ON CERTAIN PAIRS OF MATRICES WHICH DO NOT GENERATE A FREE GROUP[1]

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A complex number λ will be said to be free if the multiplicative group F_λ generated by the two matrices

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

is a free group, and non-free, otherwise. Very little is known about the distribution of free and non-free numbers [1]. It is, for instance, unknown whether the domain

$$D = \{\lambda : |\lambda| < 1 \text{ or } |\lambda + 1| < 1 \text{ or } |\lambda - 1| < 1\}$$

contains an open set which consists of only free points.

In this note, it will be shown, among other things, that the open segment joining -2 and 2 and the open segment joining $-\sqrt{-1}$ and $\sqrt{-1}$ are contained in an open domain (in the complex plane) in which non-free points are densely distributed.

For the history and motivation of the problem considered here and for the ramifications of the problem, see [1] and the references therein. The main result in [2], which appeared later than [1], is much weaker than that of [1].

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[1] This paper was written while the author held a Research Associateship of the Office of Naval Research, U. S. Navy.

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THEOREM 1. If

$$(1) \qquad G = \begin{pmatrix} p(\lambda) & q(\lambda) \\ r(\lambda) & s(\lambda) \end{pmatrix}$$

is any word generated by A and B, where $p(\lambda)$, $q(\lambda)$, $r(\lambda)$, and $s(\lambda)$ are polynomials in λ, and if $q(\lambda)$ is not identically zero, then non-free points are densely distributed in the domain (in the complex plane) defined by $|\lambda q(\lambda)| \leq 1$.

Proof. Define the words $G_1, G_2, \ldots, G_n, \ldots$, inductively by $G_1 = G$, and $G_{n+1} = G_n B G_n^{-1} B^{-1}$. If

$$G_n = \begin{pmatrix} p_n(\lambda) & q_n(\lambda) \\ r_n(\lambda) & s_n(\lambda) \end{pmatrix},$$

then

$$G_n^{-1} = \begin{pmatrix} s_n(\lambda) & -q_n(\lambda) \\ -r_n(\lambda) & p_n(\lambda) \end{pmatrix}$$

since the determinant of G_n is 1. Hence we can compute G_{n+1} and check easily that

$$q_{n+1}(\lambda) = -\lambda q_n^2(\lambda); \qquad \operatorname{tr} G_{n+1} = 2 + \lambda_n^2 q_n^2(\lambda).$$

It follows from these that

$$\operatorname{tr} G_{n+1} = 2 + (\lambda q(\lambda))^{2^n}.$$

Now, fix n, take θ arbitrarily such that $0 < \theta < 2\pi$, $\theta \neq \pi$, and let λ be any complex number satisfying

$$(2) \qquad 2 + (\lambda q(\lambda))^{2^n} = e^{i\theta} + e^{-i\theta}.$$

Then, since $\det G_{n+1} = 1$, the characteristic roots of G_{n+1} are $e^{i\theta}$ and $e^{-i\theta}$; since $e^{i\theta} \neq e^{-i\theta}$, G_{n+1} can be diagonalized, and if, moreover, θ is a rational multiple of π, then G_{n+1} has a finite period and hence λ is non-free. Since the rational multiples of π are densely distributed in $[0, 2\pi]$ and since n can be made arbitrarily large, it is clear from (2) that any complex number λ satisfying $|\lambda q(\lambda)| \leq 1$ is a limit of non-free points. Thus the theorem is proved.

By taking $G = A$, we obtain the following

COROLLARY 1. Non-free points are densely distributed in the circle $|\lambda| \leq \frac{1}{2}$.

In order to obtain some other $q(\lambda)$'s, let

$$(AB)^n = \begin{pmatrix} 2\lambda + 1 & 2 \\ \lambda & 1 \end{pmatrix}^n = \begin{pmatrix} p_n & 2q_n \\ r_n & s_n \end{pmatrix}.$$

Then we have

$$(3) \quad q_{n+2} = \mu q_{n+1} - q_n \quad (n = 1, 2, \ldots), \; q_1 = 1, \; q_2 = \mu,$$

where $\mu = 2\lambda + 2$. The polynomials $f_n(\mu)$ defined by $\sin n\theta / \sin\theta = f_n(2\cos\theta)$ satisfy the relations (3). Hence we have $q_n = f_n$ for all n.

Now any real number λ in $(-2, 0)$ can be expressed as $\lambda = \cos\theta - 1$, $0 < \theta < \pi$, and for some integer n we have

$$\left|(\cos\theta - 1)\frac{\sin n\theta}{\sin\theta}\right| < 1,$$

or

$$|\lambda q_n(\lambda)| < 1.$$

Since λ is free if and only if $-\lambda$ is free, we may conclude by theorem 1 that the open segment joining -2 and 2 is contained in an open domain in the complex plane in which non-free points are densely distributed.

Using $(ABA^{-1}B^{-1})^n$ instead of $(AB)^n$, above, and arguing similarly, we may obtain a similar conclusion about the open segment joining $-\sqrt{-1}$ and $\sqrt{-1}$. Thus we have proved

COROLLARY 2. The open segment joining -2 and 2 and the open segment joining $-\sqrt{-1}$ and $\sqrt{-1}$ are contained in an open domain in the complex plane in which non-free points are densely distributed.

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