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# ESSENTIALLY DEFINED DERIVATIONS ON SEMISIMPLE BANACH ALGEBRAS

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We prove that every partially defined derivation on a semisimple complex Banach algebra whose domain is a (non necessarily closed) essential ideal is closable. In particular, we show that every derivation defined on any nonzero ideal of a prime $C^*$-algebra is continuous.

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## 1. Introduction

Johnson and Sinclair obtained in [1] the continuity of everywhere defined derivations on semisimple Banach algebras by building suitable sequences which dissolve the continuity problem. Since then the spirit of these sequences has been successfully exploited and now we know it as the sliding hump procedure. However, in the $C^*$-algebraic formulation of quantum physics, partially defined derivations on $C^*$-algebras appear generally. On the other hand it is known that partially defined derivations, even on $C^*$-algebras, may be not closable. In this paper we obtain the automatic closability of partially defined derivations on semisimple Banach algebras whose domain is an essential ideal of the algebra, and further we obtain the continuity when the algebra is actually a prime $C^*$-algebra.

## 2. Automatic closability

Throughout this section, $A$ denotes a semisimple complex Banach algebra and $D$ stands for a complex linear map from an *essential* ideal $I$ of $A$ (i.e. $I \cap J \neq 0$ for every nonzero ideal $J$ of $A$) into $A$ satisfying $D(ab) = D(a)b + aD(b)$, for all $a, b \in I$. Such a mapping is said to be an *essentially defined derivation* on $A$. It is worth pointing out that $I$ is not assumed to be closed nor to be dense in $A$.

Let us denote by $\mathcal{P}$ the set of those primitive ideals $P$ of $A$ for which $I \not\subset P$. It is clear that $I \cap (\cap_{P \in \mathcal{P}} P) \subset Rad(A) = 0$ and therefore $\cap_{P \in \mathcal{P}} P = 0$. It is well known that every primitive ideal $P$ can be obtained as the kernel of a continuous irreducible representation of $A$ on a complex Banach space $X_P$, actually $\|ax\| \leq \|a\| \|x\|$ for all $a \in A$, $x \in X_P$.

We measure the closability of $D$ by considering the subspace $\mathcal{S}(D)$ of those $a \in A$ for which there is a sequence $\{a_n\}$ in $I$ with $\lim a_n = 0$ and $\lim D(a_n) = a$. It is well known that $D$ is closable if, and only if, $\mathcal{S}(D) = 0$. It is easy to check that $I\mathcal{S}(D) + \mathcal{S}(D)I \subset \mathcal{S}(D)$. Write $\mathcal{P}_C = \{P \in \mathcal{P} : \mathcal{S}(D) \subset P\}$ and $\mathcal{P}_E = \{P \in \mathcal{P} : \mathcal{S}(D) \not\subset P\}$. Note that $\mathcal{S}(D) \subset P_C = \cap_{P \in \mathcal{P}_C} P$. Our method consists of showing that $P_C = 0$.

**Lemma 1.** *Let $P \in \mathcal{P}$ and $J$ any non necessarily closed ideal of $A$ satisfying $J \not\subset P$. Then one of the following assertions holds:*

(i) *The ideal of those elements $b \in J$ with $\dim bX_P < \infty$ acts irreducibly on $X_P$. Accordingly, given $x, y \in X_P$ with $x \neq 0$ there is $b \in J$ with $\dim bX_P = 1$ and $bx = y$.*

(ii) *There exist sequences $\{b_n\}$ in $J$ and $\{x_n\}$ in $X_P$ satisfying $b_n \dots b_1 x_n \neq 0$ and $b_{n+1} \dots b_1 x_n = 0$ for every $n \in \mathbb{N}$.*

**Proof.** First we note that $J$ acts irreducibly on $X_P$. Now we observe that Lemma B.13 in [3] provides an algebra norm on the centralizer of $X_P$ as $J$-module and therefore equals $\mathbb{C}$ by the Gelfand-Mazur theorem.

Let $F(J)$ denote the ideal of those $b \in J$ with $\dim bX_P < \infty$. If $F(J) \not\subset P$, then $F(J)$ acts irreducibly on $X_P$ and the first assertion follows.

Otherwise, for every $b \in J$ with $bX_P \neq 0$ it is satisfied that $\dim bX_P = \infty$. In this situation we take $b_1 \in J$ and $x_1 \in X_P$ with $b_1 x_1 \neq 0$. Suppose that $b_1, \dots, b_n \in J$ and $x_1, \dots, x_n \in X_P$ have been chosen satisfying $b_j \dots b_1 x_{j-1} = 0$ and $b_j \dots b_1 x_j \neq 0$ for $j = 2, \dots, n$. Since $(b_n \dots b_1)X_P \neq 0$, $\dim(b_n \dots b_1)X_P = \infty$. Therefore there is $x_{n+1} \in X_P$ such that $b_n \dots b_1 x_n$ and $b_n \dots b_1 x_{n+1}$ are linearly independent. Consequently there exists $b_{n+1} \in J$ such that $b_{n+1} \dots b_1 x_{n+1} \neq 0$ and $b_{n+1} \dots b_1 x_n = 0$. The sequences $\{b_n\}$ and $\{x_n\}$ satisfy the requirements in the second assertion. □

Let $\{P_n\}$ be a sequence in $\mathcal{P}$. A sequence $\{b_n\}$ in $I$ is said to be a *sliding hump sequence* for $\{P_n\}$ if for every $n \in \mathbb{N}$ there exists $x_n \in X_{P_n}$ such that

$$b_n \dots b_1 x_n \neq 0 \quad \text{and} \quad b_{n+1} \dots b_1 x_n = 0.$$

**Lemma 2.** *If there exists a sliding hump sequence for a sequence $\{P_n\}$ in $\mathcal{P}$, then there is a natural number $n$ for which $\mathcal{S}(D) \subset P_n$. In particular, $\mathcal{S}(D) \subset P$ if $P_n = P$ for every $n \in \mathbb{N}$.*

**Proof.** Let $\{b_n\}$ be a sliding hump sequence for $\{P_n\}$ and, for every $n \in \mathbb{N}$, let $x_n \in X_{P_n}$ for which the sliding hump condition holds. We can certainly assume that $\|b_n\| = \|x_n\| = 1$ for every $n \in \mathbb{N}$.

We claim that there exist $n \in \mathbb{N}$ and a nonzero $x \in X_{P_n}$ such that the map $a \mapsto D(a)x$ from $I$ into $X_{P_n}$ is continuous. If the claim fails, then all the maps $a \mapsto D(a)b_n \dots b_1 x_n$ from $I$ into $X_{P_n}$ are discontinuous and we can construct inductively a sequence $\{a_n\}$ in $I$ satisfying

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$$\|a_n\| \le 2^{-n}\min\{(1+\|D(b_k\ldots b_1)\|)^{-1} : k=1,\ldots,n\} \quad \text{and}$$

$$\|D(a_n)b_n\ldots b_1x_n\| \ge n + \left\|\sum_{k=1}^{n-1} D(a_kb_k\ldots b_1)x_n\right\|.$$

Now we consider the element $a \in A$ given by $c = \sum_{n=1}^{\infty} a_nb_n\ldots b_1$ and, for every $n \in \mathbb{N}$, we write $c_n = a_n + \sum_{k=n+1}^{\infty} a_kb_k\ldots b_{n+1}$. Note that all the elements $c$ and $c_n$ lie in $I$ since $c = c_1b_1$ and $c_n = a_n + c_{n+1}b_{n+1}$. Then we have

$$D(c)x_n = \sum_{k=1}^{n-1} D(a_kb_k\ldots b_1)x_n + D(a_n)b_n\ldots b_1x_n + a_nD(b_n\ldots b_1)x_n + c_nD(b_{n+1}\ldots b_1)x_n$$

and $\|D(c)\| \ge \|D(c)x_n\| \ge n-3$, for every $n \in \mathbb{N}$ (see the proof of [1, Theorem 2.2]). This contradiction proves our claim.

Let $m \in \mathbb{N}$ such that the map $a \mapsto D(a)x$ from $I$ into $X_{P_m}$ is continuous for some nonzero $x \in X_{P_m}$ and let $X$ be the set of all $x \in X_{P_m}$ satisfying this property. $X$ is a nonzero $I$-submodule of $X_{P_m}$. Therefore we conclude that $X = X_{P_m}$. Let $a \in \mathcal{S}(D)$ then $a = \lim D(a_n)$ for a suitable sequence $\{a_n\}$ in $I$ with $\lim a_n = 0$. Then $ax = \lim D(a_n)x = 0$ for every $x \in X_{P_m}$ and therefore $a \in P_m$, which is the desired conclusion. □

**Lemma 3.** *Let $P \in \mathcal{P}$ and $J$ any subspace of $A$ satisfying $IJ + JI \subset J$ and $J \not\subset P$. Then $Jx = X_P$ for every nonzero $x \in X_P$.*

**Proof.** The set $\{x \in X_P : Jx = 0\}$ is an $I$-submodule of $X_P$ different from $X_P$ and therefore equals zero, since $I$ acts irreducibly on $X_P$. Hence, for every nonzero $x \in X_P$, $Jx$ is a nonzero $I$-submodule of $X_P$ and consequently equals $X_P$. □

**Lemma 4.** *Let $P \in \mathcal{P}$ and $J$ any non necessarily closed ideal of $A$ contained in $I$. If there is an element $b \in J$ with $b \notin P$ and* $\dim bJb < \infty$. *Then $\mathcal{S}(D) \subset P$.*

**Proof.** Note that the map $a \mapsto D(bab)$ from $J$ into $A$ is continuous. Let $a \in \mathcal{S}(D)$, then $a = \lim D(a_n)$ for a suitable sequence $\{a_n\}$ in $I$ with $\lim a_n = 0$. By continuity, $0 = \lim D(ba_nb) = bab$ and therefore $b\mathcal{S}(D)b = 0$.

Since $b \notin P$, we have $bX_P \neq 0$. If $\mathcal{S}(D) \not\subset P$, then from Lemma 3 it may be concluded that $\mathcal{S}(D)bX_P = X_P$. Hence $0 = b\mathcal{S}(D)bX_P = bX_P$ which gives $b \in P$ and this contradiction completes the proof. □

We can now formulate our main result.

**Theorem 5.** *D is closable.*

**Proof.** If the theorem fails, then $P_C \neq 0$ and $\mathcal{P}_E \neq \emptyset$. Let $J_0 = I \cap P_C$. We set $P_1 \in \mathcal{P}_E$ and we write $J_1 = J_0 \cap P_1$. On account of Lemmas 1 and 2, we may choose

$b_1 \in J_0$ and $x_1 \in X_{P_1}$ satisfying $\dim b_1 X_{P_1} = 1$ and $b_1 x_1 = x_1$. It should be noted that $\dim(b_1 + P_1)(A/P_1)(b_1 + P_1) = 1$ and therefore, for every $a \in A$, there exists a complex number $\lambda(a)$ such that $b_1 a b_1 - \lambda(a) b_1 \in P_1$.

Now we claim that there exists $P_2 \in \mathcal{P}_E$ such that $J_1 \not\subset P_2$ and $b_1 \notin P_2$. Otherwise, for every $a \in A$, we have $b_1 a b_1 - \lambda(a) b_1 \in \cap_{P \in \mathcal{P}_E, J_1 \not\subset P} P$ and so $b_1 a b_1 - \lambda(a) b_1 \in J_0 \cap (\cap_{P \in \mathcal{P}_E, J_1 \subset P} P) \cap (\cap_{P \in \mathcal{P}_E, J_1 \not\subset P} P) = 0$. Consequently $\dim b_1 A b_1 = 1$ and Lemma 4, applied to $J_0$ and $P_1$, shows that $\mathcal{S}(D) \subset P_1$. This contradiction proves our claim.

Since $b_1 \notin P_2$, there is $x_2 \in X_2$ such that $b_1 x_2 \neq 0$. Further, from Lemma 2 it may be concluded that there is no sliding hump sequences for $P_2$. Lemma 1 applied to $J_1$ now gives that there exists $b_2 \in J_1$ such that $b_2 b_1 x_2 = x_2$ and $\dim b_2 b_1 X_2 = 1$ which gives $\dim(b_2 b_1 + P_2)(A/P_2)(b_2 b_1 + P_2) = 1$.

Suppose that $P_1, \ldots, P_n$, $b_1, \ldots, b_n$, and $x_1, \ldots, x_n$ have been chosen satisfying

(i) $P_1, \ldots, P_n \in \mathcal{P}_E$,

(ii) $J_{k-1} \not\subset P_k$, for $k = 2, \ldots, n$,

(iii) $b_k \in J_{k-1} = J_0 \cap P_1 \cap \ldots \cap P_{k-1}$, for $k = 2, \ldots, n$,

(iv) $\dim(b_k \ldots b_1 + P_k)(A/P_k)(b_k \ldots b_1 + P_k) = 1$, for $k = 2, \ldots, n$,

(v) $x_k \in X_{P_k}$, for $k = 1, \ldots, n$,

(vi) $b_k \ldots b_1 x_k = x_k$, for $k = 1, \ldots, n$.

For abbreviation, we write $b$ instead of $b_n \ldots b_1$. Since $\dim(b + P_n)(A/P_n)(b + P_n) = 1$, for every $a \in A$, there exists a complex number $\lambda(a)$ such that $bab - \lambda(a) b \in P_n$. Now we claim that there exists $P_{n+1} \in \mathcal{P}_E$ satisfying $J_n \not\subset P_{n+1}$ and $b \notin P_{n+1}$. Otherwise, for each $a \in A$, we have $bab - \lambda(a) b \in \cap_{P \in \mathcal{P}_E, J_n \not\subset P} P$ and therefore $bab - \lambda(a) b \in J_n \cap (\cap_{P \in \mathcal{P}_E, J_n \subset P} P) \cap (\cap_{P \in \mathcal{P}_E, J_n \not\subset P} P) = 0$. Accordingly $\dim bAb = 1$ and from Lemma 4, applied to $J_{n-1}$ and $P_n$, we deduce that $\mathcal{S}(D) \subset P_n$. This contradiction proves the preceding claim.

Now we choose $x_{n+1} \in X_{P_{n+1}}$ with $bx_{n+1} \neq 0$. From Lemma 2 it follows that there is no sliding hump sequences for $P_{n+1}$. Lemma 1 applied to $J_n$ now gives that there exists $b_{n+1} \in J_n$ such that $b_{n+1} b x_{n+1} = x_{n+1}$ and $\dim b_{n+1} b X_{n+1} = 1$ which gives $\dim(b_{n+1} b + P_{n+1})(A/P_{n+1})(b_{n+1} b + P_{n+1}) = 1$.

Finally we note that conditions (iii) and (vi) give that the sequence $\{b_n\}$ is a sliding hump sequence for $\{P_n\}$ which, according to Lemma 2, gives a contradiction. □

## 3. Automatic continuity

A Banach algebra $A$ is said to be *ultraprime* if $k = \inf \{ \|M_{a,b}\| : a, b \in A, \|a\| = \|b\| = 1 \} > 0$, where $M_{a,b}$ is the two-sided multiplication operator on $A$ defined by $M_{a,b} x = axb$. It was proved in [2, Proposition 2.3] that every prime $C^*$-algebra is an ultraprime Banach algebra, actually in this case $k = 1$.

**Theorem 6.** *Let $D$ be a closable derivation defined on a nonzero ideal $I$ of an ultraprime Banach algebra $A$. Then $D$ is continuous.*

**Proof.** Fix $b \in I$ with $\|b\| = 1$ and consider the operator $x \mapsto D(xb)$ from $A$ into itself. Given a sequence $\{x_n\}$ in $A$ converging to zero with $\{D(x_n b)\}$ converging to $y \in A$, we have that the elements $x_n b$ lie in $I$ and therefore $y = 0$. By the closed graph theorem, this operator is continuous and we denote by $M$ its operator norm.

For all $a \in I$ and $x \in A$ we have $D(a)xb = D(axb) - aD(xb)$ and so $k\|D(a)\| \leq M_{D(a),b}\| \leq 2M\|a\|$. Consequently $D$ is continuous. □

From Theorems 5 and 6 we can now state the following.

**Corollary 7.** *Every essentially defined derivation on an ultraprime semi-simple complex Banach algebra is continuous. Accordingly, every derivation defined on a nonzero ideal of a prime $C^*$-algebra is continuous.*

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