The Co-symmedian System of Tetrahedra inscribed in a Sphere.

By Dr W. L. MARR.

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This paper deals with a system of tetrahedra in a sphere corresponding to the co-symmedian system of triangles in a circle. Such a system of tetrahedra, so far as the writer knows, has not been hitherto discussed. The condition that a tetrahedron may have a symmedian point is given in Wolstenholme's "Problems" (1878).

1. When has a tetrahedron a symmedian point, *i.e.* when are the lines joining the vertices to the poles of the opposite faces with respect to the circumsphere concurrent?

Let $ABCD$ be a tetrahedron, a the pole of BCD, c the pole of ABD. Suppose that Cc and Aa meet in K.

The points a and c lie on the tangent planes at B and D, and therefore ac is the intersection of these tangent planes. But ac and AC are coplanar, therefore the tangent planes at B and D meet on AC.

But the tangent plane at D meets AC where the tangent at D with respect to the circle ACD meets AC at X (say), and the tangent plane at B meets AC where the tangent at B with respect to the circle ABC meets AC also at X.

$$\frac{XA}{XC} = \frac{AD^2}{DC^2} = \frac{AB^2}{BC^2}$$

therefore $$AD \, . \, BC = AB \, . \, DC.$$

Hence, that there may be a symmedian point, the tetrahedron must be equianharmonic (*v.* Harkness and Morley).

2. *The symmedian point of an equianharmonic tetrahedron is the point of intersection of the lines joining the vertices to the symmedian points of the opposite faces.*

The line from A to the pole of BD (circle ABD) is a symmedian line of ABD. But the pole of BD lies on ca, therefore the

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symmedian line of ABD drawn from A lies in the plane of Aca, *i.e.* ACK. The symmedian line of ABD from B lies in the plane of BCK.

Therefore, if k_3 be the symmedian point of ABD, it will lie on the intersection of ACK and BCK, *i.e.* C, K, k_3 are collinear.

Further, since k_1, k_2, etc., lie within their triangles, K will lie within the tetrahedron.

3. *An equianharmonic tetrahedron has two Isodynamic points.*

Let H be an isodynamic point, then from the foregoing the following relations are possible :—

$$HA \,.\, BC = \quad HB \,.\, CA \quad = \quad HC \,.\, AB$$
$$HA \,.\, DC = \text{etc.}\,; \; HB \,.\, AD = \text{etc.}\,; \; HB \,.\, CD = \text{etc.},$$

i.e. H is the vertex of four harmonic tetrahedra, the opposite faces being the faces of the original tetrahedron. H lies on the circle which is the intersection of the three Apollonian spheres of BDC and on the circle which is the intersection of the three Apollonian spheres of ADC. These two circles lie on the Apollonian sphere of DC and cut in two real—as will be proved presently—points, H_1 and H_2.

These points are common to the six Apollonian spheres.

4. *The circumcentre is collinear with the two isodynamic points.*

The tangents from O, the circumcentre, to the Apollonian spheres are all equal to the radius of the circumsphere, and since these six spheres cut in H_1 and H_2, the three points O, H_1, H_2 must be collinear.

5. *The isodynamic points, the circumcentre, and the symmedian point are collinear and form a harmonic ratio.*

Consider the Apollonian sphere through A and C. The centre of this sphere will lie on BD, viz., L. Since $A\ (LDk_3B)$ is harmonic, k_3 lies on the polar plane of L with respect to the circumsphere. But LA and LC are tangents to the sphere, and therefore AC lies on the polar plane.

Hence ACk_3, *i.e.* ACK, is the polar plane of L with respect to the circumsphere. Since the circum- and Apollonian spheres cut orthogonally, ACK is the plane of intersection of the two spheres.

Let AK meet the circumsphere again in A', etc., then

$$AK \, . \, A'K = BK \, . \, B'K = \text{etc.}$$

But A, A', etc., are points on the Apollonian spheres, hence K must lie on the line of intersection.

Since ACK is the polar plane of O with respect to the Apollonian sphere whose centre is L, OH_1KH_2 is harmonic. As K is within the circumsphere, it must also lie within the Apollonian sphere, and hence H_1 and H_2 are real points.

6. Since the points $A'B'C'D'$, as above defined, have the same circum- and Apollonian spheres, $A'B'C'D'$ will have the same symmedian points.

7. *The faces of $ABCD$ and $A'B'C'D'$ touch an ellipsoid whose section perpendicular to OK is circular.*

If $\qquad BC = a, \qquad CA = b, \qquad AB = c,$

and $\qquad DA = \dfrac{k}{a}, \qquad DB = \dfrac{k}{b}, \qquad DC = \dfrac{k}{c},$

and if $m = \dfrac{abc}{k}$ it can be readily shewn that the ellipsoid is :—

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} + \frac{\omega^2}{m^4} - \frac{yz}{b^2c^2} - \frac{\omega\, x}{m^2a^2} \dots = 0,$$

where x stands for the more usual $\dfrac{x}{p_1}$, y for $\dfrac{y}{p_2}$, p being the perpendicular on a face from a vertex.

The plane of section with the sphere is $\dfrac{x}{a^2} + \dots = 0$, which is the polar plane of K, and hence the ellipsoid has a circular section perpendicular to OK.

For further convenience, let x, which stands for $\dfrac{x}{p_1}$, now stand for $\dfrac{x}{a^2p_1}$.

8. *An infinite system of tetrahedra with a common symmedian point.*

The equation of the sphere is now $\Sigma yz = 0$, of the ellipsoid $\Sigma x^2 - \Sigma yz = 0$. The tangent cone from D to the ellipsoid is

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 0.$$

Where this tangent cone meets the plane of ABC, we have an ellipse, and round this ellipse we have the triangle ABC, which in turn is inscribed in a circle. We can thus have a poristic series of triangles. Take one of these triangles and call it PQR. We have now a new tetrahedron, $DPQR$.

Let

$$QR \text{ be } lx + my + nz = 0$$
$$RP \text{ be } mx + ny + lz = 0$$
$$PQ \text{ be } nx + ly + mz = 0$$

with the condition $\Sigma mn = 0$.

The pole of DQR is given by

$$x = \tfrac{1}{3}(m + n - 2l)$$
$$y = \tfrac{1}{3}(n + l - 2m)$$
$$z = \tfrac{1}{3}(l + m - 2n)$$
$$\omega = \tfrac{1}{3}(l + m + n)$$

and P is given by $x = l$, $y = m$, $z = n$, $\omega = 0$.

As the addition of the corresponding coordinates of the pole of DQR and of P give the same result, viz., $\frac{1}{3}(l + m + n)$, the point given by (1, 1, 1, 1), which is K, lies on the line joining P to the pole of DQR.

Thus K is the symmedian point of $DPQR$.

9. A further extension.

In the last paragraph we fixed D and moved round the tangent cone in the plane of ABC—getting a new tetrahedron, $DPQR$.

Start from this and fix P and change QRD into LMN (say). Proceeding in this manner, we finally get away from $ABCD$ and reach an infinite series of tetrahedra with the same four points, O, K, H_1, and H_2.

10. *The constant of the system.*

Consider $DABC$ and $DPQR$. ABC and PQR are co-Brocardal.

Therefore
$$\frac{a^2 + b^2 + c^2}{abc} = \frac{a_1{}^2 + b_1{}^2 + c_1{}^2}{a_1 b_1 c_1}, \text{ where } a_1 = QR.$$

The areas of ABC and PQR are in the ratio of abc to $a_1b_1c_1$. The volumes of $DABC$ and $DPQR$ are also as abc to $a_1b_1c_1$, *i.e.*

$$\frac{V}{V_1} = \frac{abc}{a_1 b_1 c_1}.$$

But if $ad = k$ and $a_1 d_1 = k_1$

$$\frac{k^2}{V} = \frac{k_1^2}{V_1}$$

(See Salmon's, 4th ed., p. 37).

$$\therefore \quad \frac{abc}{k^2} = \frac{a_1 b_1 c_1}{k_1^2}.$$

Consider next

$$\frac{a_1^2 + b_1^2 + c_1^2 + m_1^2}{a_1 b_1 c_1}.$$

This expression is equal to

$$\frac{a^2 + b^2 + c^2 + m^2}{abc}.$$

Further, it is symmetrical with respect to the triangles

$$DQR,\ DPQ,\ PQR,$$

Since $$\frac{a^2 + b^2 + c^2}{abc} + \frac{a^2 + \dfrac{k^2}{b^2} + \dfrac{k^2}{c^2}}{\dfrac{k^2 a}{bc}} + \ldots = 3\,\frac{a^2 + b^2 + c^2 + m^2}{abc}.$$

Proceeding in the manner of the last paragraph, we find that this expression, viz.,

$$\frac{a^2 + b^2 + c^2 + m}{abc},$$

is constant for any tetrahedron in the infinite system.

11. *Interpretation of Constant.*

The absolute values of the coordinates of K are

$$\frac{a^2 p_1}{a^2 + b^2 + c^2 + m^2}, \text{ etc.}$$

If these values be substituted for the variables in the l.h.s. of the equation of sphere (*v.* Salmon) the result, viz.,

$$\frac{6\, a^2 b^2 c^2}{(a^2 + b^2 + c^2 + m^2)^2}$$

is equal to the rectangle $AK \,.\, A'K$.

Mr R. F. Davis, M.A., of London, has kindly contributed the following note, which shows that if a regular tetrahedron be

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inverted, the origin of inversion is an isodynamic point of the new tetrahedron :—

1. Let $\alpha\beta\gamma\delta$ be a regular tetrahedron having six equal edges $\beta\gamma$, $\gamma\alpha$, $\alpha\beta$, $\delta\alpha$, $\delta\beta$, $\delta\gamma$, and four equal equilateral faces, $\alpha\beta\gamma$, $\delta\beta\gamma$, $\delta\gamma\alpha$, $\delta\alpha\beta$.

Take any point H whatsoever within the tetrahedron. Join $H\alpha$, $H\beta$, $H\gamma$, $H\delta$, and produce them respectively to A, B, C, D in such a manner that

$$HA \cdot H\alpha = HB \cdot H\beta = HC \cdot H\gamma = HD \cdot H\delta = K^2 ,$$

so that $ABCD$ is the figure inverse to $\alpha\beta\gamma\delta$ when H is the origin and K^2 the constant of inversion.

Then $\quad BC : \beta\gamma = HB : H\gamma \quad$ (for $B\beta\gamma C$ are concyclic)

$$= HB \cdot HC : H\gamma \cdot HC = HB \cdot HC/K^2$$

and

$$BC = (\beta\gamma/K^2)\, HB \cdot HC$$

$$CA = (\ldots\ldots\ldots)\, HC \cdot HA,$$

and so on for all six pairs of corresponding edges.

Notice $\quad BC \cdot DA = (\ldots\ldots)^2\, HA \cdot HB \cdot HC \cdot HD$

$$= CA \cdot DB = AB \cdot DC, \quad \text{by symmetry.}$$

2. Conversely, it may be assumed that a tetrahedron $ABCD$ cannot be inverted into a regular tetrahedron $\alpha\beta\gamma\delta$ unless

$$BC \cdot DA = CA \cdot DB = AB \cdot DC$$

$$\text{or} \quad ad = be = cf,$$

where $\quad BC = a,\ CA = b,\ AB = c,\ DA = d,\ BD = e,\ DC = f.$

In this case the tetrahedron $ABCD$ is said to be harmonic; and if a suitable centre H of inversion be taken, we have four harmonic tetrahedra within a harmonic tetrahedron, namely,

$HABC$	with	$HA \cdot BC = HB \cdot CA = HC \cdot AB$
$HDBC$	,,	$HD \cdot BC = HB \cdot CA = HC \cdot DB$
$HDCA$	,,	$HD \cdot CA = HC \cdot AD = HA \cdot DC$
$HDAB$	,,	$HD \cdot AB = HA \cdot BD = HB \cdot DA.$

If we put $bc/d = ca/e = ab/f = \mu$, all these relations are included in the one formula

$$a \cdot HA = b \cdot HB = c \cdot HC = \mu \cdot HD,$$

which also shows that the position of H is determined as either of the two common points of intersection of four spheres corresponding to the Apollonian circles.