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BOUNDEDNESS OF MULTIPLICATIVE LINEAR FUNCTIONALS

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Let A be a complex sequentially complete commutative locally m-convex topological algebra which is symmetric with continuous involution. The purpose of this note is to prove that every multiplicative linear functional on A is bounded (Theorem 3). In fact, we prove a more general result for operators on real algebras (Theorem 1) from which we derive the above result.

Let A denote a real sequentially complete commutative locally m-convex topological algebra with the family of seminorms $\{\| \cdots \|_\alpha, \alpha \in D\}$ [1]. Let E be a real commutative Banach algebra with the norm $\| \cdots \|$ such that for any sequence $\{x_n\}$ in E, $\|x_n\| \geq 1$, there exists $\varepsilon > 0$ and a sequence of real-valued multiplicative linear functionals $f_n (n \geq 1)$ on E satisfying $\inf_n |f_n(x_n)| \geq \varepsilon$. It is not difficult to see (Thanks to referee) that such an algebra with identity can be regarded as a subalgebra of $C_R(M)$, the algebra of continuous real functions on a compact Hausdorff space M with sup norm topology.

First we prove the following main result:

THEOREM 1. *Let A and E be as mentioned above. If T is a linear operator which maps A into E such that $T(x^2) = T(x)^2$, then T is bounded* (*i.e. takes bounded sets into bounded sets*).

Proof. Suppose that T is not bounded. Then there is a bounded sequence $\{x_n\}$ in A such that $\|Tx_n\| \geq n$ for all $n \geq 1$. Since $\|Tx_n/n\| \geq 1$, there exists $\varepsilon > 0$ and a sequence of real-valued multiplicative linear functionals f_n on E such that $\inf_n |f_n(y_n)| \geq \varepsilon$, where $y_n = Tx_n/n = T(x_n/n)$.

Let $z_n = (\gamma y_n/\varepsilon)^2$, where $\gamma > 1$ is fixed, then $f_n(z_n) \geq \gamma^2 > 1$, and $f_m(z_n) = f_m(\gamma y_n/\varepsilon)^2 \geq 0$ for all $m, n \geq 1$. Put $a_n = (\gamma x_n/\varepsilon)^2$, clearly $\{a_n\}$ is a bounded sequence in A, and $z_n = T(a_n/n^2)$. Therefore for each $\alpha \in D$, there is a constant C_α such that $\|a_n\|_\alpha < C_\alpha$ for all n.

Define $D_k = \{\alpha \in D : \|a_n\|_\alpha < k \text{ for all } n\}$, then $D_1 \subseteq D_2 \subseteq \cdots$ and $D = \bigcup_{k \geq 1} D_k$. We now employ a technique in [2], and define recursively a subsequence $\{b_k\}$ of $\{a_n/n^2\}$ as follows: Let $b_1 = a_1$; if $b_1, \ldots, b_{k-1}$ are defined, then one can choose $b_k = a_{n_k}/n_k^2$ for sufficiently large k such that

$$(*) \qquad \|B_k^{(j)} - B_{k-1}^{(j)}\|_\alpha \leq 2^{-k} \quad \text{for all } \alpha \in D_k \quad \text{and} \quad 1 \leq j \leq k-1,$$

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where

$$B_k^{(j)} = b_j + (B_{j+1} + (\cdots + (b_{k-1} + b_k^2)^2 \cdots)^2)^2.$$

We see that $B_k^{(j)} - B_{k-1}^{(j)}$ is a multinomial with positive integral coefficients, because $B_{k-1}^{(j)}$ is a part of $B_k^{(j)}$ in the expansion. Since the seminorms are submultiplicative, we have

$$\|B_k^{(j)} - B_{k-1}^{(j)}\|_\alpha \leq \{\|b_j\|_\alpha + (\|b_{j+1}\|_\alpha + (\cdots + (\|b_{k-1}\|_\alpha + \|b_k\|_\alpha^2)^2 \cdots)^2)^2\}$$
$$- \{\|b_j\|_\alpha + (\|b_{j+1}\|_\alpha + (\cdots + (\|b_{k-2}\|_\alpha + \|b_{k-1}\|_\alpha^2)^2 \cdots)^2)^2\}$$

and since $\|a_i\|_\alpha \leq k$ for all $\alpha \in D_k$, and for sufficiently large m, $\|b_m\|_\alpha = \|{}^\alpha n_m/n_m^2\|_\alpha \leq k/n_m^2$ for all $\alpha \in D_k$, $m=j, \ldots$, i.e. $\|b_k\|_\alpha$ is sufficiently small, we can ensure that the inequality (*) holds.

Furthermore, for any $\alpha \in D$, $\alpha \in D_k$ for some k, and if $p > q \geq k$, $\alpha \in D_{k+1}, \ldots, D_p$. By (*) we obtain, $\|B_p^{(j)} - B_q^{(j)}\|_\alpha = \|\sum_{k=q+1}^p (B_k^{(j)} - B_{k-1}^{(j)})\|_\alpha \leq \sum_{k=q+1}^p \|B_k^{(j)} - B_{k-1}^{(j)}\|_\alpha \leq \sum_{k=q+1}^p 2^{-k}$ for all j, $1 \leq j \leq q-1$. Hence for each $j \geq 1$, $\{B_k^{(j)}\}_{k>j}$ is a Cauchy sequence in A.

Let $c_j = \lim_{k\to\infty} B_k^{(j)}$. By construction, $c_1 = b_1 + c_2^2, \ldots, c_j = b_j + c^2_+$. Since $b_j = {}^\alpha n_j/n_j^2$, we have $f_{n_j}(Tb_j) = f_{n_j}(T({}^\alpha n_j/n_j^2)) = f_{n_j}(z_{n_j}) \geq \gamma^2$, $f_{n_j}(Tc_{j+1}^2) = f_{n_j}(Tc_{j+1})^2 \geq 0$, and $f_{n_j}(Tb_i) = f_{n_j}(z_{n_i}) \geq 0$, $i = 1, \ldots, j-1$. Whence by induction,

$$f_{n_j}(Tc_1) = f_{n_j}(T(b_1 + (b_2 + (\cdots + (b_j + c_{j+1}^2)^2)^2)))$$
$$= f_{n_j}(Tb_1) + (f_{n_j}(Tb_2) + (\cdots + (f_{n_j}(Tb_j) + f_{n_j}(Tc_{j+1}^2))^2)^2) \geq \gamma^{2j},$$

for all $j \geq 1$. This is impossible, since $|f_{n_j}(Tc_1)| \leq \|Tc_1\| < \infty$ for all j.

Replacing E in Theorem 1 by the algebra of real numbers R, we obtain:

THEOREM 2. *If A is a real sequentially complete commutative locally m-convex topological algebra, then every real-valued multiplicative linear functional on A is bounded.*

Now let A be a sequentially complete commutative locally m-convex topological algebra over the complex field. An *involution* on A is, a function $x \to x^*$ from A into A that is conjugate linear and satisfies $(xy)^* = y^*x^*$, $x^{**} = x$ for all x, $y \in A$. The set H of all hermitian elements (those elements x satisfying $x^* = x$) is a real subalgebra of A. Consider A as a real vector space, then it is the direct sum of subspaces H and iH (i.e. $A = H + iH$). Moreover, if A has a continuous involution, then H is sequentially complete, and any bounded set B in A is contained in $B_H + iB'_H$ (i.e. for all $x \in B$, there exists $y \in B_H$ and $z \in B'_H$ such that $x = y + iz$), where B_H and B'_H are bounded sets in H. We say that A is *symmetric* if $-x^*x$ is advertible (i.e. invertible for the composition $\circ$ defined by $x \circ y = x + y - xy$) for all $x \in A$. A familiar argument [1, Lemma 6.4] shows that if A is symmetric, then every multiplicative linear functional on A is real-valued on H. The following theorem generalizes [1, Th. 12.6].

THEOREM 3. *If A is a commutative sequentially complete locally m-convex topological algebra over the complex field that is symmetric with a continuous involution, then every multiplicative linear functional on A is bounded.*

Proof. Let f be a multiplicative linear functional on A. A is the direct sum of real subspaces H and iH, where H is the set of all hermitian elements in A. Moreover, H is a real subalgebra of A. For any bounded set B in A, $B \subseteq B_H + iB'_H$ where B_H and B'_H are bounded sets in H. As the restriction f_H of f to H is real-valued, $f_H(B_H)$ and $f_H(B'_H)$ are bounded by Theorem 2, and hence $f(B)$ is bounded.

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