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ON THE CONJUGACY CLASSES IN AN INTEGRAL GROUP RING

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1. **Introduction.** Let G be a periodic group and ZG its integral group ring. The elements $\pm g\,(g \in G)$ are called the trivial units of ZG. In [1], S. D. Berman has shown that if G is finite, then every unit of finite order is trivial if and only if G is abelian or the direct product of a quaternion group of order 8 and an elementary abelin 2-group. By comparison, Losey in [7] has shown that if ZG contains one non-trivial unit of finite order, then it contains infinitely many.

If we set about the task of constructing non-trivial units of finite order, one way is to take conjugates of the elements of G in the group ring ZG. This raises the question as to when such a procedure will work. It is a consequence of a result of Sehgal and Zassenhaus [8] that at least one element of G has infinite conjugacy class in ZG, unless of course G is abelian or the direct product of a quaternion group of order 8 and an elementary abelian 2-group. In this paper we prove the following:

THEOREM 1. *Let G be a periodic group. An element x in G has finite conjugacy class in ZG if and only if either*

(*i*) *x is central in G, or*

(*ii*) *x has order 4 and is contained in an abelian subgroup H of index 2 in G where $G = \langle H, c : c^2 = x^2$ and $h^c = h^{-1}$ for all $h \in H\rangle$.*

In fact the proof shows that if x is not central, then there are an infinite number of non-trivial conjugates unless x has order 4 and G has the structure described in (ii). This may be compared with results of Bovdi: Let N be a normal periodic divisor subgroup of $U(ZG)$, the group of units of ZG. It is easy to show that N consists only of trivial units (Theorem 1 of [3]). In Theorem 11 of [4], Bovdi shows that $U(ZG)$ contains a non-central abelian normal subgroup if and only if G has the structure described in (ii) above.

2. **Some lemmas.** In this section we collect together various results on which our proof of Theorem 1 depends. Note that Lemmas 1, 2, and 3 are well-known, they are to be found in the work of Berman [1] and Bovdi [3, 4]. For brevity, for $y \in G$, whenever we write $\sum y^i$, it is to be understood that the sum is taken over all the elements of $\langle y\rangle$.

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LEMMA 1 (cf. p. 260 of [1]). *Suppose that* $x \in G$ *and that there is an element* $a \in ZG$ *such that* $a^2 = 0$ *and* $ax \neq xa$. *Then the set* $\{(1+ka)x(1-ka) : k \in Z\}$ *forms an infinite set of conjugates of x.*

Proof. Observe that $1+ka$ is a unit in ZG with inverse $1-ka$. Furthermore, since $(1+ka)x(1-ka) = x + k[(ax-xa) - kaxa]$ and $ax - xa$ is non-zero, an infinite number of these conjugates are distinct.

LEMMA 2. *Let g and h be elements of finite order in the group G. Then* $(\sum g^i)(h-h^g)(\sum g^i) = 0$.

Proof. Observe that $(\sum g^i)g^{-1}hg(\sum g^i) = (\sum g^i)h(\sum g^i)$.

LEMMA 3 (cf. p. 497 of [3]). *Let* $x \in G$ *and suppose that x does not normalize some cyclic subgroup* $\langle c \rangle$. *Then there are an infinite number of conjugates of x in ZG.*

Proof. Define $a = (x - x^c)\sum c^i$. This is a sum of $2\,|\langle c \rangle|$ distinct elements of G; for if $xc^i = x^c c^j$ (some i, j), then $x^{-1}cx \in \langle c \rangle$, which is not so. Now $x^{-1}ax = (1-[x, c])\sum c^i x$. This cannot equal a, for then $x \sum c^i = \sum c^i x$ and x would normalize $\langle c \rangle$. By Lemma 2, $a^2 = 0$ and use of Lemma 1 gives the desired result.

There is another result similar to Lemma 3:

LEMMA 4. *Let x, y, and c be elements in the periodic group G. Suppose that c does not normalize* $\langle y \rangle$ *and that x has finite conjugacy class in ZG. Then* $[x, c] \in \langle y \rangle$.

Proof. Define $a = (c - c^y)\sum y^i$. This is a sum of $2\,|\langle y \rangle|$ distinct terms, because c does not normalize $\langle y \rangle$. By Lemma 2 $a^2 = 0$, and as x normalizes $\langle y \rangle$ by Lemma 3, we have that $x^{-1}ax = (x^{-1}cx - x^{-1}c^y x)\sum y^i$. This must equal a (by Lemma 1). Hence $x^{-1}cx \sum y^i = c \sum y^i$ and $[x, c] \in \langle y \rangle$ as required.

COROLLARY 1. *Let x be an element in the periodic group G with finite conjugacy class in ZG. Then x normalizes every subgroup of G and* $\langle x \rangle \lhd G$.

Proof. By Lemma 3 it suffices to show that $\langle x \rangle \lhd G$. Suppose then, that $\langle x \rangle$ is not normal in G and let $c \in G \backslash N_G(\langle x \rangle)$. By Lemma 4 it follows that $[x, c] \in \langle x \rangle$, which is impossible.

COROLLARY 2. *Let the finite group G contain a non-central element x with finite conjugacy class in ZG. Then every subgroup of G of prime order is a normal subgroup of G.*

Proof. Let y have prime order and suppose that $\langle y \rangle$ is not normal in G. We have that $N_G(\langle y \rangle) < G$ and $C_G(x) < G$, so $|N_G(\langle y \rangle) \cup C_G(x)| < |G|$ and there is an element c in G which neither centralizes x nor normalizes $\langle y \rangle$. By Lemma

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4, $1 \neq [x, c] \in \langle y \rangle$. Since y has prime order, it follows that $y \in \langle [x, c] \rangle$. However x normalizes $\langle c \rangle$ (Corollary 1) and so $y \in \langle c \rangle$, which contradicts our choice of c.

Given a group G containing at least one non-normal subgroup we can define $R(G)$ to be the intersection of all the non-normal subgroups. In [2], Blackburn has classified all finite groups in which $R(G) \neq 1$. It turns out that we need his classification for p-groups, which is as follows:

THEOREM (Theorem 1 of [2]). *Let G be a finite p-group. Suppose that G contains at least one non-normal subgroup and that $R(G) \neq 1$. Then $p = 2$ and one of the following holds:*

(1) *G is the direct product of a quaternion group of order 8, a cyclic group of order 4 and an elementary abelian 2-group.*

(2) *G is the direct product of two quaternion groups of order 8 and an elementary abelian 2-group.*

(3) *G contains an abelian subgroup A of index 2 where A is not elementary abelian. G is generated by A and t where $t^{-1}at = a^{-1}$ $(a \in A)$ and $t^2 \in A$ is of order 2.*

During the proof of Theorem 1, we need to consider a group which is the direct product of a quaternion group of order 8 and a group of odd prime order. For convenience we deal with this rather special case here:

LEMMA 5. *Let x be a non-central element in a group G which is the direct product of a quaternion group of order 8 and a cyclic group of odd prime order p. Then x has an infinite conjugacy class in ZG.*

Proof. Let $G = \langle u, v, y : u^v = u^{-1},\ u^2 = v^2,\ u^4 = 1,\ y^p = 1,\ [u, y] = 1 = [v, y] \rangle$. The non-central elements in G have order 4 or $4p$. Without loss of generality we may assume that either $x = v$ or $x = vy$. Let us write w for uy which has order $4p$. Use the notation that if H is a group, then $U(ZH)$ is the group of units in ZH. By Dirichlet's Unit Theorem (see 5.3.10 and 7.6.1 of [9]), the rank of $U(Z\langle w \rangle)$ is $\frac{1}{2}\phi(4p) - 1 = p - 2$ and the rank of $U(Z\langle w^2 \rangle)$ is $\frac{1}{2}\phi(2p) - 1 = \frac{1}{2}(p-3)$. Let us define a ring-homomorphism θ from $U(Z\langle w \rangle)$ into $U(Z\langle w^2 \rangle)$ such that $w\theta = w^2$.

Now rank $U(Z\langle w \rangle)/\ker\theta + \operatorname{rank}\ker\theta = \operatorname{rank} U(Z\langle w \rangle)$. So the rank of $\ker\theta$ is at least $p - 2 - \frac{1}{2}(p-3)$. As this is at least $\frac{1}{2}(p-1)$, there is a unit f in $\ker\theta$ of infinite order such that $\langle f \rangle \cap U(Z\langle w^2 \rangle) = 1$. Since $f\theta = 1$, the unit f has form $1 + (1 - w^{2p}) \sum_1^{2p-1} a_i w^i$ $(a_i \in Z)$.

Let us write $f = f_1 + f_2$ where $f_1 \in Z\langle w \rangle$ is the sum of those terms of f involving odd powers of w, and $f_2 \in Z\langle w^2 \rangle$ is the corresponding sum involving even powers. We see that $f_2 v = v f_2$ since w^2 is central in G, but that $f_1 v = -v f_1$ on using the fact that $wv = vw^{2p+1}$.

Consider fvf^{-1}: it equals $(f_1 + f_2)v(g_1 + g_2)$ where g_1 is the sum of those terms of f^{-1} involving odd powers of w, and g_2 is the corresponding sum involving even powers. Hence $fvf^{-1}=v(f_2-f_1)(g_1+g_2)$. But from $(f_1+f_2)(g_1+g_2)=1=(f_1g_1+f_2g_2)+(f_1g_2+f_2g_1)$, it follows that $f_1g_1+f_2g_2=1$ and $f_1g_2+f_2g_1=0$, as the first term involves even powers of w only and the second term odd powers.

Hence $fvf^{-1}=v(f_2g_2-f_1g_1+2f_2g_1)$ is a non-trivial unit unless $f_2g_2-f_1g_1=w^i$ (for some i) and $f_2g_1=0=f_1g_2$. But in this case, as $f_1g_1+f_2g_2=1$ we deduce that $2f_2g_2=1+w^i$ and so $w^i=1$. Therefore $f_2g_2=1$ and $f_1g_1=0$. But from above, $f_1g_2=0$, so $f_1(g_1+g_2)=0$ and $f_1=0$ since g_1+g_2 is a unit. Hence $f=f_2$ is a unit in $Z\langle w^2\rangle$ contrary to hypothesis. An identical argument shows that $\{f^kvf^{-k}: k\in Z\}$ forms an infinite set of distinct conjugates of v. Furthermore, $\{f^kvyf^{-k}\}$ forms an infinite set of distinct conjugates of vy, as w centralizes y.

3. **Proof of Theorem 1.** Let x be a non-central element in the periodic group G with a finite conjugacy class in ZG. The Corollaries 1 and 2 restrict very much the possible structure of G. As a preliminary to proving the main theorem, we use these to prove:

PROPOSITION. *Let x be a non-central element in the periodic group G with finite conjugacy class in ZG. Then x has order* 4 *and for any element c in G not centralizing x, the group generated by x and c is isomorphic to a quaternion group of order* 8.

Proof. Suppose that the result is false and that the group G is a counterexample. So G contains a non-central element x with finite conjugacy class in ZG and an element $c\in G\backslash C_G(x)$ such that $\langle x, c\rangle\not\cong Q_8$, a quaternion group of order 8. We may suppose that $G=\langle x, c\rangle$ and that G is a minimal counterexample. Note that x normalizes $\langle c\rangle$ and c normalizes $\langle x\rangle$ (Corollary 1).

Consider first the possibility that every subgroup of G is normal. It is well-known that a finite group with this property belongs to one of the following types: Q_8, $Q_8\times A$, $Q_8\times B$ or $Q_8\times A\times B$ where A is an elementary abelian 2-group and B is an abelian group of odd order (see Theorem 10.2.5 of [5]). In our case, G is a 2-generated finite group and so G must be of type $Q_8\times B$. By Lemma 5, B has composite order. Let p be a prime dividing $|B|$; at least one of the elements x and c has order divisible by p-suppose x. By the minimality of G, the group $\langle x^p, c\rangle$ must be such that x^p is centralized by c, which is not the case. We arrive at a similar contradiction if we assume that c has order divisible by p.

Therefore there is a non-normal cyclic subgroup $\langle y\rangle$ in G. By Corollary 1, $\langle y\rangle$ is normalized by x and so $\langle y\rangle$ is not normalized by c. Again by Corollary 1, $[x, c]\in\langle y\rangle$, and so $[x, c]\in\cap\langle y\rangle$ where the intersection runs through all the non-normal cyclic subgroups of G. As $[x, c]\neq 1$, it follows that $R(G)\neq 1$. We could use Blackburn's classification at this point, but it is easier if we first show that G is a p-group. To do this, we show in turn that both x and c have prime-power order. Because $\langle x\rangle \cap \langle c\rangle \neq 1$, this is sufficient.

Assume first that x does not have prime power order. Let $\{p_i\}$ be the set of distinct primes dividing the order of x. We may choose p_i such that $\langle x^{p_i}, c\rangle$ is non-abelian (such is possible, for if c centralizes each x^{p_i}, then c would centralize x). As G is a minimal counterexample, $\langle x^{p_i}, c\rangle \cong Q_8$. Certainly $p_i \neq 2$, but if p_i is an odd prime it follows that $G \cong Q_8 \times B$ where B here is the cyclic group of order p_i generated by x^4 and Q_8 is generated by x^{p_i} and c. This contradicts Lemma 5. An identical argument works in the case where c does not have prime-power order.

We are now in a position to use Blackburn's Theorem applied to p-groups, quoted in the Introduction. The group G must be a 2-group, and the only 2-generated 2-groups with $R(G)\neq 1$ are of type (3): $G=\langle A, t\rangle$ where A is abelian, $t^{-1}at=a^{-1}$ for all $a\in A$ and $t^2\in A$ has order 2.

Now if x lies in A, then $t^{-1}xtx^{-1}=x^{-1}x^{-1}=x^{-2}$. But x normalizes every subgroup of G (Corollary 1) and so $xtx^{-1}=t^{-1}$ (x does not centralize t, since then x would be central in G). Hence $x^2=t^2$ and x has order 4. On the other hand, any element in $G\backslash A$ has order 4, for $(at)^2=t(t^{-1}at)at=t^2$. So x has order 4.

Let c have order 2^{r+1}. Clearly $\langle x\rangle\cap\langle c\rangle=\langle x^2\rangle=\langle c^{2^r}\rangle$ and so $x^2=c^{2^r}$. From the fact that $c^{-1}xc=x^{-1}$ it follows that $x^{-1}cx=c^{1+2^r}$. Hence c^2 is central in G, and as $r>1$, $c^{2^{r-1}}$ is also central. But now consider that $(xc^{2^{r-1}})^2=x^2c^{2^r}=x^4=1$ and by Corollary 2 we deduce that $xc^{2^{r-1}}$ is central in G. This is impossible, for $c^{-1}(xc^{2^{r-1}})c=x^{-1}c^{2^{r-1}}$. This completes the proof of the Proposition.

We now complete the proof of Theorem 1. Let x be a non-central element in the periodic group G with finite conjugacy class in ZG. By the Proposition, x has order 4 and for any element $g\in G\backslash C_G(x)$ we have that $\langle x, g\rangle\cong Q_8$. Put $H=C_G(x)$ and choose $c\in G\backslash C_G(x)$. For arbitrary $h\in H$, $hc\in G\backslash C_G(x)$ and so $hchc=x^2=c^2$. Therefore $c^{-1}hc=h^{-1}$ and it follows (cf. ex. 1 of Chapter 2 of [5]) that H is abelian, as required.

Conversely, if x is central in G, then clearly it has finite conjugacy class in ZG. Whereas if x has order 4 and is contained in an abelian subgroup H of index 2 in G, where G is generated by H and an element c such that $c^2=x^2$ and $h^c=h^{-1}$ for arbitrary $h\in H$, then Bovdi has shown (Theorem 11, [4]) that x is conjugate only to x and x^{-1}.

In conclusion it should be noted that conjugates of the elements $\pm g$ $(g\in G)$ are not, in general, the only units of finite order in ZG. For example the element $-3a+3a^2+b-3ab+3ba$ is a unit of order 3 in ZS_3 (where $S_3=\langle a, b : a^3=1=b^2, a^b=a^2\rangle$), but is not conjugate to a trivial unit. (For a full description of the group of units of ZS_3, see [6].) It would be interesting to know which groups have the property that any unit of finite order is conjugate to a trivial unit.

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