ANALYTIC HOPF SURFACES

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1. Definitions. The topological concept of H-space **(7)** has an analytic counterpart which so far has not been considered in the literature. We define: A complex-analytic manifold S will be called an analytic H-space if it is capable of carrying a continuous binary composition

$$f : S \times S \to S$$

with the following properties (i) and (ii).

(i) All maps $S \to S$ defined by

$l_a(z) = f(a, z)$ and $r_a(z) = f(z, a)$

are analytic (not necessarily 1:1) self-mappings of S for every fixed $a \in S$.

(ii) There exists an analytic homotopy unit, that is, an element $u \in S$ such that

$(\text{ii})_1\, f(u, u) = u$, and

$(\text{ii})_2$ the two maps $l_u(z)$ and $r_u(z)$ are analytically homotopic with the identity map of S relative to u.

Condition $(\text{ii})_2$ means explicitly: There exist two continuous maps

$$h_t(z),\ k_t(z) : I \times S \to S\ (I = \textit{unit interval}\ 0 \leqslant t \leqslant 1),$$

such that for every fixed $t \in I$ both are analytic maps $S \to S$, and

$$h_0(z) = l_u(z),\ k_o(z) = r_u(z),\ h_1(z) = k_1(z) = z,\ h_t(u) = k_t(u) = u,$$

for all $z \in S$ and all $t \in I$.

We speak of an analytic homotopy inversion if, in addition to (i) and (ii), the following condition is satisfied.

(iii) There exists an analytic map $\zeta(z) : S \to S$ such that the correspondence $z \to f[z, \zeta(z)]$ is analytically null-homotopic in S.

A group structure satisfying (i), (ii), (iii) will be termed analytic.

Because of the arcwise connectedness of S it follows from (i) and (ii) that all maps $l_a(z)$ and $r_a(z)$ for different $a \in S$ are analytically homotopic to the identity map of S. Instead of postulating (ii) it would be more natural to require more generally that all $l_a(z)$ be non null-homotopic and analytically homotopic among each other; similarly for $r_a(z)$, without assuming these two homotopy classes to be necessarily identical; but we will not consider these "analytic Γ-manifolds," cf. **(3)**.

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Thus an analytic H-space is a topological H-space; conversely, however, we have on the one hand a greater variety of complex-analytic structures which a given topological H-space may carry, and on the other hand a restriction of the possible analytic H-compositions due to the fact that not every homotopy class of continuous maps contains an analytic map.

We illustrate this phenomenon by determining the simplest types of analytic H-spaces, viz. the Riemann surfaces capable of carrying an analytic H-structure. The problem of higher dimensional analytic H-spaces remains open.

2. The fundamental groups. The fundamental group of an H-space is abelian **(4)**; from this remark one deduces that every topological H-surface is homeomorphic to one of the following four types: plane, circular cylinder, torus, and Moebius strip, cf. **(2)**. The latter can be discarded here because of its non-orientability. The first three topological types split up into the following conformally distinct Riemann surfaces **(6)**: Euclidean plane, hyperbolic plane; punctured Euclidean plane, punctured hyperbolic plane, all annuli of finite conformal modulus; all conformal types of tori. (The latter two items each represent an infinity of conformally distinct Riemann surfaces.) We consider these cases separately in the next sections.

3. Determination of all analytic H-surfaces. (a) The Euclidean plane E and the punctured Euclidean plane $\dot{E}$ carry the groups of complex numbers under addition and multiplication, which are obviously analytic H-structures.

(b) Similarly the hyperbolic plane H represented by the Poincaré model $|z| < 1$ carries, for example, the analytic H-structure

$$f(z_1, z_2) = \tfrac{1}{2}(z_1 + z_2) \,,$$

with homotopy unit $u = 0$ and analytic homotopy inversion $\zeta(z) = 0$.

It is interesting to observe, however, that H cannot carry a binary composition $f(z_1, z_2)$ satisfying condition (i) with a unit u in the ordinary sense (in particular no analytic group structure exists on H). Indeed, if we define

$$z = L(w) = \frac{w + u}{1 + \bar{u}w} \,,$$

then

$$g(w_1, w_2) = L^{-1}\{f[L(w_1), L(w_2)]\}$$

would be an analytic composition in H: $|w| < 1$ with unit $w = 0$. According to Hartogs' Theorem **(1)** the function $G(w) = g(w, w)$ would be regular in H. Because of $G(0) = 0$ and $|G(w)| \leqslant 1$ the Schwarz lemma yields

$$(1) \qquad |G'(0)| \leqslant 1.$$

But from $g(w, 0) = g(0, w) = w$ we conclude

$$\frac{\partial g}{\partial w_1}(0, 0) = \frac{\partial g}{\partial w_2}(0, 0) = 1:$$

hence

$$G'(0) = \frac{\partial g}{\partial w_1}(0, 0) + \frac{\partial g}{\partial w_2}(0, 0) = 2,$$

contrary to (1).

(c) The composition in E

$$F(z_1, z_2) = z_1 + z_2$$

is invariant with respect to any discrete group of translations in E; hence it covers an analytic group structure f on every type of conformal torus T defined as quotient space of E with respect to a group

$$w = z + n_1 w_1 + n_2 w_2,\ \Im\left(\frac{w_2}{w_1}\right) \neq 0.$$

(If $l_a(z)$ is lifted to $L_a(z)$ in E then $L_a'(z)$ is an entire elliptic function, hence a constant; from this it is easy to determine *all* analytic H-structures on T.)

4. Exclusion of $\dot{H}$ and A. Although the remaining Riemann surfaces of § 2 are topological H-surfaces we are going to show by contradiction that none can carry an *analytic* H-structure.

(d) Let the punctured hyperbolic plane $\dot{H}$ be conformally represented by $|z| < 1$, $z \neq 0$. Following **(5)** we classify an analytic self-mapping $g(z)$ of $\dot{H}$ homotopically by means of its winding number

$$W[g(z)] = \frac{1}{2\pi i}\oint \frac{g'(z)}{g(z)}\, dz, \tag{2}$$

where we integrate along the positively oriented circle $|z| = \frac{1}{2}$.

Since $|g(z)| < 1$ and $g(z) \neq 0$ for $z \neq 0$ every $g(z)$ can be regularly extended to $z = 0$; hence it has a representation

$$g(z) = z^j e^{h(z)}$$

with j a non-negative integer, and $h(z)$ regular in the whole of H. Substitution in (2) yields $W[g(z)] = j$. Obviously two analytically homotopic maps $g_0(z)$ and $g_1(z)$ of $\dot{H}$ into itself have the same winding numbers. Conversely, if $W[g_0(z)] = W[g_1(z)] = j$, then

$$g_0(z) = z^j e^{h_0(z)} \textit{ and } g_1(z) = z^j e^{h_1(z)}$$

can be connected by the following analytic homotopy in $\dot{H}$:

$$g_t(z) = z^j e^{t[h_1(z) - h_0(z)] + h_0(z)},\ z \in \dot{H},\ 0 \leqslant t \leqslant 1.$$

This is actually an analytic self-mapping of $\dot{H}$ for every fixed t; for

$$\rho(t) = |g_t(z)| = |z^j e^{h_0(z)}| e^{t\Re[h_1(z) - h_0(z)]}$$

is a monotonic function for $0 \leqslant t \leqslant 1$ and fixed z; hence $\rho(t)$ is between $\rho(0)$ and $\rho(1)$.

Hence analytically null-homotopic maps of $\dot{H}$ have vanishing winding numbers, and maps which are analytically homotopic to the identity are characterized by a simple zero at the origin. In particular, since $r_u(z)$ is homotopic to the identity we have $r_u(0) = 0$. Since also $r_u(u) = u$ the map $r_u(z)$ has two different fixpoints $0, u$ in H; in virtue of the Schwarz lemma we conclude

$$r_u(z) = f(z, u) = z. \tag{3}$$

Since $l_a(z) = f(a, z)$ must be analytically homotopic to the identity map of $\dot{H}$ for every $a \in \dot{H}$ we have also

$$l_a(0) = 0. \tag{4}$$

From (3) we obtain

$$l_a(u) = f(a, u) = r_u(a) = a. \tag{5}$$

Because of (4) the Schwarz lemma is applicable to $l_a(z)$ and yields

$$|l_a(z)| \leqslant |z| \text{ for all } z \in H. \tag{6}$$

Applying (6) to $z = u$ and making use of (5) we find

$$|a| = |l_a(u)| \leqslant |u| \text{ for all } a \in \dot{H},$$

which is absurd.

(e) The annulus $A : 1 < |z| < r$.
According to **(5)** only three among the infinitely many homotopy classes of continuous self-mappings of A contain analytic maps, viz. the class of null-homotopic maps, those homotopic to the identity, and those homotopic to the map $w = rz^{-1}$. The analytic maps contained in the last two classes are all conformal automorphisms of A, hence of the form $w = e^{i\alpha}z$ and $w = re^{i\alpha}z^{-1}$ (α real).

In particular, since $f(z_1, z_2)$ must be homotopic to the identity $I(z)$ for fixed z_2 we conclude

$$f(z_1, z_2) = e^{i\alpha(z_0)}z_1.$$

Here $\alpha(z_2)$ must be real and analytic for $z_2 \in A$, hence constant. This contradicts the assumption that $f(z_1, z_2)$ should also be homotopic to $I(z)$ for fixed z_1.

Thus we proved the following:

THEOREM. *Every analytic H-surface is conformally equivalent to one of the following Riemann surfaces: The (finite) Euclidean plane, the punctured Euclidean plane, the hyperbolic plane, and all conformal tori. The hyperbolic plane can be provided with an analytic commutative H-structure with analytic homotopy inversion, but not with an ordinary unit. The other analytic H-surfaces can carry analytic abelian group structures.*

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