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NOTE ON STRONGLY REGULAR NEAR-RINGS

by MOTOSHI HONGAN

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Let S be a semigroup. An element a of S is called right (resp. left) regular if $a=a^2x$ (resp. $a=xa^2$) for some $x\in S$. If a is regular and right (resp. left) regular, a is called strongly right (resp. left) regular. As is well known, if a is strongly regular (i.e., right and left regular) then it is regular, more precisely, there exists uniquely an element x such that $a=a^2x$, $x=x^2a$ and $ax=xa$, and a is contained in a subgroup of S (and conversely).

Following M. Petrich [**2**], a semigroup S is called weakly commutative if for each pair of elements x, y in S there exists a positive integer m and an element z in S such that $(xy)^m=yzx$. Let a be an element of a weakly commutative semigroup S. Then a is strongly regular if (and only if) $a\in\langle a\rangle^2$, where $\langle a\rangle$ is the ideal of S generated by a, or equivalently, $a=uavaw$ for some $u,v,w\in S^1$, the semigroup obtained from S by adjoining an identity. Actually, there exist positive integers k, l and $x,y\in S^1$ such that $(uav)^k=avxu$ and $(vxuaw^k)^l=aw^kyvxu$. Hence $a=uavaw=(uav)^kaw^k=avxuaw^k=a(vxuaw^k)^l=a^2w^kyvxu$; similarly, a is left regular.

An element a of S is called π-regular if there exists a positive integer n such that a^n is regular, and right (resp. left) π-regular if there exists a positive integer n such that a^n is right (resp. left) regular; a is strongly π-regular if a is both right and left π-regular. And, a is called strongly right (resp. left) π-regular if there exists a positive integer n such that a^n is strongly right (resp. left) regular, namely, $a^n=a^{2n}xa^n$ (resp. $a^n=a^nxa^{2n}$) for some $x\in S$. The semigroup S is called π-regular if every elements of S is π-regular, and right (resp. left) π-regular if every element of S is right (resp. left) π-regular; S is strongly π-regular if every element of S is strongly π-regular. Similarly, S is called strongly right (resp. left) π-regular if every element of S is strongly right (resp. left) π-regular. As is easily seen, S is strongly right (resp. left) π-regular if and only if S is π-regular and right (resp. left) π-regular.

In view of [**2**, Theorem IV.1.6] (see also Theorem 2 below), every strongly right (or left) regular semigroup is strongly regular. As an application of this result, we shall prove the following which includes [**3**, Theorem 12 and Proposition 13]:

Theorem 1. *Let N be a (left) near-ring. Then the following are equivalent:*

(1) *N is strongly regular.*

(2) *N is right regular.*

(3) *N is left regular and right π-regular.*

(4) *N is strongly right regular.*

(5) *N is strongly left regular.*

(6) *N is regular and $ae=eae$ for any idempotent e and any element a in N.*

In advance of proving Theorem 1, we state the following:

Lemma 1. *Let N be a right (resp. left) regular near-ring, and let a and b be elements of N.*

(1) *If $ab=0$ then $ba=0a$.*

(2) *If $ab=0$ and $b^n=0b$ for some $n>1$, then $b=0$. In particular, N contains no non-zero nilpotent element.*

Proof. (1) There exists $x\in N$ such that $ba=(ba)^2x=0ax$ (resp. $ba=x(ba)^2=0a$). Then $0a=aba=a0ax=0ax=ba$.

(2) There exists $y\in N$ such that $b=b^2y=b^ny^{n-1}=0by^{n-1}$ (resp. $b=yb^2=y^{n-1}b^n=y^{n-1}0b=0b$). Then $0b=b^n=b^{n-1}0by^{n-1}=0by^{n-1}=b$, and so $b=0b=a0b=ab=0$.

Proof of Theorem 1. Obviously, (1) implies (2)−(6) (see Lemma 1 (2)), and [**2**, Theorem IV.1.6] shows that (1), (4) and (5) are equivalent.

(3)⇒(2). Let a be an arbitrary element of N. Then there exists a positive integer n and $x\in N$ such that $a^n=a^{n+1}x$. If $n>1$ then $a^{n-1}(a^{n-1}-a^nx)=0$. By Lemma 1 (1), $(a^{n-1}-a^nx)a^{n-1}=0a^{n-1}$ and $(a^{n-1}-a^nx)a^nx=0a^nx$, and hence $(a^{n-1}-a^nx)^2=0(a^{n-1}-a^nx)$. Then, Lemma 1 (2) proves that $a^{n-1}=a^nx$. Continuing this procedure, we obtain eventually $a=a^2x$.

(2)⇒(4). Let a be an arbitrary element of N, and $a=a^2x$. Since $a(a-axa)=0=axa(a-axa)$, we have $(a-axa)a=0a$ and $(a-axa)axa=0axa$ (Lemma 1 (1)), and hence $(a-axa)^2=0(a-axa)$. Then Lemma 1 (2) shows that $a=axa$.

(6)⇒(2). Given $a\in N$, there exists $x\in N$ such that $a=axa$. Note that ax and xa are idempotents. Then, by (6), we have $a=axa=a(x\cdot ax)a=a(ax\cdot x\cdot ax)a=a^2(x^2ax)a$.

Remark. In [3], a near-ring N is called right (resp. left) regular if for every a there is an x in N such that $a=a^2x$ (resp. $a=xa^2$) and $a=axa$, and N is called right (resp. left) strongly regular if N is right (resp. left) regular in our sense. Obviously, if N is right (resp. left) regular in the sense of [3] then it is strongly right (resp. left) regular.

In view of a theorem of Zöschinger–Dischinger (see, e.g., [**1**, Proposition 2]), every right (or left) π-regular ring is strongly π-regular. It seems difficult to extend this result to semigroups without any restriction. We shall give the following generalization of [**2**, Theorem IV.1.6.].

Theorem 2. *A semigroup S is strongly π-regular if it is strongly right (or left) π-regular.*

Proof. It suffices to show that if $a=a^2xa$ then a is left regular. There exists a positive integer n and $y \in S$ such that $(ax)^n = (ax)^{2n} y (ax)^n$. Then $a = a \cdot a \cdot xa = a^n \cdot a \cdot (xa)^n = a^n (ax)^n a = a^n (ax)^{2n} y (ax)^n a = \{a^n (ax)^n a\} x (ax)^{n-1} y (ax)^n a = ax(ax)^{n-1} y (ax)^n a = (ax)^n y (ax)^n a = (ax)^{2n} y \{(ax)^n y (ax)^n a\} = (ax)^{2n} ya$. Since $ax = a(ax)^2 = a^{2n-1}(ax)^{2n}$, we see that $a = (ax)^{2n-1} \cdot ax \cdot ya = (ax)^{2n-1} a^{2n-1} \{(ax)^{2n} ya\} = (ax)^{2n-1} a^{2n}$.

Corollary 1. *Let S be a subsemigroup of a left (resp. right) π-regular semigroup T. If S is right (resp. left) π-regular, then it is strongly π-regular.*

Proof. Given $a \in S$, there exists a positive integer $n, s \in S$ and $t \in T$ such that $a^{2n}s = a^n = ta^{2n}$. Since $a^n s = ta^{2n}s = ta^n$, we see that $a^n = ta^{2n} = a^n s a^n$. Hence, S is strongly π-regular, by Theorem 2.

REFERENCES

1. Y. Hirano, Some studies on strongly π-regular rings, *Math. J. Okayama Univ.* **20** (1978), 141–149.

2. M. Petrich, *Introduction to semigroups* (Merrill, Columbus, Ohio, 1973).

3. Y. V. Reddy and C. V. L. N. Murty, On strongly regular near-rings, *Proc. Edinburgh Math. Soc.* **27** (1984), 61–64.

Tsuyama College of Technology
Numa, Tsuyama, Okayama
Japan