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GROUP ALGEBRAS WHOSE UNIT GROUP IS LOCALLY NILPOTENT

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Abstract

We present a complete list of groups G and fields F for which: (i) the group of normalized units $V(FG)$ of the group algebra FG is locally nilpotent; (ii) the set of nontrivial nilpotent elements of FG is finite and nonempty, and $V(FG)$ is an Engel group.

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1. Introduction and results

Let $V(FG)$ be the normalized subgroup of the group of units $U(FG)$ of the group algebra FG of a group G over a field F of characteristic $\mathrm{char}(F) = p \geq 0$. It is well known that $U(FG) = V(FG) \times U(F)$, where $U(F) = F \setminus \{0\}$. The group of normalized units $V(FG)$ of a modular group algebra FG has a complicated structure and has been studied in several papers. For an overview we recommend the survey paper [3].

A group G is said to be *Engel* if for any $x, y \in G$ the equation $(x, y, y, \dots, y) = 1$ holds, where y is repeated in the commutator sufficiently many times depending on x and y. We use the right-normed simple commutator notation $(x_1, x_2) = x_1^{-1}x_2^{-1}x_1x_2$ and

$$(x_1, \dots, x_n) = ((x_1, \dots, x_{n-1}), x_n) \quad (x_1, \dots, x_n \in G).$$

A group is called *locally nilpotent* if all its finitely generated subgroups are nilpotent. Such a group is always Engel (see [25, Introduction, page 520]). The set of elements of finite orders of a group G (which is not necessarily a subgroup) is called *the torsion part* of G and is denoted by $\mathfrak{t}(G)$. We use the notion and results from the book [2] and the survey papers [3, 25].

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An explicit list of groups G and rings K for which $V(KG)$ are nilpotent was obtained by Khripta (see [15] for the modular case and [16] for the nonmodular case). In [4] it was completely determined when $V(FG)$ is solvable. It is still a challenging problem whether $V(FG)$ is an Engel group. This question has a long history (see [5–9, 23]). The nonmodular case was solved by Bovdi (see [5, Theorem 1.1, page 174]). For the modular case there is no complete solution (see [5, Theorem 3.2, page 175]), but there is a full description of FG when $V(FG)$ is a bounded Engel group (see [5, Theorem 3.3, page 176]).

It is well known (for example, see [25, Chs. 1–2]) that the Engel property of a group is close to its local nilpotency. However these classes of groups do not coincide (see Golod's counterexample in [10]). The following results are classical (see [25, Ch. 2, pages 522–525, Ch. 3, page 528]): each profinite Engel group (see [26]), each compact Engel group (see [17]), each linear Engel group (see [24]), each 3-Engel and 4-Engel group (see [14] and [13]) and all Engel groups satisfying max (see [1]) are locally nilpotent.

In several articles, Ramezan-Nassab attempted to describe the structure of groups G for which $V(FG)$ are Engel (locally nilpotent) groups in the case when FG have only a finite number of nilpotent elements (see [20, Theorem 1.5], [19, Theorems 1.2 and 1.3] and [21, Theorem 1.3]). The following theorem gives a complete answer.

THEOREM 1.1. *Let FG be the group algebra of a group G such that the set of nonzero nilpotent elements of FG is nontrivial and finite. Then F is a finite field of $\mathrm{char}(F) = p$. Additionally, $V(FG)$ is an Engel group if and only if G is a finite group such that $G = S_p \times A$, where $S_p \neq \langle 1\rangle$ is the Sylow p-subgroup of G and A is a central subgroup of G.*

The next two theorems completely describe groups G with $V(FG)$ locally nilpotent. Some special cases of Theorem 1.2 were proved by Khripta (see [16]) and Ramezan-Nassab (see [19, Theorem 1.2] and [20, Corollary 1.3 and Theorem 1.4]).

THEOREM 1.2. *Let FG be a modular group algebra of a group G over the field F of positive characteristic p. The group $V(FG)$ is locally nilpotent if and only if G is locally nilpotent and G' is a p-group.*

THEOREM 1.3. *Let FG be a nonmodular group algebra of characteristic $p \geq 0$. The group $V(FG)$ is locally nilpotent if and only if G is a locally nilpotent group, the set $\mathfrak{t}(G)$ of elements of finite order of the group G is an abelian group and one of the following conditions holds:*

(i) *$\mathfrak{t}(G)$ is a central subgroup;*

(ii) *F is a prime field of characteristic $p = 2^t - 1$, the exponent of $\mathfrak{t}(G)$ divides $p^2 - 1$ and $g^{-1}ag = a^p$ for all $a \in \mathfrak{t}(G)$ and $g \in G \setminus C_G(\mathfrak{t}(G))$.*

2. Proofs

Note that in a group G the subset $t(G)$ of elements of finite order of G is not a group in the general case. A subgroup $H \leq G$ such that $H \subseteq t(G)$ is called a torsion subgroup of G.

We start our proof with the following important result.

Lemma 2.1. *Let FG be the group algebra of a group G such that $V(FG)$ is an Engel group. If H is a torsion subgroup of G and H does not contain an element of order* $\mathrm{char}(F)$*, then H is abelian, each subgroup of H is normal in G and each idempotent of FH is central in FG.*

Proof. If $a \in \mathrm{t}(G)$ and $g \in G \setminus N_G(\langle a \rangle)$, then $x = \widehat{a}g(a-1) \neq 0$ and $1 + x \in V(FG)$, where $\widehat{a} = 1 + a + \cdots + a^{|a|-1}$. A straightforward calculation shows that the triple commutator

$$(1 + x, a, m) = 1 + x(a-1)^m \quad (m \geq 1).$$

Since $V(FG)$ is an Engel group, $x(a-1)^s \neq 0$ and $x(a-1)^{s+1} = 0$ for a suitable $s \in \mathbb{N}$. It follows that $x(a-1)^s a^i = x(a-1)^s$ for each $i \geq 0$ and

$$x(a-1)^s \cdot |a| = x(a-1)^s(1 + a + a^2 + \cdots + a^{|a|-1}) = 0.$$

Since $\mathrm{char}(F)$ does not divide $|a|$, we have that $x(a-1)^s = 0$, which is a contradiction. Hence, every finite cyclic subgroup of H is normal in G, so H is either abelian or hamiltonian.

If H is a hamiltonian group, then, using the same proof as in the second part of [9, Lemma 1.1, page 122], we obtain a contradiction.

Hence, H is abelian and each of its subgroups is normal in G.

We claim that all idempotents of LT are central in $L\langle T, g\rangle$, where T is a finite abelian subgroup of H, $g \in G$ and L is a prime subfield of the field F.

Assume on the contrary that there exist a primitive idempotent $e \in LT$ and $g \in G$ such that $geg^{-1} \neq e$. The element $b = g^{-1}a^{-1}ga \neq 1$ for some $a \in T$ and

$$W = \langle c^{-1}(g^{-1}cg) \mid c \in T\rangle = \langle T, g\rangle' \lhd \langle T, g\rangle$$

is a nontrivial finite abelian subgroup of T.

Obviously each subgroup of $T \leq H$ is normal in $\langle T, g\rangle$, so the idempotent $f = (1/|W|)\widehat{\langle W\rangle} \in L[T]$ is central in $L\langle T, g\rangle$ and can be expressed as $f = f_1 + \cdots + f_s$ in which $f_1, \ldots, f_s$ are primitive and mutually orthogonal idempotents of the finite-dimensional semisimple algebra $L[T]$.

The idempotent e does not appear in the decomposition of f. Indeed, otherwise we have $ef = e$. If $e = \sum_i \alpha_i t_i$, where $\alpha_i \in L$ and $t_i \in T$, then

$$g^{-1}(ef)g = g^{-1}egf = \sum_i \alpha_i g^{-1} t_i g f = \sum_i \alpha_i t_i (t_i, g) f = ef,$$

so $e = fe$ is central, which is a contradiction.

If $ef \neq 0$, then $ef = e$, which again is a contradiction. Thus, $ef = 0$.

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Consider $f_* = (1/|b|)\widehat{\langle b\rangle} \in L[T]$. Since $\langle b\rangle \subseteq W$, the idempotent f_* appears in the decomposition of f, so $ef_* = 0$. Furthermore, $geg^{-1} \in L[T]$ is primitive as an automorphic image of the primitive idempotent e, so $ege = e(geg^{-1})g = 0$. Evidently $(1+eg)^{-1} = 1 - eg$ and

$$\begin{aligned}(1+eg, a) &= (1-eg)(1+ea^{-1}ga)\\ &= (1-eg)(1+egb) = 1 + eg(b-1).\end{aligned}$$

Now an easy induction on $n \geq 1$ shows that $(1+eg, a, n) = 1 + eg(b-1)^n$. However, $V(FG)$ is Engel, so there exists m such that

$$eg(b-1)^m \neq 0 \quad \text{and} \quad eg(b-1)^{m+1} = 0.$$

Thus, $eg(b-1)^m b^i = eg(b-1)^m$ for any $i \geq 0$, which leads to the contradiction

$$eg(b-1)^m = eg(b-1)^m f_* = eg(b-1)^{m-1}((b-1)f_*) = 0.$$

Therefore, all idempotents of FH are central in FG. □

Proof of Theorem 1.1. Assume that the set N_* of all nilpotent elements $FG \setminus 0$ is finite and nonempty. If $x \in N_*$, then the subset of nilpotent elements $\{\lambda x \mid \lambda \in F\}$ is finite, so F is a finite field of $\mathrm{char}(F) = p$.

Assume that there exist $u \in N_*$ and $g \in G$ such that $gu = ug$. Let us show that g has finite order. First notice that we cannot have $ug^i \neq ug^j$ whenever $i \neq j$ as this would imply that $\{ug^i | i \in \mathbb{Z}\}$ is an infinite subset of N_*, contradicting the assumptions of our theorem. Thus, we must have $ug^i = ug^j$ for some $i < j$. Let $b = g^{j-i}$, so $ub = u$. Let

$$u = \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_k g_k,$$

where $g_i \in G$ form a group basis of a vector space. We must have that the multiplication from the right by b permutes $g_1, g_2, \ldots, g_k$ and thus $b^{k!}$ acts trivially on g_1 and thus $1 = b^{k!} = g^{(i-j)\cdot k!}$ and g is of finite order.

Let $0 \neq u \in N_*$ and $g^k u \neq ug^k$ for any $g \in G \setminus \mathfrak{t}(G)$ and $k \in \mathbb{Z} \setminus \{0\}$. Clearly $g^{-i}ug^i \neq 0$ and $g^{-i}ug^i \in N_*$ for any $g \in G \setminus \mathfrak{t}(G)$ and $i \in \mathbb{Z}$. Since N_* is a finite set, at least for one $i \in \mathbb{Z}$ there exists $j \in \mathbb{Z}$ $(j \neq i)$ such that $g^{-i}ug^i = g^{-j}ug^j$, so $g^{j-i}u = ug^{j-i}$, which is a contradiction. Consequently $G = \mathfrak{t}(G)$, that is, G is a torsion group.

If $t(G)$ does not contain a p-element, then $\mathfrak{t}(G)(=G)$ is abelian by Lemma 2.1 and the commutative nonmodular algebra FG does not contain nonzero nilpotent elements. It follows that $N_* = \varnothing$, which is impossible.

By assumption, $V(FG)$ is an Engel group. Let Π be the set of all p-elements of the Engel group G. As $\{g - 1 \mid g \in \Pi\} \subseteq N_*$ and N_* is finite, Π is also finite. Let P be a maximal p-group such that $P \subseteq \Pi$ (such P always exists because Π is finite). Assume that P is not normal in G. Then $N_G(P) \neq G$. Since P is a finite p-group, it is nilpotent. Hence, some element of P must have a conjugate $w \in N_G(P) \setminus P$ (see [18, Lemma 5.4.1.3, page 307]). Consequently $P \lneqq \langle P, w\rangle$ and $\langle P, w\rangle$ is a finite p-group, which is a contradiction.

Hence, P is a normal subgroup in G and the subset of nilpotent elements

$$\{(h-1)g\lambda \mid h \in P, g \in G, \lambda \in F\}$$

is finite, so $G < V(FG)$ is a finite Engel group as well. Since every finite Engel group is nilpotent by Zorn's theorem (see [22, 12.3.4, page 372]), G is a finite nilpotent group which is a direct product of its Sylow subgroups (see [11, 10.3.4, page 176]). Each Sylow q-subgroup S_q ($q \neq p$) is abelian by Lemma 2.1. Consequently the finite group $G = P \times A$ in which $A = \bigcup_{q \neq p} S_q$ is a central subgroup of G and $G' \leq P$. Since FG is a finite algebra, $V(FG)$ is nilpotent (see [22, 12.3.4, page 372]). □

Proof of Theorem 1.2. Since $V(FG)$ is a locally nilpotent group, G is also locally nilpotent. Let $S = \langle f_1, \ldots, f_s \mid f_i \in V(FG)\rangle$ be a finitely generated (f.g.) subgroup of $V(FG)$. Clearly

$$H = \langle \mathrm{supp}(f_1), \ldots, \mathrm{supp}(f_s)\rangle$$

is an f.g. nilpotent subgroup of G and $S \leq V(FH) < V(FG)$. Hence, we may restrict our attention to the subgroup $V(FH)$, where H is an f.g. nilpotent subgroup of G containing a p-element; overwise we can add to H a p-element from G.

Let H be an f.g. nilpotent group containing a p-element and let $g, h \in H$ such that $(g, h) \neq 1$. Obviously $\mathfrak{t}(H)$ is finite (see [12, 7.7, page 29]) and the Sylow p-subgroup S_p of $\mathfrak{t}(H)$ is a direct factor of $\mathfrak{t}(H)$ (see [11, 10.3.4, page 176]). Consequently there exists $c \in \zeta(H)$ of order p, and $\widehat{c} = \sum_{i=0}^{p-1} c^i$ is a square-zero central element of FH. Since $L = \langle g, h, 1 + g\widehat{c}\rangle$ is an f.g. nilpotent subgroup of $V(FG)$, L is Engel. Clearly there exists $m \in \mathbb{N}$ such that the nilpotency class $\mathrm{cl}(V)$ of $V(FH)$ is at most p^m.

Let us show that (g, h) is a p-element. Indeed, if $q = p^m$, then

$$1 = (1 + g\widehat{c}, h, q) = 1 + \widehat{c}\sum_{i=0}^{q}(-1)^i\binom{q}{i}g^{h^{q-i}} = 1 + \widehat{c}(g^{h^q} - g)$$

because $\binom{q}{i} \equiv 0 \mod p$ for $0 < i < q$. It follows that $\widehat{c}(g^{h^q} - g) = 0$ and $(g, h^q) \in \mathrm{supp}(\widehat{c})$, which yields that $g^{h^q} = c^i g$ for some $0 \leq i < p$. Hence,

$$(\underbrace{h^{-q}\cdots h^{-q}}_{p})g(\underbrace{h^q \cdots h^q}_{p}) = (c^i)^p g = g$$

and $h^{p^{m+1}} \in C_G(g)$, so $h^{p^{m+1}} \in \zeta(L)$ for all $h \in L$ and L' is a finite p-group by a theorem of Schur (see [22, 10.1.4, page 287]). Since each f.g. subgroup of G is a p-group, G' is a p-group too.

Conversely, it is sufficient to prove that if H is an f.g. nilpotent subgroup of G such that H' is a p-group, then $V(FH)$ is nilpotent. All subgroups and factor groups of H are also f.g. groups [22, 5.2.17, page 137]. Moreover $\mathfrak{t}(H) = P \times D$ is finite (see [22, 12.1.1, page 356]) and the f.g. group $H/\mathfrak{t}(H)$ is a direct product of a finite number of infinite cyclic groups. Since H' is a p-group, $H' \leq P \leq \mathfrak{t}(H)$ is a finite p-group, so $V(FH)$ is nilpotent (see [15, Theorem 1]). □

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Proof of Theorem 1.3. If $V(FG)$ is a locally nilpotent group, then it is Engel, so, by [5, Theorem 1.1], the locally nilpotent group G satisfies parts (i)–(ii) of our theorem. Since a locally nilpotent group is a u.p. group, the converse of our theorem follows from [16, Theorem 2] and [5, Theorem 1.1]. □

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