Proceedings of the Edinburgh Mathematical Society (2013) **56**, 501–508
DOI:10.1017/S0013091512000296

A VANISHING THEOREM IN TWISTED DE RHAM COHOMOLOGY

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(Received 30 May 2011)

Abstract We prove a vanishing theorem for the twisted de Rham cohomology of a compact manifold.

Keywords: twisted de Rham cohomology; connections with skew torsion; Dirac operators

2010 *Mathematics subject classification:* Primary 53C21
Secondary 58A14

1. Introduction

The concept of twisted cohomology was first introduced in 1986 by Rohm and Witten in the appendix of [**12**]. It has played a significant role in physics, in particular in string theory, since the Ramond–Ramond fields and their charges in type-II theories lie in the twisted cohomology of space-time [**6**].

From the mathematics point of view, twisted de Rham cohomology, or simply d_H cohomology, has been studied in the context of both K-theory and Poisson geometry. The link with K-theory was first considered by Atiyah in [**2**]. The precise definition is given by Bouwknegt *et al.* in [**5**].

From a different approach, d_H cohomology has been present in Poisson geometry since Severa and Weinstein's introduction of Courant algebroids in [**14**]. Roytenberg connected this Courant bracket with a homological vector field in his doctoral thesis [**13**] and Kosmann-Schwartzbach spelled this out in differential geometric terms in [**9**].

Further, basic properties of d_H cohomology and its relation to formality were obtained by Cavalcanti in his doctoral thesis [**7**], where it was shown that the different differentials in the spectral sequence correspond to Massey products, a result obtained independently by Atiyah and Segal in [**3**].

Twisted de Rham cohomology continues to be a topic of research interest. In a very recent paper [**11**], Mathai and Wu have considered the notion analytic torsion for twisted complexes; they generalize the classical construction of the Ray–Singer torsion to the twisted de Rham complex with an odd-degree differential form and with coefficients in a flat vector bundle.

In this paper, we present a crossover between Riemannian geometry and differential topology. We show how to use connections with skew torsion to identify the operator

$(d_H) + (d_H)^*$, where H is a 3-form, with a cubic Dirac operator. In the compact case, if H is closed, we prove a vanishing theorem for twisted de Rham cohomology by means of a Lichnerowicz formula. As an application, we prove that for a compact non-abelian Lie group the cohomology of the complex defined by $\mathrm{d} + H$, where H is the 3-form defined by the Lie bracket, vanishes. This is a similar result to that of Cavalcanti [**7**], which is reobtained here by purely Riemannian geometric methods.

2. The Dirac operator

Let (M, g) be a Riemannian manifold. Suppose that ∇ is a connection on the tangent bundle of M and let T be its (1,2) torsion tensor. If we contract T with the metric we get a $(0, 3)$ tensor, which we will still call the torsion of ∇. If T is a 3-form, then we say that ∇ is a connection with skew-symmetric torsion. Given any 3-form H on M, there exists a unique metric connection with skew torsion H defined explicitly by

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) + \tfrac{1}{2} H(X, Y, Z),$$

where ∇^g is the Levi-Civita connection.

Fix a 3-form H and consider the one-parameter family of affine connections

$$\nabla^s := \nabla^g + 2sH.$$

(Notice that if $s = \frac{1}{4}$, we recover the connection with torsion H.) If M is spin, these connections lift to the spin bundle $\$$ of M as

$$\nabla^s_X(\varphi) := \nabla^g_X(\varphi) + s(i_X H)\varphi,$$

where X is a vector field, φ is a spinor field and $i_X H$ is acting by Clifford multiplication.

We may define the Dirac operator $\not{D}$ on $\$$ with respect to ∇ by means of the following composition:

$$\Gamma(M, \$) \to \Gamma(M, T^*M \otimes \$) \to \Gamma(M, TM \otimes \$) \to \Gamma(M, \$),$$

where the first arrow is given by the connection, the second by the metric and the third by the Clifford action. Suppose now that we have a complex vector bundle $\mathcal{W}$; we can form the tensor product $\$ \otimes \mathcal{W}$, which is usually called a twisted spinor bundle or a spinor bundle with values in $\mathcal{W}$. If $\mathcal{W}$ is equipped with a Hermitian connection $\nabla^{\mathcal{W}}$, we can consider the tensor product connection $\nabla \otimes 1 + 1 \otimes \nabla^{\mathcal{W}}$, again denoted by ∇, on $\$ \otimes \mathcal{W}$. We can define a Dirac operator on this twisted spinor bundle associated with the connection ∇ by the same formula, where the action of the tangent bundle by Clifford multiplication is only on the left factor.

We will need to make use of a Lichnerowicz-type formula for the square of the Dirac operator. Such a formula first appeared in the literature in [**4**] (for the case $s = 1$) and was subsequently proved in full generality in [**1**].

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Theorem 2.1 (Bismut [4]; Agricola-Friedrich [1]). *The rough Laplacian* $\Delta^s = \nabla^{s*}\nabla^s$ *and the square of the Dirac operator* $D^{s/3}$ *are related by*

$$(D^{s/3})^2 = \Delta^s + F^{\mathcal{W}} + \tfrac{1}{4}\kappa + s\,\mathrm{d}H - 2s^2\|H\|^2,$$

where κ *is the Riemannian scalar curvature and* F *is the curvature of the twisting bundle acting as* $\sum_{i<j} F^{\mathcal{W}}(e_i, e_j)e_ie_j$ *on* $\slashed{S} \otimes \mathcal{W}$.

Notice that this formula relates the square of the Dirac operator $D^{s/3}$ and the Laplacian Δ^s. The Dirac operator $D^{1/3}$ is usually referred to as the cubic Dirac operator.

3. Twisted cohomology

Consider the spinor bundle with values in itself, that is, $\slashed{S} \otimes \slashed{S}$. Recall that for this we do not need a global spin structure. We have, in even dimensions, the following chain of isomorphisms

$$\slashed{S} \otimes \slashed{S} \simeq \slashed{S}^* \otimes \slashed{S} \simeq \mathrm{End}(\slashed{S}) \simeq \mathrm{Cl} \simeq \Lambda,$$

where Cl denotes the Clifford bundle and Λ denotes the bundle of exterior forms.

If we take the induced Levi-Civita connection ∇^g on both factors of $\slashed{S} \otimes \slashed{S}$ and consider the tensor product connection $\nabla^g \otimes 1 + 1 \otimes \nabla^g$, we obtain the induced Levi-Civita connection, again denoted by ∇^g, on Λ. If we consider the associated Dirac operator D^g on $\slashed{S} \otimes \slashed{S}$, we get a familiar operator on Λ. In fact,

$$D^g = \mathrm{d} + \mathrm{d}^*,$$

where d is the exterior differential and d^* is its formal adjoint [**10**].

The same fact can be claimed for an odd-dimensional manifold. Consider the inclusion $M \hookrightarrow \mathbb{R} \times M$, and the half spinor bundles $\slashed{S}^+$ and $\slashed{S}^-$, of $\mathbb{R} \times M$. The Clifford action by e_0, where e_0 is a unit vector field of $\mathbb{R}$, gives an isomorphism between $\slashed{S}^+$ and $\slashed{S}^-$, and thus we can regard $\slashed{S}^+ \simeq \slashed{S}^-$ as the spinor bundle of M. Under this identification, the Dirac operator associated to the Levi-Civita connection becomes

$$\slashed{S}^+ \xrightarrow{D^g} \slashed{S}^- \xrightarrow{e_0} \slashed{S}^+$$

where 'e_0' denotes multiplication by e_0. Consider also the Levi-Civita connection on $\slashed{S}$ and the twisted Dirac operator

$$\slashed{S}^+ \otimes \slashed{S} \xrightarrow{D^g} \slashed{S}^- \otimes \slashed{S} \xrightarrow{e_0} \slashed{S}^+ \otimes \slashed{S}.$$

Notice that the exterior bundle of M is $\Lambda \simeq \mathrm{Cl} \simeq \slashed{S}^+ \otimes \slashed{S}$, and so the twisted Dirac operator above is, in terms of differential forms, the restriction of the operator $\mathrm{d} + \mathrm{d}^*$ on $\mathbb{R} \times M$ to forms that are independent of the coordinate t of $\mathbb{R}$, and can therefore be seen as $\mathrm{d} + \mathrm{d}^*$ on M.

We may now ask ourselves what happens if we introduce connections with skew torsion in this setting.

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Theorem 3.1. *Let H be a 3-form, and suppose that the left and right spinor factors are, respectively, equipped with the connections $\nabla^g + \frac{1}{12}H$ and $\nabla^g - \frac{1}{4}H$. Consider the tensor product of these two connections on $\not{S} \otimes \not{S}$. The corresponding Dirac operator on Λ is given by*

$$D = (\mathrm{d} + H) + (\mathrm{d} + H)^*,$$

where H is acting by exterior multiplication and $(\mathrm{d}+H)^*$ is the formal adjoint of $\mathrm{d}+H$ with respect to the metric, namely, $\mathrm{d}^* + (-1)^{n(p+1)} * H *$ on Λ^p.

Proof. Let us consider first an even-dimensional manifold. Take a p-form θ and identify it with

$$\varphi = \sum_r \varphi_r^+ \otimes \varphi_r^- \in \Gamma(M, \not{S} \otimes \not{S}).$$

Then the Clifford left and right actions of a vector field e are given, respectively, by

$$e\varphi = \sum_r e\varphi_r^+ \otimes \varphi_r^- = e \wedge \theta - e \lrcorner \theta,$$

$$\varphi e = \sum_r \varphi_r^+ \otimes e\varphi_r^- = (-1)^p (e \wedge \theta + e \lrcorner \theta).$$

Using the summation convention, we have

$$\begin{aligned} D(\varphi) &= e_i \nabla^g_{e_i} \varphi_r^+ \otimes \varphi_r^- + e_i \varphi_1 \otimes \nabla^g_{e_i} \varphi_2 \\ &\qquad + \tfrac{1}{12} e_i (e_i \lrcorner H) \varphi_r^+ \otimes \varphi_r^- - \tfrac{1}{4} e_i \varphi_r^+ \otimes (e_i \lrcorner H) \varphi_r^- \\ &= e_i \nabla^g_{e_i}(\varphi) + \tfrac{1}{12} e_i (e_i \lrcorner H)\varphi + \tfrac{1}{4} e_i \varphi (e_i \lrcorner H). \end{aligned}$$

Since $D^g(\varphi) = e_i \nabla^g_{e_i}(\varphi)$ corresponds to $(\mathrm{d}+\mathrm{d}^*)\theta$, it remains to see that $\frac{1}{12} e_i (e_i \lrcorner H)\varphi + \frac{1}{4} e_i \varphi (e_i \lrcorner \varphi)$ can be identified with $(H + H^*)\theta$.

Write $H = H_{abc} e_a \wedge e_b \wedge e_c$ and observe that

$$H_{abc} e_a \wedge e_b \wedge e_c \wedge \theta + H_{abc} e_c \lrcorner (e_b \lrcorner (e_a \lrcorner \theta))$$

is the same as $(H + H^*)\theta$, since the formal adjoint of exterior multiplication is interior multiplication. It is simple to see that $e_i(e_i \lrcorner H)\varphi = 3H\varphi$ and that the action of H is given by

$$H_{abc}(e_a \wedge e_b \wedge e_c \wedge \theta + e_a \wedge e_b \wedge (e_c \lrcorner \theta) + e_a \wedge (e_b \lrcorner (e_c \lrcorner \theta) + \cdots))$$

and that $e_i \varphi (e_i \lrcorner H)$ is such that when we add

$$\tfrac{1}{12} e_i (e_i \lrcorner H)\theta = \tfrac{1}{4} H\theta \quad \text{and} \quad \tfrac{1}{4} e_i \theta (e_i \lrcorner H),$$

the mixed terms cancel and it amounts to

$$\tfrac{1}{4} H_{abc} [e_a \wedge e_b \wedge e_c \wedge \theta + e_c \lrcorner (e_b \lrcorner (e_a \lrcorner \theta))]$$

plus

$$\tfrac{3}{4}H_{abc}[e_a \wedge e_b \wedge e_c \wedge \theta + e_c \lrcorner (e_b \lrcorner (e_a \lrcorner \theta))]$$

which is then $(H+H^*)\theta$. The proof in the odd-dimensional case is perfectly analogous. □

Remark 3.2. Notice that these are lifts of the metric connections on the tangent bundle with torsion $\frac{1}{3}H$ and $-H$. It is interesting to observe that these weights $\frac{1}{3}$ and -1 also appear in Bismut's proof of the local index theorem for non-Kähler manifolds [**4**].

Suppose now that H is a closed 3-form. On the de Rham complex of differential forms Ω we can define the operator $d_H = \mathrm{d} + H$. Note that

$$(\mathrm{d} + H)^2 = \mathrm{d}^2 + \mathrm{d}H + H\mathrm{d} + H^2 = 0$$

since H is closed and of odd degree. The operator d_H does not preserve form degrees but preserves the $\mathbb{Z}_2$-grading. We then have a two-step chain complex, and the cohomology of this complex is then the twisted de Rham cohomology.

The twisted de Rham complex is an elliptic complex so, on a compact manifold, Hodge theory applies. If H^+ and H^- are the cohomology groups, then

$$H^{\pm} \simeq \{\theta \in \Omega^{\pm} : \ (\mathrm{d} + H)\theta = 0 \text{ and } (\mathrm{d} + H)^*\theta = 0\},$$

or, in other words, each cohomology class has a unique representative in the kernel of D^2, where

$$D = (\mathrm{d} + H) + (\mathrm{d} + H)^*.$$

4. A vanishing theorem

We can use the Lichnerowicz formula of Theorem 2.1 and also Theorem 3.1 to prove the following.

Theorem 4.1. *Let M be a compact spin manifold and let H be a closed 3-form. Consider the Dirac operator $D^{1/12}$ on $\$ \otimes \$$ associated with the connection*

$$\nabla = \nabla^{1/12} \otimes 1 + 1 \otimes \nabla^{-1/4},$$

let $F^{-1/4}$ be the curvature of $\nabla^{-1/4}$ on $\$$ and let κ be the Riemannian scalar curvature of M. If

$$F^{-1/4} + \tfrac{1}{4}\kappa - \tfrac{1}{8}\|H\|^2$$

acts as a positive endomorphism, then the twisted de Rham cohomology for $\mathrm{d} + H$ vanishes.

Proof. We start by observing that we need only to prove that the kernel of the operator $D^{1/12}$ is zero. Consider ψ, a smooth section of $\$ \otimes \$$. Since $\mathrm{d}H = 0$, the Lichnerowicz formula gives

$$(D^{1/12})^2\psi = \Delta^{1/4}\psi + F^{-1/4}\psi + \tfrac{1}{4}\kappa\psi - \tfrac{1}{8}\|H\|^2\psi.$$

Now take the inner product of this with ψ. Since the Dirac operator is self-adjoint and the Laplacian Δ is given by $\nabla^*\nabla$, we get

$$\int_M \|D^{1/12}\psi\|^2 \,\mathrm{dVol} = \int_M \|\nabla^{1/4}\psi\|^2 + (F^{-1/4}\psi,\psi) + \tfrac{1}{4}\kappa\|\psi\|^2 - \tfrac{1}{8}\|H\|^2\|\psi\|^2 \,\mathrm{dVol}.$$

Using the hypothesis that

$$F^{-1/4} + \tfrac{1}{4}\kappa - \tfrac{1}{8}\|H\|^2$$

is a positive endomorphism, we conclude that $D^{1/12}\psi = 0$ if and only if $\psi = 0$. □

5. An example

Let G be a compact, non-abelian Lie group equipped with a bi-invariant metric. Consider the one-parameter family of connections $\nabla^t_X(Y) = t[X,Y]$. Given t, the torsion of ∇^t is $(2t-1)[X,Y]$. Notice that since the metric is ad-invariant, it means that these are metric connections and also that their torsion is skew symmetric. Note also that if $t=\frac{1}{2}$, we get the Levi-Civita connection, since the torsion vanishes. The curvature of ∇^t is given by

$$R^{\nabla^t}(X,Y)Z = t^2[X,[Y,Z]] - t^2[Y,[X,Z]] - t[[X,Y],Z] = (t^2-t)[[X,Y],Z],$$

by means of the Jacobi identity. For $t=0$ and $t=1$, we get two flat connections. These correspond, respectively, to the left and right invariant trivializations of the tangent bundle [**8**].

Let us write the above one-parameter family of connections as

$$\nabla^{2s}_X(Y) = \nabla^g_X(Y) + 2s[X,Y].$$

Notice that the Levi-Civita connection now corresponds to the parameters $s=0$, while the two flat connections correspond to $s=\pm\frac{1}{4}$.

Consider the lift of these connections to the spinor bundle $\$$ of G. Take the connection $\nabla^{1/12}\otimes 1 + 1\otimes\nabla^{-1/4}$ on $\Gamma(M,\$\otimes\$)$. We know from Theorem 3.1 that the Dirac operator $D^{1/12}$ then corresponds to $(\mathrm{d}+H)+(\mathrm{d}+H)^*$ on ΛG, where H is given by $H(X,Y,Z) = ([X,Y],Z)$. Note that H, being a bi-invariant form, is closed.

We need the following auxiliary lemma, which can be proved by direct computation.

Lemma 5.1. *Let G be a non-abelian Lie group equipped with a bi-invariant metric. Then the scalar curvature κ of G is given by*

$$\kappa = \tfrac{1}{4}\sum_{ij}\|[e_i,e_j]\|^2,$$

where $\{e_i\}$ is an orthonormal basis of the Lie algebra of G.

Theorem 5.2. *Let G be a compact, non-abelian Lie group equipped with a bi-invariant metric and let $H(X,Y,Z) = ([X,Y],Z)$ be the associated bi-invariant 3-form. Then the twisted de Rham cohomology of $\mathrm{d}+H$ vanishes.*

Proof. Since $F^{-1/4} = 0$, by means of Theorem 4.1 we only need to show that the constant $\rho = \frac{1}{4}\kappa - \frac{1}{8}\|H\|^2$ is positive. We have already computed κ in Lemma 5.1, so if we take the same orthonormal basis we get that

$$\|H\|^2 = \frac{1}{6}\sum_{ijk} |([e_i, e_j], e_k)|^2,$$

and, using the Cauchy–Schwarz inequality,

$$\|H\|^2 \leqslant \frac{1}{6}\sum_{ijk} \|[e_i, e_j]\|^2 \|e_k\|^2 = \frac{1}{6}\sum_{ij} \|[e_i, e_j]\|^2.$$

So $\rho > (\frac{1}{16} - \frac{1}{48})\sum_{ij} \|[e_i, e_j]\|^2 > 0$. □

Remark 5.3. To see this result for connected, compact, simple groups in a different way (see also [**7**, Example 1.2]), note that it is well known that by averaging, each cohomology class of G can be represented by a bi-invariant form. The de Rham cohomology ring $H^*(G)$ is an exterior algebra (more precisely $H^*(G)$ is an exterior algebra on generators in degree $2d_i - 1$, where each d_i is the degree of generators of invariant polynomials on the Lie algebra of G). The Killing form gives $H^3(G) = \mathbb{R}$. Consider now the twisted de Rham operator $\mathrm{d} + H$. Since H is bi-invariant, the twisted cohomology classes can also be represented by bi-invariant forms. Since bi-invariant forms are closed, $(\mathrm{d} + H)\alpha = H \wedge \alpha$. So if $H \wedge \alpha = 0$, since H is a generator, then $H \wedge \alpha = 0$ implies that $\alpha = H \wedge \beta$ for some β. Therefore, the twisted cohomology vanishes.

Acknowledgements. The author thanks Nigel Hitchin for pointing her to this topic, and for the many helpful conversations that have ensued. This research was partly supported by the Portuguese Foundation for Science and Technology (FCT) through the POPH–QREN Scholarship Program and by the Research Center of Mathematics of the University of Minho through the FCT Pluriannual Funding Program.

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