.

LEMMA 1. *The order of $\hat{S}$ is 2^{t-2}.*

PROOF. Each odd prime which ramifies in K determines a character. If 2 ramifies in K then it determines either 3 or 1 characters depending on whether 2 totally ramifies or not. In the former case only two of these characters are independent. These t characters and their products are only restricted by the conditions $\Pi_{\chi \in \hat{H}_i} \chi = 1$ for $i = 1, 2, 3$. However, any one product condition follows from the other two. Hence there are 2^{t-2} characters in $\hat{S}$.

LEMMA 2. *The number t is determined by*

$$t_1 + t_2 + t_3 = \begin{cases} 2t - 1 & \text{if 2 is totally ramified in } K, \\ 2t & \text{otherwise.} \end{cases}$$

Moreover,

$$r_a = \begin{cases} t_1 + t_2 + t_3 - 4 & \text{if a prime } q \equiv 3 (\text{mod } 4) \textit{ divides } d_2, \\ t_1 + t_2 + t_3 - 3 & \text{otherwise.} \end{cases}$$

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Consequently,

$$r_a = \begin{cases} 2t-5 & \text{if 2 is totally ramified and } q|d_2 \text{ for some prime } q \equiv 3 (\text{mod } 4), \\ 2t-3 & \text{if 2 is not totally ramified and no prime } q \equiv 3 (\text{mod } 4) \text{ divides } d_2, \\ 2t-4 & \text{otherwise.} \end{cases}$$

PROOF. Each odd prime which ramifies in K, ramifies in exactly two of the subfields k_i for $i = 1, 2, 3$. The same is true for 2 unless it is totally ramified in K. The 2-rank of H_i is $t_i - 1$ unless $i = 2$ and some prime $q \equiv 3 (\text{mod } 4)$ divides d_2. In that case the 2-rank of H_2 is $t_2 - 2$. The final expressions for r_a are immediate.

LEMMA 3. *The order of S is given by $|S| = \frac{h_1 h_2 h_3}{2^{r_a}} \cdot 2^{t-2} = \frac{h_1 h_2 h_3}{2^{s_0}}$ where*

$$s_0 = \begin{cases} t-3 & \text{if } r_a = 2t-5, \\ t-2 & \text{if } r_a = 2t-4, \\ t-1 & \text{if } r_a = 2t-3. \end{cases}$$

PROOF. The order of $\hat{H}_1 \times \hat{H}_2 \times \hat{H}_3 = 2^{r_a}$. Now the same number of classes of $H_1 \times H_2 \times H_3$ belong to each character value of $\hat{H}_1 \times \hat{H}_2 \times \hat{H}_3$. Since there are 2^{t-2} character values in $\hat{S}$, the first result follows. Since $s_0 = r_a - t + 2$, the final result is immediate from Lemma 2.

LEMMA 4. *The homomorphism θ induces an isomorphism $S/S \cap \ker \simeq H^2$.*

PROOF. Let $C = (C_1, C_2, C_3) \in S$. Then the characters on C_i in $\hat{H}_i$ are consistent with one another for $i = 1, 2, 3$. Hence there exists a prime p which satisfies these character values. Now p splits completely in K and has a prime divisor P_0 in K such that $\wp_i = P_0 \cap k_i = P_0 P_i$ where $(p) = P_0 P_1 P_2 P_3$ in K. Now $(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3) \in S$ with $\tilde{\wp}_i$ and C_i being in the same genus of k_i. The image of $(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3)$ in H is

$$\tilde{\wp}_1 \tilde{\wp}_2 \tilde{\wp}_3 = \tilde{P}_0^2 \tilde{p} = \tilde{P}_0^2 \in H^2.$$

Moreover, $\tilde{\wp}_i C_i^{-1}$ is in the principal genus of k_i for $i = 1, 2, 3$. Thus $\tilde{\wp}_i C_i^{-1} = B_i^2$ for some class B_i of k_i. Hence

$$(B_1^2, B_2^2, B_3^2) = (\tilde{\wp}_1 C_1^{-1}, \tilde{\wp}_2 C_2^{-1}, \tilde{\wp}_3 C_3^{-1})$$

so

$$\begin{aligned} (B_1 B_2 B_3)^2 &= (\tilde{\wp}_1 \tilde{\wp}_2 \tilde{\wp}_3)(C_1 C_2 C_3)^{-1} \\ &= \tilde{P}_0^2 (C_1 C_2 C_3)^{-1}. \end{aligned}$$

Thus $C_1 C_2 C_3 \in H^2$.

Conversely, let $C^2 \in H^2$ and $P_0 \in C$ be a prime ideal which is of degree 1 and index 1 over Q and let $\wp_i = P_0 \cap k_i$ for $i = 1, 2, 3$. Then $\wp_1 = P_0 P_1$, $\wp_2 = P_0 P_2$ and $\wp_3 = P_0 P_3$ where $P_0 \cap Q = (p) = P_0 P_1 P_2 P_3$. Now $(\tilde{\wp}_1, \tilde{\wp}_2, \tilde{\wp}_3) \in S$ and $\tilde{\wp}_1 \tilde{\wp}_2 \tilde{\wp}_3 = \tilde{P}_0^2 = C^2$. Thus $S/S \cap \ker \simeq H^2$.

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THEOREM 5. *The 2-rank of* $H = t - 2 + \log_2[H_1 \times H_2 \times H_3 : S \cdot \ker]$.

PROOF. Now $r_2 = \log_2[H : H^2] = \log_2[H : H_0] + \log_2[H_0 : H^2] = t - 2 + \log_2[H_0 : H^2]$ where the last equality follows from Kubota [7]. But $H_0/H^2 \simeq \frac{(H_1 \times H_2 \times H_3)/\ker}{S/S\cap\ker}$ and $S/S \cap \ker \simeq S \cdot \ker/\ker$ so $[H_0 : H^2] = [H_1 \times H_2 \times H_3 : S \cdot \ker]$. Thus $r_2 = t - 2 + \log_2[H_1 \times H_2 \times H_3 : S \cdot \ker]$.

The unit index of K or simply the unit index refers to the index of the subgroup generated by the units of the three quadratic subfields in the group of units of K. This index is well known to be either 1 or 2; see Kubota [7].

COROLLARY 1. *If the unit index is 2 then* $s_0 \leq r_2 \leq r_a$. *In any case* $t - 2 \leq r_2 \leq r_a$.

PROOF. From Kubota [7], $|\ker| = 2^{t-2}$ or 2^{t-1} according as the unit index of K is 2 or 1. Let $2^e = |\ker|$. Then

$$\begin{aligned}[H_1 \times H_2 \times H_3 : S \cdot \ker] &= \frac{|H_1 \times H_2 \times H_3|}{|S||\ker|}|S \cap \ker| \\ &= 2^{s_0 - e}|S \cap \ker|.\end{aligned}$$

The results now follow from Lemma 3 and Theorem 5.

COROLLARY 2. *In Theorem 5, the term* $[H_1 \times H_2 \times H_3 : S \cdot \ker]$ *can be replaced with* $[(H_1 \times H_2 \times H_3)^{(2)} : S^{(2)} \cdot \ker]$.

PROOF. Note that all elements of odd order in $H_1 \times H_2 \times H_3$ are in S since they belong to the principal genus of each k_i. Thus $(H_1 \times H_2 \times H_3)^{(2)} \cdot S \cdot \ker = H_1 \times H_2 \times H_3$. Moreover, since ker is a 2-group $(H_1 \times H_2 \times H_3)^{(2)} \cap S \cdot \ker = S^{(2)} \cdot \ker$. Hence $H_1 \times H_2 \times H_3/S \cdot \ker \simeq (H_1 \times H_2 \times H_3)^{(2)}/S^{(2)} \cdot \ker$.

COROLLARY 3. *If* $(H_1 \times H_2 \times H_3)^{(2)}$ *has no cyclic factors of order 2 then* $r_2 = r_a$.

PROOF. Since Kubota [7] shows ker is an elementary 2-group, the hypothesis of the corollary requires that ker be contained in the direct product of the principal genera of the k_i's. Since the trivial character system clearly belongs to $\hat{S}$, it follows that $\ker \subseteq S$. Hence $r_2 = r_a$.

THEOREM 6. *If m denotes the 2-rank of* $\hat{S} \cdot \widehat{\ker}$ *then*

$$r_2 = r_a + t - 2 - m = \begin{cases} 3t - 7 - m & \text{if } r_a = 2t - 5, \\ 3t - 6 - m & \text{if } r_a = 2t - 4, \\ 3t - 5 - m & \text{if } r_a = 2t - 3. \end{cases}$$

PROOF. Let $\phi\colon (H_1 \times H_2 \times H_3)^{(2)} \longrightarrow \hat{H}_1 \times \hat{H}_2 \times \hat{H}_3$ be the mapping determined by taking a class C_i of H_i to its character system in $\hat{H}_i$. then

$$\frac{(H_1 \times H_2 \times H_3)^{(2)}}{\ker \phi \cdot (S^{(2)} \cdot \ker)} \simeq \frac{\hat{H}_1 \times \hat{H}_2 \times \hat{H}_3}{\phi(S^{(2)} \cdot \ker)}.$$

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But $\phi(S^{(2)} \cdot \ker) = \phi(S^{(2)}) \cdot \phi(\ker) = \hat{S} \cdot \widehat{\ker}$. Moreover, $\ker\phi$ is the direct product of the principal genera of k_1, k_2 and k_3. But this is clearly contained in $S^{(2)}$. Thus

$$\frac{(H_1 \times H_2 \times H_3)^{(2)}}{S^{(2)} \cdot \ker} \simeq \frac{\hat{H}_1 \times \hat{H}_2 \times \hat{H}_3}{\hat{S} \cdot \widehat{\ker}}$$

and the result follows from Theorem 5.

In order to determine r_2, all that remains to do is to compute $m = \operatorname{rank}(\hat{S} \cdot \widehat{\ker})$. Recall that $\operatorname{rank}\hat{S} = t - 2$ and $\operatorname{rank}\widehat{\ker} = t - 1$ or $t - 2$ according as the unit index of K is 1 or 2. Let $n = \operatorname{rank}\hat{S} + \operatorname{rank}\widehat{\ker}$ then m is the rank of a $n \times r_a Z_2$-matrix M whose rows correspond to generators of $\hat{S} \cdot \widehat{\ker}$ by means of the isomorphism ψ. The following example illustrates the technique.

EXAMPLE. Let $d_1 = -lqrs, d_2 = lq$ and $d_3 = -rs$ with $l \equiv q \equiv r \equiv 1 (\operatorname{mod} 4)$ and $s \equiv 3 (\operatorname{mod} 4)$. Here the unit index is 1, $t = 4$ and $r_a = 5$. Moreover, $\ker = (l, 1, 1), (q, 1, 1), (r, 1, r)$.

The table of consistent character systems is:

l	q	r	s
+	+	+	+
+	+	−	−
−	−	+	+
−	−	−	−

Since the product of all of the characters for each quadratic field is +1, we may delete one character for each subfield. In the following matrix the first three columns correspond to characters for k_1 determined by primes l, r, and s in that order. The fourth column corresponds to a character of k_2, determined by l, and the last column to a character of k_3, determined by r. The first two rows are determined by two of the nontrivial elements of $\hat{S}$ and the last three rows by the elements of ker in the order stated above.

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ x+y+z & y & z & x & 0 \\ x & w & u & x & 0 \\ y & y+w+v & v & 0 & v \end{bmatrix}.$$

Using row and column operations over Z_2, M reduces to the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & z & z & w \\ 0 & 0 & u & 0 & w \\ 0 & 0 & 0 & y & y+w \end{bmatrix}.$$

It is easily seen that $m = 2, 3, 4$ or 5 depending on the values of the variables u, w, y

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and z. The corresponding values of r_2 are 5,4,3 and 2. The exact conditions for each value will be determined in Section 5.

4. **Determination of r_2 when $t = 3$.** When $t = 2, r_a \leq 1$ and the 2-Sylow subgroup of H is cyclic. All such fields have been determined in our recent article [8]. In this section we determine r_2 when $t = 3$.

THEOREM 7. *Let 2 be totally ramified in $K, t = 3, r_a = 2$ and l be the unique odd prime which ramifies in K. Then $r_2 = 1$ or 2 according as $l \equiv 5 \pmod 8$ or $l \equiv 1 \pmod 8$.*

PROOF. Since 2 is totally ramified in K, the only possible values for d_1, d_2 and d_3 are $-l$, $2l$, -2; $-2l$, $2l$, -1 and $-2l$, 2, $-l$ with $l \equiv 1 \pmod 4$. In each case $\hat{S}$ has the form $\pm\pm$. Moreover ker = (2,1,1), (1,2,1), (1,1,2).

Thus $\hat{S}\cdot\widehat{\ker}$ is generated by character values corresponding to the matrix $M = \begin{bmatrix} 1 & 1 \\ x & 0 \\ 0 & x \end{bmatrix}$ over Z_2 which reduces to $\begin{bmatrix} 1 & 0 \\ 0 & x \\ 0 & 0 \end{bmatrix}$. Now the rank $\hat{S}\cdot\widehat{\ker}$ = rank M, so Theorem 6 shows r_2 = 1 or 2 according as x = 1 or 0; *i.e.* according as $l \equiv 5$ or $1 \pmod 8$.

THEOREM 8. *Let $t = 3, r_a = 2$ and 2 not be totally ramified in K. Assume the 2-Sylow subgroup of each H_i is cyclic. If the unit index of K is 1 then $r_2 = 1$ or 2 according as $h_i \equiv 2 \pmod 4$ for some i or not. If the unit index of K is 2 then $r_2 = 2$.*

PROOF. Since $r_a = 2, (H_1 \times H_2 \times H_3)^{(2)} \simeq Z_{2^a} \times Z_{2^b}$. When the unit index is 1, $|\ker|$ = 4 and $\ker \simeq (2^{a-1}, 0), (0, 2^{b-1})$. Here $S^{(2)} \simeq (2i, 2j), (2i+1, 2j+1)$. Thus if either $a = 1$ or $b = 1$ then $S^{(2)} \cdot \ker \simeq Z_{2^a} \times Z_{2^b}$ and Theorem 5 shows $r_2 = 1$. If $a > 1$, $b > 1$ then $S^{(2)} \cdot \ker = S^{(2)}$ and Theorem 5 shows $r_2 = 2$. If the unit index is 2 then $S^{(2)}$ is the same as above, but $|\ker|$ = 2. In order for the unit index to be 2, either d_1 and d_3 are principal divisors of k_2 or $d_3 = -1$ and 2 is a principal divisor of k_2. In the first case $\ker \simeq (2^{a-1}, 2^{b-1})$. In the second case, since 2 is not totally ramified in K, it must be unramified in k_1. Hence ker is the same as above. Theorem 5 shows $r_2 = 2$ if and only if $\ker \subseteq S^{(2)}$ if and only if $a = b = 1$ or $a > 1, b > 1$. To complete the proof we show $a = 1$ if and only if $b = 1$. Since the unit index is 2, the discriminant of k_2 is divisible by all three primes which ramify in K. Using the symmetry of k_1 and k_3 we may assume $H_1^{(2)} \simeq Z_{2^a}$ and $H_2^{(2)} \simeq Z_{2^b}$. Thus the discriminant of k_1 is divisible by exactly two primes l and r with $l \not\equiv r \pmod 4$, where we choose l to be odd. Now $a > 1$ if and only if $(\frac{r}{l}) = +1$. From above we see r is not a principal divisor of k_2, so $b > 1$ if and only if $(\frac{r}{l}) = +1$.

COROLLARY. *For each of the following values of d_1, d_2 and $d_3, r_2 = 2$. Moreover, if $r_2 = 2$ and K satisfies the hypothesis of Theorem 7 or 8 then K is listed below. Here $l, q,$*

and r denote primes with $l \equiv 1, q \equiv r \equiv 3 \pmod 4$.

d_1	d_2	d_3	conditions
$-2l$	2	$-l$	$l \equiv 1 \pmod 8$
$-2l$	$2l$	-1	$l \equiv 1 \pmod 8$
$-l$	$2l$	-2	$l \equiv 1 \pmod 8$
$-lq$	qr	$-l$	$(\frac{l}{q}) = (\frac{l}{r}) = +1$
$-lq$	lqr	$-r$	$(\frac{l}{r}) = +1$
$-2q$	qr	$-2r$	$(\frac{2}{q}) = (\frac{2}{r}) = +1$
$-lq$	q	$-l$	$l \equiv 1 \pmod 8, (\frac{l}{q}) = +1$
$-lq$	lq	-1	$l \equiv 1 \pmod 8$
$-l$	lq	$-q$	$(\frac{l}{q}) = +1$
$-2l$	$2q$	$-l$	$l \equiv 1 \pmod 8, (\frac{l}{q}) = +1$
$-2l$	$2lq$	$-q$	$(\frac{l}{q}) = +1$
$-lq$	$2lq$	-2	$(\frac{2}{l}) = +1$
$-2q$	$2qr$	$-r$	$(\frac{2}{r}) = +1$

PROOF. The results follow from Theorems 7 and 8 and conditions for $h_i \equiv 0 \pmod 4$, see Brown [2,3,4].

THEOREM 9. *If 2 is not totally ramified in* K, $t = 3$, $r_a = 2$, *and exactly one quadratic subfield of* K *has even class number then* $r_2 = 2$ *for precisely the values listed below where* l, q *and* r *are primes with* $l \equiv q \equiv r \equiv 3 \pmod 4$:

d_1	d_2	d_3	conditions
$-lqr$	qr	$-l$	$(\frac{l}{q}) = (\frac{l}{r}) = -1$
$-2lq$	lq	-2	$(\frac{2}{l}) = (\frac{2}{q}) = -1$
$-2lq$	$2q$	$-l$	$(\frac{2}{l}) = +1, (\frac{l}{q}) = -1$
$-lq$	q	$-l$	$(\frac{2}{l}) = +1, (\frac{l}{q}) = -1$

When these conditions are not satisfied $r_2 = 1$.

PROOF. Since $r_a = 2$ and 2 is not totally ramified in $K, q | d_2$ for some prime $q \equiv 3 \pmod 4$. Since $t = 3$, 2-rank $H_2 \leq 1$, so 2-rank $H_2 = 0$. The only possible values for d_1, d_2 and d_3 other than those listed above are $-lq, lq, -1$. In all cases ker $= (l, 1, 1), (r, 1, 1)$ where $r = 2$ when it is not explicitly defined. In the first two lines the generating matrix for $\hat{S} \cdot \widehat{\ker}$ is $M = \begin{bmatrix} 0 & 1 \\ x+y & x \\ y+1 & w+1 \end{bmatrix}$, where the columns correspond to l and q respectively. The matrix reduces to $\begin{bmatrix} 0 & 1 \\ y+1 & 0 \\ x+1 & 0 \end{bmatrix}$. In the last two lines the generating matrices are $\begin{bmatrix} 0 & 1 \\ x+y+1 & x \\ y & w \end{bmatrix}$, with columns corresponding to l and q respectively,

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and $\begin{bmatrix} 0 & 1 \\ x+1 & x \\ y & w \end{bmatrix}$, where the columns correspond to l and -1. Both of these reduce to $\begin{bmatrix} 0 & 1 \\ x+1 & 0 \\ y & 0 \end{bmatrix}$. The results follow. In the case $-lq, lq, -1, \ker = (l,1,1), (2,1,1)$ and the generating matrix is $\begin{bmatrix} 0 & 1 \\ 1 & x+1 \\ y+w & y \end{bmatrix}$, with columns corresponding to -1 and l. The matrix reduces to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ y+w & 0 \end{bmatrix}$. Thus $r_2 = 1$ in this case.

Next we turn to the case where $t = 3$ and $r_a = 3$. Here 2 is not totally ramified in K and d_2 is divisible by no prime congruent to 3 modulo 4. We consider separately the cases where each H_i is cyclic and where some H_i is noncyclic for $i = 1, 2, 3$. In the following theorems k_1, k_2 and k_3 will be defined by two ordered n-tuples where the first n-tuple defines d_1, d_2 and d_3 and the second n-tuple specifies congruence conditions on l, q, r *etc.* modulo 4.

THEOREM 10. *If $t = 3$ and $r_a = 3$ with each H_i cyclic then K and r_2 are determined as follows:*

a) $(-lq, lr, -qr), (1, 2 \text{ or } 3, 1 \text{ or } 2), r_2 = 3 - w - x - y + wxy$.

b) $(-l, lr, -r), (1, 2, 1), r_2 = 3 - w - x - y + wx + wxy$.

PROOF. Since each H_i is cyclic and d_2 is divisible by no prime congruent to 3 modulo 4, the discriminant of each k_i is divisible by exactly two primes. Since 2 is not totally ramified the only possible values are those listed above. In case (a) ker is generated by $(l,l,1), (q,1,q)$ and the matrix M reduces to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & y & w \\ 0 & x & x+w \end{bmatrix}$. In case (b) ker $=$ $(1,l,1), (2,1,2)$ and M reduces $\begin{bmatrix} 1 & 0 & 0 \\ 0 & y & 0 \\ 0 & x & x+w \end{bmatrix}$. The results follow.

THEOREM 11. *If $t = 3$ and $r_a = 3$ with some H_i noncyclic then K and r_2 are determined as follows:*

a) $(-lqr, lr, -q), (1, 2 \text{ or } 3, 1 \text{ or } 2), r_2 = 3 - x - w$.

b) $(-lr, lr, -1), (1, 2, 1), r_2 = 3 - x - w + xw$.

c) $(-lr, l, -r), (1, 2, 1)$ or $(-lqr, l, -qr), (1 \text{ or } 2, 2 \text{ or } 3, 1 \text{ or } 2), r_2 = 3 - x - y$.

PROOF. Since 2 is not totally ramified in K and no prime congruent to 3 (modulo 4) divides d_2, it is seen that H_2 is cyclic and we may assume H_1 is noncyclic. It follows that only the three cases listed above can occur. In each case we list generators for ker and a matrix to which M reduces. The results follow immediately.

a) $(l,l,1), (q,1,1)$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & w \end{bmatrix}$

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b) $(l,l,1),(2,1,1),\begin{bmatrix}1&0&0\\0&x&w\\0&0&0\end{bmatrix}$

c) $(l,1,1),(q,1,q),\begin{bmatrix}1&0&0\\0&y&0\\0&0&x\end{bmatrix}$

5. **Determination of r_2 when $t=4$.** When $t=4$, $3\le r_a\le 5$ and $2\le r_2\le r_a$. In this section, we determine r_2 for all fields K with $t=4$. The notation here is similar to that used in Theorems 10 and 11.

THEOREM 12. *When $r_a=3$, the fields K and the values of r_2 are listed below:*

a) $(-2lq,q,-2l),(1,3),r_2=3-x-y+xy$.
b) $(-2lq,q,-2l),(3,3),r_2=3-\bar{x}-y+\bar{x}y$.
c) $(-2lq,2q,-l),(1,3),r_2=3-x-y+xy$.
d) $(-lq,2q,-2l),(3,3),r_2=3-\bar{x}-y+\bar{x}y$.
e) $(-2lq,lq,-2),(1,3),r_2=3-y$.
f) $(-2q,lq,-2l),(1,3),r_2=3-x-y+xy$.
g) $(-2lq,2lq,-1),(1,3),r_2=3-y$.
h) $(-2lq,2lq,-1),(3,3),r_2=2$.
i) $(-lq,2lq,-2),(3,3),r_2=3-\bar{w}-\bar{y}+\overline{wy}$.
j) $(-2q,2lq,-l),(1,3),r_2=3-x-y+xy$.

PROOF. In order that $r_a=3, 2$ must be totally ramified in K and there exists a prime $q\equiv 3 \pmod 4$ which divides d_2. Hence there is exactly one other odd prime l which ramifies in K. The only possible fields are those listed above. In part (a), ker $=$ $(2,1,1),(l,1,1),(1,1,2)$ and the matrix M reduces to $\begin{bmatrix}1&0&0\\0&1&0\\0&0&y\\0&0&x\\0&0&0\end{bmatrix}$. Thus $m=2+x+y-xy$ and the formula for r_2 follows from Theorem 6. The remaining parts are done similarly. Also, each case has been computer checked.

THEOREM 13. *When $r_a=4$, the fields K and the values of r_2 are listed below:*

(a) $(-lqr,lqrs,-s)$, (1 or 2, 2 or 3, 1, 2 or 3), $r_2=4-v-z$.
(3, 2 or 3, 3, 3), $r_2=4-u-v-z+uvz$.
$(3,3,3,2)$, $r_2=4-\bar{u}-\bar{v}-\bar{z}+\overline{uvz}$.

(b) $(-lq,lqrs,-rs)$, (1, 3, 1 or 2, 2 or 3), $r_2=4-w-y-z+wyz$.

(c) $(-lqr,lqr,-1)$, $(1,3,1)$, $r_2=4-v-z+vz$.
$(1,3,3)$, $r_2=3-z$.
$(3,3,3)$, $r_2=2$.
$(1,1,2)$, $r_2=4-w-y+wy$.

(d) $(-lqrs,qrs,-l)$, (2 or 3, 3, 1 or 2, 3), $r_2=4-\bar{x}-y-\bar{z}+\overline{xz}$.
$(3,3,1,2)$, $r_2=4-\bar{x}-y-z+\bar{x}z$.

(e) $(-lq,lqr,-r)$, (1, 1, 2 or 3), $r_2=4-w-y$.

$(1,3,1)$, $\quad r_2 = 4 - w - y - z + wyz.$

$(3,3,3)$, $\quad r_2 = 4 - v - w - y + vwy.$

$(3,3,1)$, $\quad r_2 = \begin{cases} 3 - v & \textit{if } w = y = 0, \\ 3 - u - z + 2uz & \textit{if } w = y = 1, \\ 2 & \textit{if } w \neq y. \end{cases}$

$(2,1,1)$, $\quad r_2 = 4 - w - x - y + wxy.$

(f) $(-lqr, qrs, -ls)$, $(1 \text{ or } 2, 2 \text{ or } 3, 1, 3)$, $\quad r_2 = 4 - x - y - v - z + xz + vxy + v\bar{x}yz.$

$(1,3,1,2)$, $\quad r_2 = 4 - x - y - v - z + xz + vyz + xyv\bar{z}.$

$(1,2,3,3)$, $\quad r_2 = 4 - u - x - y - z + yz + uxz + uxy - uxyz.$

$(3,3,3,1)$, $\quad r_2 = \begin{cases} 3 - \bar{x}y - x\bar{y} - xyz & \text{if } u + v = 2, \\ 4 - u - v - z - x - y + xy & \text{if } u + v + z \leq 1, \\ 2 & \text{if } u + v = z = 1. \end{cases}$

$(3,3,3,2)$, $\quad r_2 = \begin{cases} 4 - u - \bar{x} - \bar{y} - z + u\bar{y}z + u\bar{x}z + \overline{xy} + u\overline{xyz} & \text{if } u = v, \\ 3 - \max\{\bar{x}, \bar{y}, z\} & \text{if } u \neq v. \end{cases}$

$(2,3,3,1)$, $\quad r_2 = \begin{cases} 3 - \max\{yz, |x - y|\} & \text{if } u = v = 1, \\ 4 - u - v - z - \bar{x} - \bar{y} + \overline{xy} + \bar{x}z + x\bar{y}z - \overline{uvxz} - \overline{uv}x\bar{y}z & \text{otherwise.} \end{cases}$

$(3,2,3,1)$, $\quad r_2 = \begin{cases} 4 - u - v - \bar{y} - x + uv + \bar{y}x + uv\bar{y}x & \text{if } z = 0, \\ 2 + \overline{uvy}\bar{x} + uv\bar{y}x & \text{if } z = 1. \end{cases}$

(g) $(-lqr, qr, -l)$, $(3,3,1)$, $\quad r_2 = 3 + x - y - xz.$

$(1,2,1)$, $\quad r_2 = 4 - w - x - y + wxy.$

$(1,3,1)$, $\quad r_2 = 4 - v - x - y - z + vz + xz - vxz + vxy + v\bar{x}yz.$

$(1,3,3)$, $\quad r_2 = 3 - x - y + 2xy + 2uvxy - vxy - uxy.$

(h) $(-lqrs, qs, -lr)$, $(1 \text{ or } 2, 3, 3, 3)$, $\quad r_2 = 4 - \bar{v} - w - x - z + \bar{v}w + xz + \bar{v}wxz.$

$(1,3,3,2)$, $\quad r_2 = 4 - v - w - x - z + vw + xz + vwxz.$

$(1,3,2,3)$, $\quad r_2 = 4 - \bar{u} - \bar{v} - x - z + \overline{uv} + xz + \overline{uv}xz.$

(i) $(-lqr, qs, -lrs)$, $(2 \text{ or } 3, 3, 3, 3)$, $\quad r_2 = 4 - \bar{v} - w - \bar{x} - \bar{z} + \bar{v}w + \overline{xz} + \bar{x}w\overline{vz}.$

$(1, 3, 1, 2 \text{ or } 3)$, $\quad r_2 = 4 - v - w - x - z + vw + xz + vwxz.$

$(3,3,3,2)$, $\quad r_2 = 4 - v - w - \bar{x} - \bar{z} + vw + \bar{x}z + vw\bar{x}z.$

$(2,3,1,3)$, $\quad r_2 = 4 - v - w - x - z + vw + xz + wvxz.$

(j) $(-lq, qr, -lr)$, $(1,2,1)$, $\quad r_2 = 4 - w - x - y + wxy.$

$(1,3,1)$, $\quad r_2 = 4 - w - x - y - z + xz + yw - w\bar{x}y\bar{z}.$

$(3,3,3)$, $\quad r_2 = 2 + (\bar{x} - y)^2 - \overline{xy}|u - v|.$

$(3,3,1)$, $\quad r_2 = \begin{cases} 4 - y - |z - v| - \bar{y}zv & \text{if } w = 0, x = 1, \\ 4 - w - \bar{x} - y - z + \bar{x}z - v\bar{w} + w\bar{x}y + \bar{w}yv + xyz & \text{otherwise.} \end{cases}$

(k) $(-lqr, q, -lr)$, $(1,3,1)$, $\quad r_2 = 4 - v - w - x - z + vw + xz + v\bar{w}\bar{x}z.$

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(3, 3, 1 or 3), $r_2 = 4 - v - w - \bar{x} - z + vw + \bar{x}z + v\bar{w}\bar{x}z$.
(1, 2, 1) or (3, 2, 3), $r_2 = 4 - w - x$.

PROOF. Since $r_a = 4$, either 2 is totally ramified in K or there exists a prime $q \equiv 3 \pmod 4$ dividing d_2, but not both. The only possible fields are those listed above. In part (a) with the congruence conditions (3, 3, 3, 3), ker $= (l, l, 1), (q, q, 1), (1, s, 1)$ and the matrix M reduces to $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & z & z \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & v \end{bmatrix}$. The formula for r_2 follows from Theorem 6. The other cases are done similarly. Moreover, each case has been computer checked.

THEOREM 14. *When $r_a = 5$, the fields and the values of r_2 are listed below:*

a) $(-lqrs, lqr, -s)$, (1, 1, 1 or 2, 2 or 3), $r_2 = 5 - u - v - z$.
b) $(-lqr, lqr, -1), (1, 1, 1), r_2 = 5 - \max\{u, v, z\}$.
c) $(-lqs, lqr, -rs)$, (1 or 2, 1, 1 or 2, 2 or 3), $r_2 = 5 - u - v - w - y - z + vyz + uvw + uwyz + u\bar{v}wyz$.
d) $(-lq, lqr, -r), (1, 1, 1), r_2 = 5 - u - w - y - \bar{u}v\bar{u}z + uvw + \bar{u}vz - uvwz + \bar{u}yvz + u\bar{v}wyz$.
e) $(-lqrs, lq, -rs)$, (1, 1 or 2, 1, 3) or (1, 1, 1 or 3, 2), $r_2 = 5 - u - w - y - z + 2uwyz$.
f) $(-lqr, lq - r), (1, 1, 1), r_2 = 5 - u - w - y - z + uz + uwyz$.
g) $(-lqr, lr, -qrs)$, (1 or 2, 1 or 2, 1, 3) or (1, 2, 1, 1), $r_2 = 5 - v - w - x - y - z + vyz + wxy + 2vwxz - vwxyz$.
h) $(-lq, lr, -qr), (1, 1, 1), r_2 = 5 - v - w - x - y - \bar{v}z + vyz + wxy + vwx\bar{y}z$.
i) $(-lqrs, l, -qrs)$, (1, 1 or 3, 1 or 3, 2), (1 or 2, 1, 1, 3) or (1 or 2, 3, 3, 3), $r_2 = 5 - x - y - z$.
j) $(-lqr, l, -qr)$, (1, 1, 1) or (1, 3, 3), $r_2 = 5 - x - y - z$.

PROOF. Since 2 is not totally ramified in K, $r_a = 5$ and no prime congruent to 3 modulo 4 divides d_2, the possible fields are those listed above. In part (a), with the congruence conditions (1, 1, 1, 3), ker $= (l, l, 1), (q, q, 1), (s, 1, 1)$ and the matrix M reduces to $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & z & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & v \end{bmatrix}$. The formula for r_2 follows from Theorem 6. The other cases are done similarly. Moreover, each case has been computer checked.

6. **Fields with class group of order 4.** In this section we determine all fields K with 2-class group $Z_2 \times Z_2$. Assuming that Buell's results [6] are complete, we also determine all fields with class groups $Z_2 \times Z_2$ and Z_4. Here we used Oriat's tables [9] to determine the value of h_2 when $d_2 \leq 24,572$ and direct computation for larger values of d_2.

THEOREM 15. *The fields with 2-class group $Z_2 \times Z_2$ are exactly those listed below:*

a) $(-2lq, q, -2l), (1, 3), (\frac{2}{l}) = -1$ *and either* $(\frac{l}{q}) = -1$ *or* $(\frac{2}{q}) = -1$.
b) $(-2lq, q, -2l), (3, 3), (\frac{2}{l}) = (\frac{l}{q}) = -1, (\frac{2}{q}) = +1$.

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c) $(-2lq, 2q, -l), (1,3), (\frac{2}{l}) = -1$ *and either* $(\frac{l}{q}) = -1$ *or* $(\frac{2}{q}) = -1$.

d) $(-lq, 2q, -2l), (3,3), (\frac{2}{l}) = -1$ *and either* $(\frac{l}{q}) = -1$ *or* $(\frac{2}{q}) = -1$.

e) $(-2lq, lq, -2), (1,3), (\frac{2}{l}) = -1$ *and either* $(\frac{l}{q}) = -1$ *or* $(\frac{2}{q}) = -1$.

f) $(-2q, lq, -2l), (1,3), (\frac{l}{q}) = +1, (\frac{2}{l}) = (\frac{2}{q}) = -1$.

g) $(-2lq, 2lq, -1), (1,3), (\frac{2}{l}) = -1$ *and either* $(\frac{l}{q}) = -1$ *or* $(\frac{2}{q}) = -1$.

h) $(-2lq, 2lq, -1), (3,3), (\frac{2}{l}) = (\frac{l}{q}) = -(\frac{2}{q})$.

i) $(-lq, 2lq, -2), (3,3), (\frac{2}{l}) = (\frac{l}{q}) = -(\frac{2}{q})$.

j) $(-2q, 2lq, -l), (1,3), (\frac{2}{l}) = (\frac{2}{q}) = -1, (\frac{l}{q}) = +1$.

k) $(-lqr, qr, -l), (3,3,3), (\frac{l}{q}) = (\frac{l}{r}) = -1, h_1 \equiv 8 \pmod{16}$.

l) $(-2lq, 2q, -l), (3,3), (\frac{2}{l}) = +1, (\frac{l}{q}) = -1, h_1 \equiv 8 \pmod{16}$.

m) $(-2lq, lq, -2), (3,3), (\frac{2}{l}) = (\frac{2}{q}) = -1, h_1 \equiv 8 \pmod{16}$.

n) $(-lq, q, -l), (3,3), (\frac{2}{l}) = +1, (\frac{l}{q}) = -1, h_1 \equiv 8 \pmod{16}$.

o) $(-l, lr, -r), (1,1), (\frac{l}{r}) = (\frac{2}{l}) = (\frac{2}{r}) = -1$.

p) $(-lqr, lr, -q), (1, 2$ *or* $3, 1$ *or* $2), (\frac{l}{r}) = -1, (\frac{q}{r}) \neq (\frac{q}{l})$.

q) $(-lr, lr, -1), (1,1), (\frac{l}{r}) = -1, (\frac{2}{l}) \neq (\frac{2}{r})$.

r) $(-lr, l, -r), (1,1), (\frac{2}{r}) = (\frac{l}{r}) = -1, (\frac{2}{l}) = +1$.

s) $(-lqr, l, -qr), (1, 2$ *or* $3, 1$ *or* $2), (\frac{q}{r}) = -1, (\frac{l}{q}) \neq (\frac{l}{r})$.

t) $(-2qr, 2, -qr), (1,3), (\frac{q}{r}) = -1, (\frac{2}{q}) \neq (\frac{2}{r})$.

u) $(-lq, lqr, -r), (1,3,3), (\frac{l}{q}) = -1, (\frac{l}{r}) = +1$.

v) $(-lq, lq, -1), (1,3), (\frac{2}{l}) = +1, (\frac{l}{q}) = -1$.

w) $(-l, lq, -q), (1,3), (\frac{2}{l}) = -1, (\frac{l}{q}) = +1$.

x) $(-2l, 2lq, -q), (1,3), (\frac{2}{l}) = -1, (\frac{l}{q}) = +1$.

y) $(-lq, 2lq, -2), (1,3), (\frac{2}{l}) = +1, (\frac{l}{q}) = -1$.

z) $(-2l, 2lq, -q), (3,3), (\frac{2}{l}) = -1, (\frac{2}{q}) = +1$.

PROOF. Since $h \equiv 4 \pmod 8$ and $h = h_1h_2h_3$ or $\frac{1}{2}h_1h_2h_3$ according as the unit index of K is 2 or 1, it follows that $r_a = 2$ or 3. The only possible fields come from the Corollary to Theorem 8 and Theorems 9, 10, 11 and 12 where $r_2 = 2$. The results follow by using conditions from Brown [1,2,3,4] to determine when $h \equiv 4 \pmod 8$.

In order to determine all fields with class number 4, note the class number formula shows $h_1h_3 \leq 8$ when the unit index is 1 and $h_1h_3 \leq 4$ when the unit index is 2. If we choose $h_1 \leq h_3$ then $h_1 = 1$ or 2. Stark [13,14] has determined all such quadratic fields. For each such field k_1, we use Arno's result [1] and Buell's list [6] to obtain the possible fields k_3 and Oriat's table [9] or direct computation to determine h_2. The unit index is 1, unless all t primes ramify in k_2 and in that case is 2 exactly when $\pm d_1$ and $\pm d_3$ (or ± 2 and $\pm 2d_3$ when $d_1 = -1$) are non trivial principal divisors of k_2. The structure of the class group can be determined from Theorem 15 and Theorems 5,7 of [8]. Assuming that Buell's list of imaginary quadratic fields of class number 8 is complete, all fields K with class group $Z_2 \times Z_2$ and Z_4 are listed in Tables 1 and 2 respectively. In Table 1, the letter

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following the number of the field refers to the part of Theorem 15 which corresponds to the field. All fields listed between two letters correspond to the upper letter in the table.

Fields where $H \equiv Z_2 \times Z_2$

	d_1	d_2	d_3
1, *a*	−30	3	−10
2	−70	7	−10
3	−190	19	−10
4, *b*	−42	7	−6
5, *c*	−30	6	−5
6	−70	14	−5
7	−78	6	−13
8	−190	38	−5
9, *d*	−21	14	−6
10	−33	22	−6
11	−33	6	−22
12	−57	38	−6
13	−93	62	−6
14	−177	118	−6
15	−253	46	−22
16, *e*	−30	15	−2
17	−70	35	−2
18	−78	39	−2
19	−190	95	−2
20, *f*	−22	55	−10
21, *g*	−30	30	−1
22	−70	70	−1
23	−78	78	−1
24	−190	190	−1
25, *h*	−42	42	−1
26, *i*	−21	42	−2
27	−93	186	−2
28	−133	266	−2
29, *j*	−22	110	−5
30	−6	78	−13
31	−6	222	−37
32	−22	814	−37
33, *k*	−651	217	−3
34	−1659	553	−3
35	−1771	253	−7
36	−1947	177	−11
37	−2163	721	−3
38	−2667	889	−3

	d_1	d_2	d_3
39	−5467	781	−7
40, *l*	−154	22	−7
41	−322	46	−7
42, *m*	−66	33	−2
43	−114	57	−2
44	−258	129	−2
45	−418	209	−2
46	−498	249	−2
47, *n*	−77	11	−7
48	−301	43	−7
49, *o*	−5	65	−13
50	−5	185	−37
51	−13	481	−37
52, *p*	−195	65	−3
53	−555	185	−3
54	−715	65	−11
55	−70	10	−7
56	−78	26	−3
57	−190	10	−19
58, *q*	−85	85	−1
59, *r*	−85	17	−5
60, *s*	−195	13	−15
61	−435	29	−15
62	−555	37	−15
63	−1435	41	−35
64	−78	13	−6
65	−102	17	−6
66, *t*	−70	2	−35
67	−102	2	−51
68, *u*	−15	165	−11
69	−15	285	−19
70	−35	385	−11
71	−35	665	−19
72	−51	969	−19
73	−51	2193	−43
74	−51	3417	−67
75	−91	273	−3
76	−91	3913	−43

	d_1	d_2	d_3
77	−115	1265	−11
78	−115	2185	−19
79	−123	5289	−43
80	−123	20049	−163
81	−187	3553	−19
82	−187	8041	−43
83	−187	12529	−67
84	−235	2585	−11
85	−235	4465	−19
86	−267	2937	−11
87	−403	1209	−3
88	−403	17329	−43
89	−427	1281	−3
90	−427	69601	−163
91, *v*	−51	51	−1
92	−123	123	−1
93	−187	187	−1
94	−267	267	−1
95, *w*	−5	55	−11
96	−5	95	−19
97	−13	39	−3
98	−13	559	−43
99	−37	111	−3
100	−37	259	−7
101	−37	407	−11
102	−37	2479	−67
103, *x*	−10	110	−11
104	−10	190	−19
105	−58	406	−7
106	−58	3886	−67
107, *y*	−51	102	−2
108	−123	246	−2
109	−187	374	−2
110	−267	534	−2
111, *z*	−6	42	−7
112	−22	154	−7

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Fields where $H \equiv Z_4$

	Conductor	Qudratic Fields		
	f	d_1	d_2	d_3
1	56	−2	7	−14
2	56	−1	14	−14
3	95	−19	5	−95
4	111	−3	37	−111
5	120	−15	10	−6
6	120	−15	2	−30
7	120	−6	5	30
8	120	−3	10	−30
9	156	−13	3	−39
10	156	−1	39	−39
11	164	−1	41	−41
12	165	−15	33	−55
13	165	−3	165	−55
14	168	−6	21	−14
15	168	−3	42	−14
16	183	−3	61	−183
17	184	−2	23	−46
18	184	−1	46	−46
19	195	−39	5	−195
20	204	−17	3	−51
21	204	−3	51	−17
22	220	−5	11	−55
23	220	−1	55	−55
24	255	−15	85	−51
25	264	−11	6	−66
26	264	−3	22	−66
27	273	−91	21	−39
28	273	−7	273	−39
29	276	−1	69	−69
30	276	−3	23	−69
31	280	−7	10	−70
32	308	−11	7	−77
33	308	−1	77	−77
34	312	−2	78	−39
35	371	−7	53	−371
36	385	−35	77	−55
37	385	−7	385	−55
38	408	−34	6	−51
39	408	−3	34	−102
40	408	−3	102	−34
41	429	−11	429	−39

	Conductor	Qudratic Fields		
	f	d_1	d_2	d_3
42	440	−10	22	−55
43	440	−2	110	−55
44	452	−1	113	−113
45	455	−35	65	−91
46	456	−19	6	−114
47	456	−3	38	−114
48	465	−15	93	−155
49	465	−3	465	−155
50	476	−7	119	−17
51	520	−10	13	−130
52	520	−2	65	−130
53	548	−1	137	−137
54	552	−3	46	−138
55	552	−6	69	−46
56	552	−3	138	−46
57	552	−2	69	−138
58	564	−1	141	−141
59	564	−3	47	−141
60	568	−2	71	−142
61	579	−3	193	−579
62	583	−11	53	−583
63	595	−35	17	−595
64	595	−7	85	−595
65	609	−3	609	−203
66	615	−15	205	−123
67	616	−11	14	−154
68	616	−22	77	−14
69	616	−11	154	−14
70	616	−2	77	−154
71	620	−5	31	−155
72	620	−1	155	−155
73	651	−7	93	−651
74	712	−2	89	−178
75	741	−19	741	−39
76	748	−17	11	−187
77	748	−11	187	−17
78	795	−15	53	−795
79	795	−3	265	−795
80	812	−1	203	−203
81	868	−1	217	−217
82	816	−51	34	−6

Fields where $H \equiv Z_4$

	Conductor	Qudratic Fields		
	f	d_1	d_2	d_3
83	852	−3	71	−213
84	868	−7	31	−217
85	904	−2	113	−226
86	939	−3	313	−939
87	952	−7	238	−34
88	969	−51	57	−323
89	969	−3	969	−323
90	979	−11	89	−979
91	984	−82	6	−123
92	984	−3	246	−82
93	987	−7	141	−987
94	987	−3	329	−987
95	1032	−43	6	−258
96	1032	−3	86	−258
97	1036	−37	7	−259
98	1036	−1	259	−259
99	1043	−7	149	−1043
100	1064	−19	266	−14
101	1065	−15	213	−355
102	1065	−3	1065	−355
103	1085	−35	217	−155
104	1085	−7	1085	−155
105	1128	−3	94	−282
106	1128	−2	141	−282
107	1204	−1	301	−301
108	1204	−43	7	−301
109	1209	−403	93	−39
110	1240	−10	62	−155
111	1240	−2	310	−155
112	1252	−1	313	−313
113	1265	−115	253	−55
114	1288	−2	161	−322
115	1299	−3	433	−1299
116	1348	−1	337	−337
117	1420	−5	71	−355
118	1420	−1	355	−355
119	1435	−7	205	−1435
120	1496	−187	34	−22
121	1496	−34	22	−187
122	1496	−11	374	−34
123	1533	−7	1533	−219
124	1624	−58	14	−203
125	1624	−2	406	−203
126	1659	−7	237	−1659
127	1672	−19	22	−418
128	1672	−11	38	−418
129	1704	−6	213	−142
130	1704	−3	426	−142
131	1771	−11	161	−1771
132	1803	−3	601	−1803
133	1828	−1	457	−457
134	1864	−2	233	−466
135	1939	−7	277	−1939
136	1947	−3	649	−1947
137	1992	−3	166	−498
138	2001	−3	2001	−667
139	2024	−22	253	−46
140	2037	−7	2037	−291
141	2044	−7	511	−73
142	2072	−2	518	−259
143	2135	−35	305	−427
144	2136	−267	178	−6
145	2139	−3	713	−2139
146	2163	−7	309	−2163
147	2212	−1	553	−553
148	2248	−2	281	−562
149	2261	−7	2261	−323
150	2307	−3	769	−2307
151	2365	−43	2365	−55
152	2408	−43	602	−14
153	2409	−11	2409	−219
154	2451	−43	57	−2451
155	2451	−19	129	−2451
156	2485	−35	497	−355
157	2485	−7	2485	−355
158	2585	−235	517	−55
159	2611	−7	373	−2611
160	2613	−67	2613	−39
161	2632	−7	94	−658
162	2632	−2	329	−658
163	2667	−7	381	−2667
164	2668	−1	667	−667

Fields where $H \equiv Z_4$

	Conductor	Qudratic Fields				Conductor	Qudratic Fields		
	f	d_1	d_2	d_3		f	d_1	d_2	d_3
165	2716	−7	679	−97	206	5947	−19	313	−5947
166	2840	−2	710	−355	207	6104	−2	1526	−763
167	2865	−15	573	−955	208	6220	−5	311	−1555
168	2865	−3	2865	−955	209	6222	−1	1555	−1555
169	2947	−7	421	−2947	210	6232	−19	1558	−82
170	3009	−51	177	−1003	211	6248	−22	781	−142
171	3009	−3	3009	−1003	212	6248	−11	1562	−142
172	3052	−1	763	−763	213	6357	−163	6357	−39
173	3212	−11	803	−73	214	6665	−43	6665	−155
174	3243	−3	1081	−3243	215	6685	−35	1337	−955
175	3507	−7	501	−3507	216	7021	−7	7021	−1003
176	3507	−3	1169	−2507	217	7189	−91	553	−1027
177	3553	−187	209	−323	218	7189	−7	7189	−1027
178	3553	−11	3553	−323	219	7285	−235	1457	−155
179	3565	−115	713	−155	220	7337	−11	7337	−667
180	3608	−11	902	−82	221	7372	−19	1843	−97
181	3685	−67	3685	−55	222	7640	−10	382	−955
182	3729	−3	3729	−1243	223	7640	−2	1910	−955
183	3752	−67	938	−14	224	7912	−43	1978	−46
184	3787	−7	541	−3787	225	7953	−11	7953	−723
185	3820	−5	191	−955	226	8165	−115	1633	−355
186	3820	−1	955	−955	227	8216	−2	2054	−1027
187	3857	−19	3857	−203	228	8393	−11	8393	−763
188	3883	−11	353	−3883	229	8492	−11	2123	−193
189	3963	−3	1321	−3963	230	8589	−7	8589	−1227
190	4108	−1	1027	−1027	231	8729	−43	8729	−203
191	4123	−19	217	−4123	232	8965	−163	8965	−55
192	4123	−7	589	−4123	233	9128	−163	2282	−14
193	4233	−51	249	−1411	234	9417	−43	9417	−219
194	4233	−3	4233	−1411	235	9709	−7	9709	−1387
195	4323	−11	393	−4323	236	9877	−7	9877	−1411
196	4323	−3	1441	−4323	237	10385	−67	10385	−155
197	4521	−3	4521	−1507	238	10792	−19	2698	−142
198	4539	−51	1513	−267	239	10947	−123	3649	−267
199	4665	−15	933	−1555	240	11033	−187	649	−1003
200	4665	−3	4665	−1555	241	11033	−11	11033	−1003
201	4921	−19	4921	−259	242	11084	−163	2771	−17
202	5061	−7	5061	−723	243	11297	−11	11297	−1027
203	5336	−58	46	−667	244	12328	−67	3082	−46
204	5336	−2	1334	−667	245	12440	−10	622	−1555
205	5467	−11	497	−5467	246	12440	−2	3110	−1555

https://doi.org/10.4153/CJM-1997-014-2 Published online by Cambridge University Press

Fields where $H \equiv Z_4$

	Conductor	Qudratic Fields		
	f	d_1	d_2	d_3
247	12556	−43	3139	−73
248	12673	−19	12673	−667
249	13497	−11	13497	−1227
250	13737	−19	13737	−723
251	14497	−19	14497	−763
252	15521	−187	913	−1411
253	16685	−235	3337	−355
254	19497	−67	19497	−291
255	19513	−19	19513	−1027
256	21965	−115	4393	−955
257	21976	−67	5494	−82
258	23313	−19	23313	−1227
259	23785	−67	23785	−355
260	24424	−43	6106	−142
261	25265	−163	25265	−155
262	28681	−43	28681	−667
263	29992	−163	7498	−46
264	31089	−43	31089	−723
265	31837	−403	2449	−1027
266	33089	−163	33089	−203
267	35697	−163	35697	−219
268	35765	−115	7153	−1555
269	38056	−67	9514	−142
270	41065	−43	41065	−955
271	42217	−163	42217	−259
272	44885	−235	8977	−955

	Conductor	Qudratic Fields		
	f	d_1	d_2	d_3
273	47596	−163	11899	−73
274	51121	−67	51121	−763
275	52649	−163	52649	−323
276	52761	−43	52761	−1227
277	53449	−43	53449	−1243
278	57865	−163	57865	−355
279	59641	−43	59641	−1387
280	64801	−43	64801	−1507
281	66865	−43	66865	−1555
282	68809	−67	68809	−1027
283	73085	−235	14617	−1555
284	82209	−67	82209	−1227
285	83281	−67	83281	−1243
286	100969	−67	100969	−1507
287	104185	−67	104185	−1555
288	108721	−163	108721	−667
289	117848	−163	117849	−723
290	124369	−163	124369	−763
291	155665	−163	155665	−955
292	163489	−163	163489	−1003
293	167401	−163	167401	−1027
294	226081	−163	226081	−1387
295	229993	−163	229993	−1411
296	245641	−163	245641	−1507
297	253465	−163	253465	−1555

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