Addition of a Third of a Period to the Argument of the Elliptic Function.

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1. If I is an inflexion on a non-singular plane cubic curve, a variable line IPP' establishes a (1, 1) correspondence between points P, P' on the curve. This correspondence defines a perspective transformation of the whole plane, with I for pole, and the harmonic polar of I for axis, of perspective; for, when I is projected to infinity on the y-axis, and its harmonic polar taken for x-axis, the resulting equation

$$y^2 = ax^3 + bx^2 + cx + d$$

indicates a curve symmetrical with respect to the latter.

It follows that the cartesian coordinates of P' are linear functions of those of P. But if the elliptic parameters of I, P are $\Omega / 3$, u, (Ω being as usual a period), that of P' is $-u - \Omega / 3$. Hence $\wp(u + \Omega / 3)$, $\wp'(u + \Omega / 3)$ can be expressed *linearly* in terms of $\wp(u)$, $\wp'(u)$.

If to I, P, P' are assigned the homogeneous point-coordinates (x_0, y_0, z_0), (x, y, z), (X, Y, Z), and to the axis of perspective the equation

$$L \equiv \lambda x + \mu y + \nu z = 0,$$

the equations of the transformation are

$$X : Y : Z = x + \tau x_0 L : y + \tau y_0 L : z + \tau z_0 L, \qquad \text{.............. (1)}$$

where $\quad -2\tau^{-1} = \lambda x_0 + \mu y_0 + \nu z_0$.

Let the equation of the cubic be taken in the form

$$y^2 = 4x^3 - q_2 x - q_3 ; \qquad \text{.......................... (2)}$$

let $\wp(\Omega / 3)$, $\wp'(\Omega / 3)$, ... be denoted by α, α', ... ; then

$$\lambda = 12\, \alpha \alpha', \; \mu = \alpha'', \; \nu = 3\, \alpha' (\alpha'' - 4\alpha^2), \; \tau = -1/2\, \alpha' \alpha'' ; \qquad \text{...... (3)}$$

and we have the desired formulae

$$\wp(u+\Omega/3)=\frac{2\alpha'\alpha''\wp(u)-\alpha L}{2\alpha'\alpha''-L},\quad -\wp'(u+\Omega/3)=\frac{2\alpha'\alpha''\wp'(u)-\alpha' L}{2\alpha'\alpha''-L},\quad \ldots.(4)$$

where $L\equiv 12\alpha\alpha'\wp(u)+\alpha''\wp'(u)+3\alpha'(\alpha''-4\alpha^2)$.

Two modifications of these formulae may be noticed.

(i) From the condition for an inflexion, $\alpha''^2=\alpha'\alpha'''=12\alpha\alpha'^2$. Write

$$\beta=\alpha''/\alpha'=\pm\sqrt{(12\alpha)},\ \beta'=24\alpha';\ \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

write also

$$x_1=12\wp(u),\quad X_1=12\wp(u+\Omega/3),$$

$$y_1=24\wp'(u),\quad Y_1=24\wp'(u+\Omega/3)$$

then

$$\left.\begin{aligned} X_1&=\frac{-2(\beta'-\beta^3)x_1+\beta^2 y_1+\beta^2(3\beta'-2\beta^3)}{2\beta x_1+y_1+(\beta'-2\beta^3)}\\ -Y_1&=\frac{2\beta\beta' x_1-\beta' y_1+\beta(3\beta'-2\beta^3)}{2\beta x_1+y_1+(\beta'-2\beta^3)}\end{aligned}\right\}.\ \ldots\ldots\ldots(6)$$

(ii) Write

$$x_2=12\{\wp(u)-\wp(\Omega/3)\},\quad X_2=12\{\wp(u+\Omega/3)-\wp(\Omega/3)\},$$

$$y_2=24\{\wp'(u)+\wp'(\Omega/3)\}/2\wp'(\Omega/3),$$

$$Y_2=24\{\wp'(u+\Omega/3)+\wp'(\Omega/3)\}/2\wp'(\Omega/3);$$

then

$$X_2:Y_2:1=-x_2:y_2-1:\delta x_2+y_2,\ \ldots\ldots\ldots\ldots\ldots\ldots(7)$$

where $\delta=\beta/\beta'=\alpha/2\alpha''$.

2. New expressions for the coefficients in (6) are obtained by considering the *canonical* form of (2), say

$$x_1^3+x_2^3+x_3^3+6mx_1x_2x_3=0.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

To transform (8) into (2), write

$$x_1:x_2:x_3=x+b:mx+ny-c:mx-ny-c;\ \ldots\ldots\ldots\ldots(9)$$

then, n remaining arbitrary, b, c are determined by

$$b/6m^2=c/(1+2m^3)=1/24n^3;$$

also $\quad q_2=-m(1-m^3)/12n^4,\ q_3=(1-20m^3+8m^6)/12^3n^6,$

$$\Delta\equiv q_2^3-27q_3^2=-\{(1+8m^3)/48n^4\}^3.$$

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The two cases to be considered are:

Case A: $m < -\frac{1}{2}$, $\Delta > 0$;

Case B: $m > -\frac{1}{2}$, $\Delta < 0$.

CASE A: $\Delta > 0$.

Let $\tau_{\mu\mu'} \equiv (2\mu\omega + 2\mu'\omega')/3$, $(\mu, \mu' = 0, 1, 2)$, where 2ω, $2\omega'$ are the real and purely imaginary periods; and let $I_{\mu\mu'}$ denote the inflexion of which $\tau_{\mu\mu'}$ is the elliptic parameter. The (x_1, x_2, x_3) coordinates of the inflexions may then be taken as follows, where $\epsilon \equiv \exp.(2\pi i/3)$:

$$I_{00}(0, 1, -1),\ I_{01}(0, \epsilon, -\epsilon^2),\ I_{02}(0, \epsilon^2, -\epsilon),$$
$$I_{10}(-1, 0, 1),\ I_{11}(-\epsilon^2, 0, \epsilon),\ I_{12}(-\epsilon, 0, \epsilon^2),$$
$$I_{20}(1, -1, 0),\ I_{21}(\epsilon, -\epsilon^2, 0),\ I_{22}(\epsilon^2, -\epsilon, 0).$$

Of the nine non-congruent values of $\tau_{\mu\mu'}$, four only, in addition to τ_{00}, are effectively distinct for the present purpose, viz., τ_{10}, τ_{01}, τ_{11}, τ_{12}. The values of $\wp(\tau_{\mu\mu'})$, $\wp'(\tau_{\mu\mu'})$, briefly written $\wp_{\mu\mu'}$, $\wp'_{\mu\mu'}$, are given by the following table, in which the last column is inserted to resolve the ambiguity of sign in (5) above.*

μ	μ'	$12n^2\wp_{\mu\mu'}$	$24n^3\wp'_{\mu\mu'}$	$24n^4\wp''_{\mu\mu'}$
1	0	$(1-m)^2$	$1-2m+4m^2$	$(1-2m)(1-2m+4m^2)$
0	1	$-3m^2$	$-i(1+8m^3)/\sqrt{3}$	$m(1+8m^3)$
1	1	$\epsilon(1-\epsilon m)^2$	$1-2\epsilon m+4\epsilon^2 m^2$	$\epsilon^2(1-\epsilon m)(1-2\epsilon m+4\epsilon^2 m^2)$
1	2	$\epsilon^2(1-\epsilon^2 m)^2$	$1-2\epsilon^2 m+4\epsilon m^2$	$\epsilon(1-\epsilon^2 m)(1-2\epsilon^2 m+4\epsilon m^2)$

* *Cf.* Baker, *B. A. Report* 1910 (Sheffield), p. 528.

Now writing

$$p_{\mu\mu'} = n\wp''_{\mu\mu'} / \wp'_{\mu\mu'}, \; p'_{\mu\mu'} = 24n^3 \wp'_{\mu\mu'},$$

we have for these the table

μ	μ'	$p_{\mu\mu'}$	$p'_{\mu\mu'}$
1	0	$1 - m$	$1 - 2m + 4m^2$
0	1	$i\sqrt{3}\,.\,m$	$-i(1+8m^3)/\sqrt{3}$
1	1	$\epsilon^2(1-\epsilon m)$	$1 - 2\epsilon m + 4\epsilon^2 m^2$
1	2	$\epsilon(1-\epsilon^2 m)$	$1 - 2\epsilon^2 m + 4\epsilon m^2$

Putting further

$$12n^2\wp(u) = P, \; 12n^2\wp(u+\tau_{\mu\mu'}) = P_{\mu\mu'},$$

$$24n^3\wp'(u) = P', \; 24n^3\wp'(u+\tau_{\mu\mu'}) = P'_{\mu\mu'},$$

we have

$$P_{\mu\mu'} : -P'_{\mu\mu'} : 1 = A_{\mu\mu'} : B_{\mu\mu'} : C_{\mu\mu'}, \quad \ldots\ldots\ldots\ldots(10)$$

where

$$A_{\mu\mu'} = -2(p'_{\mu\mu'} - p^3_{\mu\mu'})P + p^2_{\mu\mu'}P' + p^2_{\mu\mu'}(3p'_{\mu\mu'} - 2p^3_{\mu\mu'}).$$

$$B_{\mu\mu'} = 2p_{\mu\mu'}p'_{\mu\mu'}P - p'_{\mu\mu'}P' + p'_{\mu\mu'}(3p'_{\mu\mu'} - 2p^3_{\mu\mu'}),$$

$$C_{\mu\mu'} = 2p_{\mu\mu'}P + P' + (p'_{\mu\mu'} - 2p^3_{\mu\mu'}).$$

Formula (7) may also now be written more explicitly as follows. Put

$$x_{\mu\mu'} = 12n^2\{\wp(u) - \wp_{\mu\mu'}\}, \; X_{\mu\mu'} = 12n^2\{\wp(u+\tau_{\mu\mu'}) - \wp_{\mu\mu'}\},$$

$$y_{\mu\mu'} = \{\wp'(u) + \wp'_{\mu\mu'}\}/2\wp'_{\mu\mu'}, \; Y_{\mu\mu'} = \{\wp'(u+\tau_{\mu\mu'}) + \wp'_{\mu\mu'}\}/2\wp'_{\mu\mu'},$$

$$\rho_{\mu\mu'} = y_{\mu\mu'}/y'_{\mu\mu'};$$

then

$$X_{\mu\mu'} : Y_{\mu\mu'} : 1 = -x_{\mu\mu'} : y_{\mu\mu'} - 1 : \rho_{\mu\mu'}x_{\mu\mu'} + y_{\mu\mu'} \ldots\ldots(11)$$

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Thus $\wp(u+\tau_{\mu\mu'})$, $\wp'(u+\tau_{\mu\mu'})$ have been expressed linearly in terms of $\wp(u)$, $\wp'(u)$, the coefficients being, for all values of μ, μ', simple rational functions of m and of the disposable constant n.

CASE B: $\Delta < 0$.

Let 2ω, $2\omega'$ denote the conjugate imaginary periods; we can assume them so chosen that $\omega_2 \equiv \omega' + \omega$, $\omega_2^1 = \omega' - \omega$, are respectively a positive real quantity and a positive pure imaginary. The discussion of case A will become valid for this case also, if we replace the ω, ω' of case A by the quantities ω_2, ω_2^1.

3. For the general cubic $\phi(x, y, 1) = 0$ let the parametric representation be given by

$$x = \epsilon(u), \quad y = \zeta(u),$$

where ϵ, ζ are elliptic functions with the same periods, and the parameter u vanishes at an inflexion. It follows, just as in §1, that $\epsilon(u+\Omega/3)$, $\zeta(u+\Omega/3)$ can be expressed linearly in terms of $\epsilon(u)$, $\zeta(u)$, where Ω is any period. Writing $x_0 = \epsilon(\Omega/3)$, $y_0 = \zeta(\Omega/3)$, $X = \epsilon(u+\Omega/3)$, $Y = \zeta(u+\Omega/3)$, we have (*cf.* (1))

$$X : Y : 1 = x + \tau x_0 L : y + \tau y_0 L : 1 + \tau L, \quad \text{..............(12)}$$

where

$$L \equiv \frac{\phi_{11}}{\phi_1} x + \frac{\phi_{22}}{\phi_2} y + \frac{\phi_{33}}{\phi_3}, \quad -2\tau^{-1} = \frac{\phi_{11}}{\phi_1} x_0 + \frac{\phi_{22}}{\phi_2} y_0 + \frac{\phi_{33}}{\phi_3},$$

ϕ_1, ϕ_{11}, ... denoting the values of $\partial\phi(x, y, z)/\partial x$, $\partial^2\phi(x, y, z)\,\partial x^2$, ... at the point $(x_0, y_0, 1)$. The form of L (the harmonic polar) is obvious from its definition as part of the degenerate polar conic of the point of inflexion. The coefficients of the substitution (12) are then simple rational functions of $\epsilon(\Omega/3)$, $\zeta(\Omega/3)$.

As a typical case we might take the cubic defined by

$$x = \text{sn}^2 u, \quad y = \text{sn}\, u \,\text{cn}\, u \,\text{dn}\, u\,;$$

if we write $v = u + 2mK/3 + 2niK'/3$, we shall have $\text{sn}^2 v$, $\text{sn}\, v\, \text{cn}\, v\, \text{dn}\, v$, expressed as linear functions of $\text{sn}^2 u$, $\text{sn}\, u\, \text{cn}\, u\, \text{dn}\, u$.