METRIZATION OF TOPOLOGICAL SPACES

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A single valued function $D(x, y)$ is a *metric* for a topological space provided that for points x, y, z of the space:

1. $D(x, y) \geqslant 0$, the equality holding if and only if $x = y$,
2. $D(x, y) = D(y, x)$ (*symmetry*),
3. $D(x, y) + D(y, z) \geqslant D(x, z)$ (*triangle inequality*),
4. x belongs to the closure of the set M if and only if $D(x, m)$ (m element of M) is not bounded from 0 (preserves limit points).

A function $D(x, y)$ is a metric for a point set R of a topological space S if it is a metric for R when R is considered as a subspace of S. A topological space or point set that can be assigned a metric is called metrizable.

If a topological space has a metric, this metric may be useful in studying the space. Determining which topological spaces can be assigned metrics leads to interesting and important problems. For example, see [3].

A regular[1] topological space is metrizable if it has a countable basis[2] [7 and 8]. However, it is not necessary that a space be separable[3] in order to be metrizable. Theorem 3 gives a necessary and sufficient condition that a space be metrizable by using a condition more general than perfect separability.

Alexandroff and Urysohn showed [1] that a necessary and sufficient condition that a topological space be metrizable is that there exist a sequence of open coverings $G_1, G_2, \ldots$ such that (a) G_{i+1} is a refinement[4] of G_i, (b) the sum of each pair of intersecting elements of G_{i+1} is a subset of an element of G_i, and (c) for each point p and each open set D containing p there is an integer n such that every element of G_n containing p is a subset of D. We call a sequence of open coverings satisfying condition (c) a development. A developable space is a topological space that has a development. In section 2 we study conditions under which developable spaces can be assigned metrics.

The results of this paper hold in a topological space as defined by Whyburn in [10] or in a Hausdorff space.

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[1]A topological space is regular if for each open set D and each point p in D there is an open set containing p whose closure lies in D.

[2]A basis for a topological space is a collection G of open sets such that each open set is the sum of a subcollection of G. In [7 and 8] a space with a countable basis was said to satisfy the second axiom of countability. More recently such spaces have been called perfectly separable.

[3]A space is separable if it has a countable dense subset.

[4]The collection G is a refinement of the collection H if each element of G is a subset of an element of H.

1. Screenable spaces. We shall use the following definitions:

Discrete. A collection of point sets is *discrete* if the closures of these point sets are mutually exclusive and any subcollection of these closures has a closed sum.

Screenable. A space is *screenable* if for each open covering H of the space, there is a sequence $H_1, H_2, \ldots$ such that H_i is a collection of mutually exclusive domains and $\sum H_i$ is a covering of the space which is a refinement of H. A space is *strongly screenable* if there exist such H_i's which are discrete collections.

Perfectly screenable. A space is *perfectly screenable* if there exists a sequence $G_1, G_2, \ldots$ such that G_i is a discrete collection of domains and for each domain D and each point p in D there is an integer $n(p, D)$ such that $G_{n(p,D)}$ contains a domain which lies in D and contains p.

Collectionwise normal. A space is *collectionwise normal* if for each discrete collection X of point sets, there is a collection Y of mutually exclusive domains covering X^* such that no element of Y intersects two elements of X. We use X^* to denote the sum of the elements in X.

The following result follows from the definitions of perfectly screenable and strongly screenable.

THEOREM 1. *A perfectly screenable space is strongly screenable.*

The following example shows that a developable space may not be screenable.

EXAMPLE A. *A locally connected*[5] *separable Moore space*[6] *S such that no space homeomorphic with the closure of any open set in S is either normal or screenable.* The points of S are the points of the plane and the open sets of S are given in terms of a development $G_1, G_2, \ldots$ which is described as follows. Let $L_1, L_2, \ldots$ be a sequence of horizontal lines whose sum is dense in the plane. Either of the following types of sets is an element of G_i: (a) the interior of a circle with diameter less than $1/i$ which does not intersect $L_1 + L_2 + \ldots + L_i$, (b) $p + I_1 + I_2$ where p is a point of some L_j and I_1, I_2 are interiors of circles of diameter less than $1/2i$ which are tangent to L_j at p on opposite sides of L_j and such that $I_1 + I_2$ does not intersect $L_1 + L_2 + \ldots + L_i$.

That S is locally connected follows from the fact that vertical lines are connected and horizontal lines other than $L_1, L_2, \ldots$ are connected. The elements of each G_i are connected. Since the plane is separable and any set dense in the plane is dense in S, S is separable. The sequence $G_1, G_2, \ldots$ satisfies the conditions of Axiom 1 of [5], so S is a Moore space.

Let S' be a space homeomorphic with the closure of an open set E in S and K be an interval in E that is a subset of $L_1 + L_2 + \ldots$. If K_1 and K_2 are

[5]A topological space is locally connected if it has a basis such that the elements of the basis are connected. A set is connected if it cannot be expressed as the sum of two non-null sets such that neither contains a point of the closure of the other.

[6]A space satisfying the first three parts of Axiom 1 of [5] is called a Moore space. It is a regular developable space.

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subsets of S' corresponding to the points of K with rational and irrational abscissas respectively, K_1 and K_2 are two mutually exclusive closed point sets. That S' is not normal follows from the fact that there do not exist two mutually exclusive domains containing K_1 and K_2 respectively.

If H is an open covering of S' such that no two points of $K_1 + K_2$ belong to the same element of H, any open covering of S' that refines H contains uncountably many elements. Since S' is separable, it does not contain an uncountable collection of mutually exclusive domains. Hence, S' is not screenable because there is not a sequence $H_1, H_2, \ldots$ such that H_1 is a collection of mutually exclusive domains and $\sum H_i$ is a covering of S' that refines H.

The proof of the following theorem may be compared with one given by Tychonoff [7] to show that any regular perfectly separable topological space is normal.

THEOREM 2. *A regular strongly screenable space is collectionwise normal.*

Proof. Suppose $\{A_\alpha\}$ is a discrete collection of closed sets. Let K be a collection of open sets covering the space such that the closure of no element of K intersects two elements of $\{A_\alpha\}$. Since the space is strongly screenable, there is a sequence $H_1, H_2, \ldots$ such that H_i is a discrete collection of domains and $\sum H_i$ is a covering of the space which is a refinement of K.

Let $U_{i\beta}$ be the sum of the elements H_i that intersect A_β and $V_{i\beta}$ be the sum of those intersecting elements of $\{A_\alpha\}$ other than A_β. If $D_\beta = U_{1\beta} + (U_{2\beta} - \overline{V}_{1\beta}) + (U_{3\beta} - [\overline{V}_{1\beta} + \overline{V}_{2\beta}]) + \ldots + (U_{i\beta} - \sum_{j=1}^{i-1} \overline{V}_{j\beta}) + \ldots$, then $\{D_\alpha\}$ is a collection of mutually exclusive domains covering $\sum A_\alpha$ such that no element of $\{D_\alpha\}$ intersects two elements of $\{A_\alpha\}$.

It cannot be concluded that a strongly screenable space is normal for a perfectly separable space may not even be regular. Example B shows us that we cannot conclude that a regular screenable space is normal.

EXAMPLE B. *A screenable, point-wise paracompact,[7] nonparacompact, nonnormal Moore space with an open covering H such that the star[8] of each point with respect to H is metrizable.*

Points are of three types: (a) elements of a countable sequence of points $p_1, p_2, \ldots$; (b) elements of the collection of all continuous functions $f_\alpha(x)$ $(0 < x < 1)$ such that $\frac{1}{2} < f_\alpha(x) < 1$; and (c) ordered triples (p_i, t, f_α) where p_i is a point of type (a), f_α is one of type (b), and t is a positive number less than one.

The open sets of this space are defined by a development $G_1, G_2, \ldots$ which is described as follows. The elements of G_n are of three sorts: p_j $(j = 1, 2, \ldots)$

[7] A topological space is point-wise paracompact if for each open covering H there is an open covering H' such that H' refines H and no point lies in infinitely many elements of H'. It is paracompact if for each open covering H there are open coverings H' and H'' such that H' refines H and no element of H'' intersects infinitely many elements of H'.

[8] The star of a point set A with respect to an open covering H is the sum of the element of of H that intersect A.

plus all points (p_j, t, f_α) $(t < 1/n)$; (p_j, t_0, f_α) plus all points (p_j, t, f_α) $(|t - t_0| < 1/n)$; an element f_α of type (b) plus all points (p_j, t, f_α) where $1 - t < 1/n$ if $j \leqslant n$ and $1 - t < [1 + nf_\alpha(1/j)]/(n + 1)$ if $j > n$.

It is convenient to think of the space as being the collection of points of type (a) plus the collection of points of type (b) plus open unit intervals joining points of type (a) to points of type (b). An element of G_n is either (i) an element of type (a) plus all points that can be joined to it by intervals of lengths less than $1/n$, or (ii) a point of an open interval plus the collection of all points of the open interval that are nearer than $1/n$ to the point or (iii) a point f of type (c) plus all points that can be joined to it by an interval of length less than $x(j)$ where p is on the interval from f_α to p_j and $x(j) = 1/n$ if $j \leqslant n$ and $x(j) = [1 + nf_\alpha(1/j)]/(n + 1)$ if $j > n$.

The above space is screenable because for any open covering K of it, there is an open covering $K_1 + K_2$ of it which is a refinement of K and such that K_1 is a collection of mutually exclusive domains covering all points of type (a) and a dense set of those of type (c), while K_2 is another such collection covering all points of type (c) not covered by K_1. The space is point-wise paracompact because for each open covering K of it there is an open covering $K_1 + K_2$ such that no point is covered by more than two elements of $K_1 + K_2$.

The space is not normal because for each domain D containing the collection of all points of type (a), there is a point of type (b) which is a limit point of D. To find such a point, let n_i be an integer such that an element of G_{n_i} contains p_i and lies in D. If f_β is a point of type (b) which satisfies $f_\beta(1/i) > 1 - 1/n_i$ $(i = 1, 2, \ldots)$, it is a point of the closure of D. Since the collection of points of type (a) and the collection of points of type (b) are closed sets, and there do not exist mutually exclusive open sets containing these two collections, the space is not normal. Since it is not normal, it is not paracompact.

To show that there is an open covering H such that the star of any point with respect to H is metrizable, let H be any G_i and M be the star of some point with respect to H. There is a connected open set M' that contains M such that M' contains only one point p_j of type (a) and only one point f_α of type (b). If R is the component of $M' - p_j$ containing f_α, $R + p_j$ has a metric $D_1(x, y)$ because $R + p_j$ is homeomorphic with a regular, perfectly separable space. Furthermore, $M' - R$ has a metric $D_2(x, y)$ described as follows: $D_2[p_j, (p_j, t, f_\alpha)] = t$; $D_2[(p_j, t_1, f_\alpha), (p_j, t_2, f_\beta)]$ is $|t_1 - t_2|$ or $t_1 + t_2$ according as f_α is or is not f_β. There is a metric $D(x, y)$ for M' where $D(x, y) = D_1(x, y)$ if x and y are points of R, $D(x, y) = D_2(x, y)$ if x and y are points of $M' - R$, and $D(x, y) = D_1(x, p_j) + D_2(p_j, y)$ if x and y belong to R and $M' - R$ respectively.

The following theorem may be compared with the result of Urysohn [8] which states that a normal perfectly separable topological space is metrizable.

THEOREM 3. *A necessary and sufficient condition that a regular topological space be metrizable is that it be perfectly screenable.*

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Proof of sufficiency. Suppose $H_1, H_2, \ldots$ is a sequence of discrete collections of domains such that for each domain D and each point p in D there is an integer $n(p, D)$ such that an element of $H_{n(p,D)}$ lies in D and contains p.

Let K_{ij} be the sum of the elements of H_j whose closures lie in an element of H_i. Since the space is normal, there is a continuous transformation F_{ij} of space into the real numbers between 0 and 1 such that the image of K_{ij} under F_{ij} is 1 and the image of the complement of H^*_i is 0 [9]. For points x, y of the space we define the distance between them to be

$$D(x, y) = \sum\sum \frac{|F_{ij}(x) + R_{ij}(x, y)F_{ij}(y)|}{2^{i+j}}$$

where $R_{ij}(x, y)$ is -1 or 1 according as y does or does not belong to an element of H_i that contains x. It may be found that $D(x, y)$ satisfies the conditions for a metric.

Proof of necessity. A. H. Stone has shown [6] that for each metric space and each positive number ϵ, there is a sequence $R_{\epsilon 1}, R_{\epsilon 2}, \ldots$ such that $\sum R_{\epsilon i}$ covers the space and $R_{\epsilon i}$ is a discrete collection of closed sets each of diameter less than ϵ. A proof of this is also found in Theorem 9 of the present paper. For each element r of $R_{\epsilon i}$ let D_r be a domain covering r such that D_r is of diameter less than ϵ and each point of D_r is more than twice as close to r as to any other element of $R_{\epsilon i}$. If $H_{\epsilon i}$ denotes the collection of all such open sets D_r, $\{H_{\epsilon i}; \epsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots \text{ and } i = 1, 2, \ldots\}$ becomes on suitable ordering a countable sequence of discrete collections of open sets which insures that a metric space is perfectly screenable.

In the following modification of Theorem 3 we dispense with the supposition that the elements of H_i are mutually exclusive. E. E. Floyd suggested that such a modification might be possible.

THEOREM 4. *A regular topological space S is metrizable if and only if there is a sequence $G_1, G_2, \ldots$ such that*

(a) *G_i is a collection of open subsets of S such that the sum of the closures of any subcollection of G_i is closed and*

(b) *if p is a point and D is an open set containing p there is an integer $n(p, D)$ such that an element of $G_{n(p,D)}$ contains p and each element of $G_{n(p,D)}$ containing p lies in D.*

Proof. Since a metric space is perfectly screenable, it contains a sequence $G_1, G_2, \ldots$ satisfying the conditions of the theorem. We complete the proof of Theorem 4 by showing that any regular topological space admitting such a sequence is perfectly screenable.

First, we show that any open subset D of S is strongly screenable. Let $H = (h_1, h_2, \ldots, h_\alpha, \ldots)$ be a well ordered collection of open sets whose sum is D. Let $V_{\alpha i}$ be the sum of the elements of G_i whose closures lie in h_α. If $U_{\alpha ij}$ denotes the sum of the elements of G_j whose closures lie in $V_{\alpha i}$ but do not intersect $\sum_{\beta<\alpha} \overline{V}_{\beta i}$ then $W_{ij} = \{U_{\gamma ij}; \gamma = 1, 2, \ldots, \alpha, \ldots\}$ is a discrete collection of open sets which is a refinement of H. To see that $\sum\sum W_{ij}$ covers D, let p be a point of D and h_β be the first element of H containing p. Then p belongs to some $V_{\beta k}$ but does not belong to $\sum_{\alpha<\beta}\overline{V}_{\alpha k}$. Then for some integer m, p lies in an element of G_m whose closure lies in $V_{\beta k}$ but does not intersect $\sum_{\alpha<\beta}\overline{V}_{\alpha k}$ and p is a point of $U_{\beta k m}$.

For each positive integer k let $X_{k1}, X_{k2}, \ldots$ be a sequence of discrete collections of open sets such that each X_{ki} is a refinement of G_k and $\sum_{i=1}^{\infty} X_{ki}$ covers G^*_k. That S is perfectly screenable follows from the fact that the elements of $\{X_{ki}; i, k = 1, 2, \ldots\}$ may be ordered in a sequence fulfilling the conditions to be satisfied by the sequence $G_1, G_2, \ldots$ mentioned in the definition of a perfectly screenable space.

We find from Example C that condition (b) of Theorem 4 could not be weakened to

(b′) *if p is a point and D is an open set containing p, there is an integer $n(p,D)$ such that an element of $G_n(p, D)$ contains p and lies in D.*

EXAMPLE C. *A regular strongly screenable topological space not satisfying the first axiom of countability.*[9] Points are the points of the plane. A neighbourhood is either (a) an open interval of a line through the origin such that this interval does not contain the origin or (b) the sum of a collection of open intervals each of which contains the origin and such that each line through the origin contains one of these open intervals.

The space is strongly screenable because for each open covering H of it there are two discrete collections H_1, and H_2 such that $H_1 + H_2$ is an open covering of the space which refines H. The space does not satisfiy the first axiom of countability because for each sequence of neighbourhoods of the origin there is a neighbourhood of the origin that does not contain any of these neighbourhoods. Let G_1 be the collection of all neighbourhoods N of type (b) containing the origin such that if p is a point of the boundary of N, then for some integer n, p is at a distance $1/n$ from the origin in the plane. Then collections $G_2, G_3, \ldots$ may be chosen to satisfy conditions (a) and (b′). However, the space is not metrizable.

2. Developable spaces. For each developable topological space there is a sequence $G_1, G_2, \ldots$ such that (a) G_i is a covering of the space with open sets, (b) G_{i+1} is a refinement of G_i, and (c) for each domain and each point p in D there is an integer $n(p, D)$ such that each element of $G_{n(p,D)}$ which contains p lies in D. Condition (b) is not necessary in defining a developable space because if there is a sequence satisfying conditions (a) and (c), there is one satisfying conditions (a), (b), and (c). In fact, in [5] the condition is imposed that G_{i+1} is a subcollection of G_i. Regular developable topological spaces have been studied extensively because a Moore space is such a space.

[9]A topological space satisfies the first axiom of countability at a point p if there is a countable collection G of neighbourhoods of p such that any neighbourhood of p is a subset of an element of G.

As seen from Examples A and D, not all developable spaces are screenable and not all screenable spaces are developable.

EXAMPLE D. *A regular, separable, strongly screenable space that is not perfectly screenable or developable.* Points belong to the x-axis and neighbourhoods are closed intervals minus their right hand end points.

The space is separable because each set of points dense on the x-axis is dense in the space. If H is an open covering of it, there is an open covering H' which refines H such that no two elements of H' intersect each other. The space is strongly screenable because each such open covering H' is a discrete collection.

The space is not perfectly separable because for each countable collection G of neighbourhoods, there is a point p that does not belong to the left end of any element of G and any neighbourhood of p with a left end at p is not the sum of a subcollection of G. Since the space is separable but not perfectly separable, it is not metrizable. It follows from Theorem 3 that it is not perfectly screenable and from Theorem 5 that it is not developable.

THEOREM 5. *A separable screenable developable space is perfectly separable.*

Proof. No separable space contains uncountably many mutually exclusive domains. Hence if H is an open covering of a separable screenable space, there is a countable open covering H' which refines H. If $G_1, G_2, \ldots$ is a development of a space S and G'_i is a countable open covering that refines G_i, then $\sum G'_i$ is a countable basis for S.

A similar argument shows that a separable perfectly screenable space is perfectly separable.

THEOREM 6. *A strongly screenable developable space is perfectly screenable.*

Proof. Let $G_1, G_2, \ldots$ be a development of the space. Since the space is strongly screenable, for each positive integer i there is a sequence $H_{i1}, H_{i2}, \ldots$ such that H_{ij} is a discrete collection of domains and $\sum_{j=1}^{\infty} H_{ij}$ covers the space and is a refinement of G_i. Then $\{H_{ij}; i, j = 1, 2, \ldots\}$ is a countable collection insuring that the space is perfectly screenable.

THEOREM 7. *A regular developable space (Moore space) is metrizable if it is strongly screenable.*

Theorem 7 follows from Theorems 3 and 6. That Theorem 7 cannot be altered by assuming screenability instead of strong screenability may be seen from Example B.

THEOREM 8. *A screenable Moore space is metrizable if it is normal.*

Proof. This result will follow from Theorem 7 if it is shown that a screenable normal developable space is strongly screenable.

Let H be a collection of mutually exclusive open sets, W be the complement of H^*, and $G_1, G_2, \ldots$ be a development of the space. Denote by X_i the sum of all points p such that no element of G_i containing p intersects W. Since

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the space is normal, there is a domain D containing X_i such that $\bar{D}$ does not intersect W. If H_i is the collection of all domains h such that h is the common part of D and an element of H, then H_i is a discrete collection of domains. Since for each collection H of mutually exclusive domains there is a collection $\sum H_i$ covering H such that H_i is a discrete collection of domains which is a refinement of H, a normal developable space is strongly screenable if it is screenable.

THEOREM 9. *For each open covering H of a developable space there is a sequence $X_1, X_2, \ldots$ such that X_i is a discrete collection of closed sets which is a refinement of both X_{i+1} and H while $\sum X_i$ covers the space.*

Proof. Suppose W is a well ordering of H and $G_1, G_2, \ldots$ is a development of the space such that G_{i+1} is a refinement of G_i. For each element h of H, let $x(h, i)$ denote the sum of all points p such that no element of H that contains p precedes h in W and each element of G_i containing p is a subset of h. If X_i denotes the collection of all such sets $x(h, i)$, X_i is a discrete collection because no element of G_i intersects two elements of X_i. If p is a point and $h(p)$ is the first element of H and W containing p, then for some integer i, $[h(p), i]$ contains p. Hence $\sum X_i$ covers the space.

THEOREM 10. *A Moore space is metrizable if it is collectionwise normal.*

Proof. If it is shown that a collectionwise normal Moore space is screenable Theorem 10 will follow from Theorem 8.

For each open covering H of the space, it follows from Theorem 9 that there is a sequence $X_1, X_2, \ldots$ such that X_i is a discrete collection of closed sets and $\sum X_i$ is a covering of the space which is a refinement of H. Collectionwise normality insures that there is a collection Y_i of mutually exclusive open sets covering X^*_i such that no element of Y_i intersects two elements of X_i but each is a subset of an element of H. Then $Y_1, Y_2, \ldots$ is a sequence such that $\sum Y_i$ is a refinement of H covering the space and Y_i is a collection of mutually exclusive domains. Hence, the space is screenable.

Question. Is there a normal Moore space which is neither screenable nor collectionwise normal? If this question could be answered, it could be determined whether or not each normal Moore space is metrizable. F. B. Jones showed [4] that such a space is metrizable if it is separable and $\aleph_1 = C$. Hence, the space mentioned in Example E is metrizable if $\aleph_1 = C$.

EXAMPLE E. *A separable normal Moore space.* The points of the space are the points of the plane which lie above the x-axis and the points of a subset X of the x-axis such that each subset of X is the common part of X and a G_δ set in the plane. The elements of G_i are of two sorts: (a) the interior of a circle of radius less then $1/i$ which lies above the x-axis and (b) $p + I$ where p is a point of X and I is the interior of a circle of diameter less than $1/i$ which is tangent to the x-axis at p from above.

If X is countable, the above space is metrizable because it is perfectly

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separable. The set X cannot have the power of the continuum because each subset Y with the power of the continuum in a separable metric space S contains a subset Y' which is not the common part of Y and a G_δ set in S.

In my paper [2] the following theorem is proved.

THEOREM 11. *A topological space is metrizable provided there exists a sequence* $H_1, H_2, \ldots$ *such that*

(a) *for each integer* i, H_i *is a collection of sets covering space.*

(b) *a point* p *is a point of the closure of the set* M *if and only if for each integer* n, *some element of* H_n *contains* p *and intersects* M, *and*

(c′) *each pair of points that is covered by either an element of* H_{i+1} *or the sum of a pair of intersecting elements of* H_{i+1} *can be covered by an element of* H_i.

In [2] it was falsely stated that (c′) could be replaced by (c) *each pair of points that is covered by the sum of a pair of intersecting elements of* H_{i+1} *can be covered by an element of* H_i. I am indebted to Dick Wick Hall for calling my attention to the fact that this replacement is not possible. This paper was being studied in one of his classes and L. K. Meals, a member of that class, pointed out that if S is a nonmetrizable space with a development $G_1, G_2, \ldots$ such that G_{i+1} is a refinement of G_i (as in Example A), then if H_i is G_i or S according as i is odd or even, then $H_1, H_2, \ldots$ satisfies conditions (a), (b), and (c).

3. Collectionwise normality. In a developable space, either full normality[10] or collectionwise normality implies metrizability. However, in general, collectionwise normality is weaker than full normality as is shown in the following theorem.

THEOREM 12. *Full normality implies collectionwise normality but not conversely.*

Proof. Let W be a discrete collection of closed sets and H be an open covering of the space such that no element of H intersects two elements of W. If the space is fully normal, there is an open covering H' of the space such that for each point p, the sum of the elements of H' containing p is a subset of an element of H. For each element w of W let D_w be the sum of the elements of H' intersecting w. If w_1 and w_2 are different elements of W, D_{w_1} does not intersect D_{w_2} in a point p or else an element of H containing p intersects both w_1 and w_2. Then the collection of all such domains D_w is a collection of mutually exclusive domains covering W such that no one of these domains intersects two elements of W.

The space described below shows that collectionwise normality does not imply full normality.

[10]A space is fully normal if for each open covering H of the space there is an open covering H' of the space such that the star of each point with respect to H' is a subset of an element of H. Stone showed [6] that the notions of full normality and paracompactness are equivalent for a topological space.

EXAMPLE F. *A collectionwise normal space which is not fully normal.* Points are the elements of an uncountable well ordered collection W such that no element of W is preceded by uncountably many elements of it. A neighbourhood is either the first element of W or the collection of all points that lie between two nonadjacent elements of W.

This space is collectionwise normal because it is normal and does not contain an infinite discrete collection of points. The space is not fully normal because if H is any collection of open sets covering the space, there is a point p such that the star of p with respect to H contains all points that follow p. There is an open covering H such that no elements of H contains all the points that follow some point in it.

THEOREM 13. *Suppose H is an open covering of a collectionwise normal space S and $H_1, H_2, \ldots$ is a sequence such that H_i is a discrete collection of closed sets and $\sum H_i$ is a refinement of H which covers S. Then there is an open covering G of S such that G is a refinement of H and for each point p of an element of H_i there is a domain D containing p such that not more than i elements of G intersect D.*

Since a metric space is developable and collectionwise normal, Theorems 9 and 13 imply that a metric space is paracompact [6]. Theorem 13 would not be true if the hypothesis that $\sum H_i$ covers S were omitted.

Proof of Theorem 13. Since S is collectionwise normal, there is a discrete collection Y_i of open sets covering the sum of the elements of H_i such that Y_i is a refinement of H and each element of Y_i intersects just one element of H_i. As S is normal, there is an open set D_i containing the sum of the elements of H_i such that Y_i covers $\bar{D}_i$. Each element of Y_1 is an element of G and if y is an element of Y_{i+1} not covered by $\bar{D}_1+\bar{D}_2+\ldots+\bar{D}_i$, $y-(\bar{D}_1+\bar{D}_2 \ldots +\bar{D}_i)$ is an element of G.

THEOREM 14. *Collectionwise normality implies normality but normality does not imply collectionwise normality.*

Proof. Since a collection consisting of two mutually exclusive closed sets is a discrete collection, collectionwise normality implies normality. We shall show that the space described in Example G is normal but not collectionwise normal.

EXAMPLE G. *A normal topological space that is not collectionwise normal.* Let P be an uncountable set, Q the set of all subsets of P, and F the set of all functions f on Q having only 1 and 0 as values. To each element p of P associate the function f_p whose value $f_p(q)$ on q is 1 or 0 according as p belongs to q or not. Let F_p be the set of all such functions f_p. The set F is topologized as follows. Any point f in $F - F_p$ is declared to be a neighbourhood of itself. Given a point f_p in F_p and a finite subset r of Q we define the r neighbourhood of f_p to be the set of all f such that $f(q) = f_p(q)$ whenever q belongs to r.

To show the space F thus topologized is normal consider two mutually exclusive closed subsets H_1 and H_2 of it. Let A_k(k = 1, 2) be the set of points common to H_k and F_p and let q_k be the associated set in P consisting of all p for which f_p belongs to A_k. We suppose that neither A_1 nor A_2 is null because if $A_1 = 0$, H_1 and $F - H_1$ are mutually exclusive domains containing H_1 and H_2 respectively. The set D_k of all f in F such that $f(q_k) = 1$ and $f(q_j) = 0$ $(j \neq k)$ is then an open set containing A_k. Moreover no point in F is common both to D_1 and D_2. Therefore the sets $(D_1 - H_2) + (H_1 - A_1)$ and $(D_2 - H_1) + (H_2 - A_2)$ are mutually exclusive open sets containing H_1 and H_2 respectively.

We now show that F is not collectionwise normal. The subset $F_p = \{f_p\}$ of F is a discrete collection of points. However, there does not exist a collection of mutually exclusive neighbourhoods $\{D_p\}$ such that D_p is a neighbourhood of f_p. For suppose, to the contrary, that there were such a collection. Let D_p be the r_p neighbourhood of f_p. Since r_p is a finite subset of Q and P is uncountable there is an integer n and an uncountable subset W of P such that r_p has exactly n elements for every p in W. For any two elements a and b of W the sets r_a and r_b have an element in common, else D_a and D_b would intersect. Hence there is an element q_1 of Q and an uncountable subset W'_1 of W such that q_1 belongs to r_p for every p in W'_1. Moreover there is a t_1 with value 1 or 0 and an uncountable subset W_1 of W'_1 such that $f_p(q_1) = t_1$ for every p in W_1. Similarly there is an element q_2 of Q different from q_1, a t_2 with value 1 or 0, and an uncountable subset W_2 of W_1 such that q_2 belongs to r_2 and $f_p(q_2) = t_2$ for every p in W_2. Continuing recursively in this fashion we get q_k, t_k, W_k for $k = 1, 2, \ldots, n$. Let r be the set consisting of $q_1, q_2, \ldots, q_n$ and D the set of all f with $f(q_k) = t_k$ for $k = 1, 2, \ldots, n$. Then $r_p = r$ and $D_p = D$ for all p in W_n in contradiction to the choice of D_p as mutually exclusive.

One might wonder if Example G could be modified so as to obtain a normal developable space which is not metrizable. A developable space could be obtained by introducing more neighbourhoods into the space F. However, a difficulty might arise in introducing enough neighbourhoods to make the resulting space developable but not enough to make it collectionwise normal.

Another method of modifying Example G is to replace points by closed sets. No non-isolated point of the space F is the intersection of a countable number of neighbourhoods. In Example H, we modify Example G by replacing the points of F by closed sets so as to get a space in which all closed sets are inner limiting (G_δ) sets.[11]

EXAMPLE H. *A normal topological space that is not collectionwise normal and in which each closed set is an inner limiting* (G_δ) *set.* We define P, Q, and F_p as in Example G but let the points of F be functions f defined on Q such

[11]A set is an inner limiting set or G_δ set if it is the intersection of a countable number of open sets.

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that $f(x)$ is a non-negative integer. Each point of $F - F_p$ is a neighbourhood. For each finite subset r of Q, each element f_p of F_p, and each positive integer n, f_p plus all points f of F such that $f(x) > n$ (x element of Q) and $f(x') = f_p(x')$ mod 2 (x' element of r) is a neighbourhood.

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